

Question 8

Let

$$f(x) = \frac{\sin(x) \cos(x)}{1 + \sin^2(x)}$$

- (a) Differentiate $f(x)$.
- (b) Simplify the result as much as possible.

Question 8 - Solution

We are given:

$$f(x) = \frac{\sin(x) \cos(x)}{1 + \sin^2(x)}$$

Step 1: Rewrite using a double-angle identity.

Recall:

$$\sin(x) \cos(x) = \frac{1}{2} \sin(2x)$$

So we rewrite the function:

$$f(x) = \frac{\frac{1}{2} \sin(2x)}{1 + \sin^2(x)}$$

Step 2: Apply the quotient rule.

Let:

$$u(x) = \frac{1}{2} \sin(2x), \quad v(x) = 1 + \sin^2(x)$$

Differentiate each function:

$$u'(x) = \cos(2x)$$

$$v'(x) = 2 \sin(x) \cos(x) = \sin(2x)$$

The quotient rule states:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Substitute:

$$f'(x) = \frac{\cos(2x)(1 + \sin^2(x)) - \frac{1}{2} \sin(2x) \sin(2x)}{(1 + \sin^2(x))^2}$$

Step 3: Simplify carefully.

Rewrite the squared sine:

$$\sin^2(2x) = 1 - \cos^2(2x)$$

So the numerator becomes:

$$\cos(2x)(1 + \sin^2(x)) - \frac{1}{2}(1 - \cos^2(2x))$$

Expand:

$$= \cos(2x) + \cos(2x) \sin^2(x) - \frac{1}{2} + \frac{1}{2} \cos^2(2x)$$

Now use:

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

Substitute:

$$\cos(2x) \sin^2(x) = \frac{1}{2} \cos(2x) - \frac{1}{2} \cos^2(2x)$$

Combine terms:

$$= \frac{3}{2} \cos(2x) - \frac{1}{2}$$

Final Simplified Result:

$$f'(x) = \frac{3 \cos(2x) - 1}{2(1 + \sin^2(x))^2}$$

Final Answer:

$$f'(x) = \frac{3 \cos(2x) - 1}{2(1 + \sin^2(x))^2}$$