

Question 7

Let $f(x) = x^2e^x$.

- (a) Find the first three derivatives of $f(x)$.
- (b) Find a general expression for $f^{(n)}(x)$.
- (c) Evaluate $f^{(3)}(0)$.

Question 7 - Solution

We are given:

$$f(x) = x^2 e^x$$

(a) Compute Derivatives:

We use the product rule: If $u = x^2$, $v = e^x$, then $f = uv$

$$f'(x) = (x^2)'e^x + x^2(e^x) = 2xe^x + x^2e^x = e^x(2x + x^2)$$

$$f''(x) = \frac{d}{dx}[e^x(2x + x^2)] = e^x(2 + 2x + 2x + x^2) = e^x(x^2 + 4x + 2)$$

$$f^{(3)}(x) = \frac{d}{dx}[e^x(x^2 + 4x + 2)] = e^x(x^2 + 4x + 2 + 2x + 4 + 0) = e^x(x^2 + 6x + 6)$$

(b) General Formula:

We notice a pattern in each derivative:

$$f^{(n)}(x) = e^x \cdot P_n(x)$$

where $P_n(x)$ is a degree-2 polynomial (since the original function had x^2).

This suggests:

$$\boxed{f^{(n)}(x) = e^x \cdot (x^2 + 2nx + n(n-1))}$$

(This can be proven by induction.)

(c) Evaluate $f^{(3)}(0)$:

From (a):

$$f^{(3)}(0) = e^0(0^2 + 6 \cdot 0 + 6) = 1 \cdot 6 = \boxed{6}$$