

Question 10

Let

$$f(x) = \frac{x^2 e^x \sin(x)}{x^2 + 1}$$

- (a) Differentiate $f(x)$ using appropriate differentiation rules.
- (b) Simplify and factor your answer as much as possible.

Question 10 - Solution

We are given:

$$f(x) = \frac{x^2 e^x \sin(x)}{x^2 + 1}$$

This is a quotient of two functions:

$$u(x) = x^2 e^x \sin(x), \quad v(x) = x^2 + 1$$

We apply the **quotient rule**:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Step 1: Differentiate the numerator

The numerator is a product of three functions:

$$u(x) = x^2 \cdot e^x \cdot \sin(x)$$

Using the product rule for three functions:

$$u'(x) = (x^2)' e^x \sin(x) + x^2 (e^x)' \sin(x) + x^2 e^x (\sin(x))'$$

Compute derivatives:

$$(x^2)' = 2x, \quad (e^x)' = e^x, \quad (\sin x)' = \cos x$$

Substitute:

$$u'(x) = 2x e^x \sin(x) + x^2 e^x \sin(x) + x^2 e^x \cos(x)$$

Factor:

$$u'(x) = e^x [\sin(x)(2x + x^2) + x^2 \cos(x)]$$

Step 2: Differentiate the denominator

$$v'(x) = 2x$$

Step 3: Apply the quotient rule

$$f'(x) = \frac{e^x [\sin(x)(2x + x^2) + x^2 \cos(x)] (x^2 + 1) - x^2 e^x \sin(x)(2x)}{(x^2 + 1)^2}$$

Step 4: Simplify

Factor out e^x :

$$f'(x) = \frac{e^x \{ (x^2 + 1) [\sin(x)(2x + x^2) + x^2 \cos(x)] - 2x^3 \sin(x) \}}{(x^2 + 1)^2}$$

This is a fully factored and simplified form.

Final Answer:

$$f'(x) = \frac{e^x [(x^2 + 1) (\sin(x)(2x + x^2) + x^2 \cos(x)) - 2x^3 \sin(x)]}{(x^2 + 1)^2}$$