

Question 4

Let the function $f(x) = e^{x \ln(\sin x)}$, for $0 < x < \pi$.

- (a) Simplify the expression $f(x)$.
- (b) Find the derivative $f'(x)$.

Question 4 - Solution

We are given:

$$f(x) = e^{x \ln(\sin x)}, \quad 0 < x < \pi$$

(a) Simplify the expression:

Recall the identity:

$$e^{x \ln(\sin x)} = (e^{\ln(\sin x)})^x = (\sin x)^x$$

So:

$$f(x) = (\sin x)^x$$

(b) Differentiate $f(x) = (\sin x)^x$ using logarithmic differentiation:

Take natural log on both sides:

$$\ln(f(x)) = x \ln(\sin x)$$

Differentiate both sides:

$$\frac{1}{f(x)} \cdot f'(x) = \frac{d}{dx}[x \ln(\sin x)]$$

Apply product rule on the right-hand side:

$$\frac{d}{dx}[x \ln(\sin x)] = \ln(\sin x) + x \cdot \frac{\cos x}{\sin x} = \ln(\sin x) + x \cot x$$

Multiply both sides by $f(x) = (\sin x)^x$:

$$f'(x) = (\sin x)^x \cdot (\ln(\sin x) + x \cot x)$$

Final Answer:

$$\boxed{f'(x) = (\sin x)^x (\ln(\sin x) + x \cot x)}$$