

Question 9

Let $a > 0$ be a real constant and define the function

$$f(x) = \ln\left(\frac{e^{ax} + 1}{e^{ax} - 1}\right).$$

- (a) Find the derivative $f'(x)$ in terms of a .
- (b) Simplify your answer as much as possible.
- (c) Determine the domain of $f(x)$.
- (d) Show that $f'(x)$ can be written in the form

$$f'(x) = -\frac{2a}{e^{ax} - e^{-ax}}.$$

- (e) Hence, or otherwise, determine whether $f(x)$ is increasing or decreasing on its domain.

Question 9 - Solution

We are given

$$f(x) = \ln\left(\frac{e^{ax} + 1}{e^{ax} - 1}\right), \quad a > 0.$$

(a) Differentiation

Rewrite the function using logarithmic identities:

$$f(x) = \ln(e^{ax} + 1) - \ln(e^{ax} - 1).$$

Differentiate term by term using the chain rule:

$$f'(x) = \frac{ae^{ax}}{e^{ax} + 1} - \frac{ae^{ax}}{e^{ax} - 1}.$$

(b) Simplification

Factor out ae^{ax} :

$$f'(x) = ae^{ax} \left(\frac{1}{e^{ax} + 1} - \frac{1}{e^{ax} - 1} \right).$$

Combine the fractions:

$$= ae^{ax} \left(\frac{(e^{ax} - 1) - (e^{ax} + 1)}{(e^{ax} + 1)(e^{ax} - 1)} \right) = ae^{ax} \left(\frac{-2}{e^{2ax} - 1} \right).$$

Thus,

$$f'(x) = -\frac{2ae^{ax}}{e^{2ax} - 1}.$$

(c) Domain of $f(x)$

The logarithm requires:

$$\frac{e^{ax} + 1}{e^{ax} - 1} > 0, \quad e^{ax} - 1 \neq 0.$$

Since $e^{ax} > 0$ for all x , the expression is positive only when

$$e^{ax} - 1 > 0 \quad \Rightarrow \quad x > 0.$$

Hence, the domain is:

$$\boxed{(0, \infty)}.$$

(d) Alternative Form

Multiply numerator and denominator of the derivative by e^{-ax} :

$$f'(x) = -\frac{2ae^{ax}}{e^{2ax} - 1} = -\frac{2a}{e^{ax} - e^{-ax}}.$$

(e) Monotonicity

For $x > 0$,

$$e^{ax} - e^{-ax} > 0, \quad a > 0.$$

Therefore,

$$f'(x) < 0 \quad \text{for all } x > 0.$$

The function $f(x)$ is strictly decreasing on its domain.