

Question 5

Let

$$f(x) = \frac{\ln(x)}{x^3}$$

- (a) Find the derivative $f'(x)$ using the quotient rule.
- (b) Determine any local extrema of $f(x)$ for $x > 0$.

Question 5 - Solution

We are given:

$$f(x) = \frac{\ln(x)}{x^3}$$

(a) Find $f'(x)$:

Use the quotient rule:

$$f'(x) = \frac{x^3 \cdot \frac{1}{x} - \ln(x) \cdot 3x^2}{x^6} = \frac{x^2 - 3x^2 \ln(x)}{x^6}$$

Factor the numerator:

$$f'(x) = \frac{x^2(1 - 3\ln(x))}{x^6} = \frac{1 - 3\ln(x)}{x^4}$$

So,

$$f'(x) = \boxed{\frac{1 - 3\ln(x)}{x^4}}$$

(b) Local Extrema:

Critical points occur where $f'(x) = 0$:

$$\frac{1 - 3\ln(x)}{x^4} = 0 \Rightarrow 1 - 3\ln(x) = 0 \Rightarrow \ln(x) = \frac{1}{3} \Rightarrow x = e^{1/3}$$

Now determine if this is a maximum or minimum by checking the sign of $f'(x)$ around $x = e^{1/3}$:

For $x < e^{1/3}$, say $x = 1$: $f'(1) = \frac{1 - 3\ln(1)}{1^4} = 1 > 0$

For $x > e^{1/3}$, say $x = 2$: $f'(2) = \frac{1 - 3\ln(2)}{2^4} \approx \frac{1 - 3(0.693)}{16} \approx \frac{-1.079}{16} < 0$

So $f(x)$ has a local maximum at $x = e^{1/3}$.

Conclusion:

$f'(x) = \frac{1 - 3\ln(x)}{x^4}$, Local maximum at $x = \boxed{e^{1/3}}$