

Question 10

Let

$$f(x) = \ln(1 + x^2).$$

- (a) Show that all odd-order derivatives of $f(x)$ evaluated at $x = 0$ are zero.
- (b) Find a closed-form expression for the $2n$ -th derivative of $f(x)$ evaluated at $x = 0$, where $n \in \mathbb{N}$.
- (c) Using your result from part (b), write the Maclaurin series for $f(x)$ in summation notation.
- (d) Determine the radius of convergence of the Maclaurin series and justify your answer.
- (e) Use the Maclaurin series to approximate $\ln(1.1)$ to within an error of 10^{-4} , and explain how you control the error.

Question 10 - Solution

We are given

$$f(x) = \ln(1 + x^2).$$

(a) Odd-order derivatives at $x = 0$

Observe that $f(x)$ is an **even function**, since

$$f(-x) = \ln(1 + (-x)^2) = \ln(1 + x^2) = f(x).$$

The derivative of an even function is odd, and the derivative of an odd function is even. Therefore:

- $f'(x)$ is odd,
- $f''(x)$ is even,
- $f^{(3)}(x)$ is odd,
- and so on.

All odd functions evaluate to zero at $x = 0$. Hence,

$$f^{(2k+1)}(0) = 0 \quad \text{for all } k \geq 0.$$

(b) Closed form for $f^{(2n)}(0)$

We begin with the known Taylor expansion:

$$\ln(1 + u) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{u^n}{n}, \quad |u| < 1.$$

Substitute $u = x^2$:

$$\ln(1 + x^2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}.$$

Comparing this with the Maclaurin series definition:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

we see that only even powers appear. Matching coefficients of x^{2n} , we obtain:

$$\frac{f^{(2n)}(0)}{(2n)!} = (-1)^{n+1} \frac{1}{n}.$$

Solving for $f^{(2n)}(0)$:

$$f^{(2n)}(0) = (-1)^{n+1} \frac{(2n)!}{n}.$$

(c) Maclaurin series

Using the result above, the Maclaurin series for $f(x)$ is:

$$\ln(1 + x^2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}, \quad |x| < 1.$$

(d) Radius of convergence

The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}$$

is a power series in x^2 .

Using the Ratio Test:

$$\lim_{n \rightarrow \infty} \left| \frac{x^{2(n+1)}/(n+1)}{x^{2n}/n} \right| = |x|^2.$$

The series converges when:

$$|x|^2 < 1 \quad \Rightarrow \quad |x| < 1.$$

Thus, the radius of convergence is:

$$\boxed{R = 1.}$$

(e) Approximating $\ln(1.1)$

We write:

$$\ln(1.1) = \ln(1 + 0.1) = \ln(1 + x^2) \quad \text{with } x^2 = 0.1.$$

Using the alternating series:

$$\ln(1.1) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.1)^n}{n}.$$

Since this is an alternating series with decreasing terms, the error after N terms is bounded by the magnitude of the next term:

$$|R_N| \leq \frac{(0.1)^{N+1}}{N+1}.$$

We test values:

$$\frac{(0.1)^3}{3} \approx 0.00033 > 10^{-4}, \quad \frac{(0.1)^4}{4} = 0.000025 < 10^{-4}.$$

Thus, using the first four terms guarantees the desired accuracy.

$$\boxed{\ln(1.1) \approx 0.1 - \frac{0.1^2}{2} + \frac{0.1^3}{3} - \frac{0.1^4}{4}}$$

