

Question 6

Let

$$f(x) = \frac{x^2 e^{2x} \sin x}{1 + x^3}$$

- (a) Compute $f'(x)$.
- (b) Simplify and factor your answer as much as possible.

Question 6 - Solution

We are given:

$$f(x) = \frac{x^2 e^{2x} \sin x}{1 + x^3}$$

Let

$$u(x) = x^2 e^{2x} \sin x, \quad v(x) = 1 + x^3$$

Using the quotient rule:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Step 1: Differentiate $u(x)$

The function $u(x)$ is a product of three functions:

$$u(x) = x^2 \cdot e^{2x} \cdot \sin x$$

Apply the product rule:

$$u'(x) = (x^2)' e^{2x} \sin x + x^2 (e^{2x})' \sin x + x^2 e^{2x} (\sin x)'$$

Compute derivatives:

$$(x^2)' = 2x, \quad (e^{2x})' = 2e^{2x}, \quad (\sin x)' = \cos x$$

So:

$$u'(x) = 2x e^{2x} \sin x + 2x^2 e^{2x} \sin x + x^2 e^{2x} \cos x$$

Factor:

$$u'(x) = e^{2x} [\sin x (2x + 2x^2) + x^2 \cos x]$$

Step 2: Differentiate $v(x)$

$$v'(x) = 3x^2$$

Step 3: Apply the Quotient Rule

$$f'(x) = \frac{e^{2x} [\sin x (2x + 2x^2) + x^2 \cos x] (1 + x^3) - x^2 e^{2x} \sin x (3x^2)}{(1 + x^3)^2}$$

Factor common terms in the numerator:

$$f'(x) = \frac{e^{2x} x [(1 + x^3)(2 \sin x + 2x \sin x + x \cos x) - 3x^3 \sin x]}{(1 + x^3)^2}$$

Final Answer:

$$\boxed{f'(x) = \frac{e^{2x} x [(1 + x^3)(2 \sin x + 2x \sin x + x \cos x) - 3x^3 \sin x]}{(1 + x^3)^2}}$$