

Question 6

Let $f(x) = \frac{1}{1-x}$.

- (a) Find the first four derivatives $f'(x), f''(x), f^{(3)}(x), f^{(4)}(x)$.
- (b) Find a general formula for $f^{(n)}(x)$.
- (c) Evaluate $f^{(n)}(0)$ and write the Taylor series for $f(x)$ centered at $x = 0$.

Question 6 - Solution

We are given:

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

(a) Compute derivatives:

Using the chain rule,

$$\frac{d}{dx}(1-x)^{-1} = -1(1-x)^{-2} \cdot (-1) = (1-x)^{-2}$$

$$f'(x) = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{d}{dx}(1-x)^{-2} = -2(1-x)^{-3} \cdot (-1) = \frac{2}{(1-x)^3}$$

$$f^{(3)}(x) = \frac{d}{dx} \left(\frac{2}{(1-x)^3} \right) = 2 \cdot 3(1-x)^{-4} = \frac{6}{(1-x)^4}$$

$$f^{(4)}(x) = \frac{d}{dx} \left(\frac{6}{(1-x)^4} \right) = 6 \cdot 4(1-x)^{-5} = \frac{24}{(1-x)^5}$$

(b) General formula:

Each derivative introduces a factorial factor and increases the power of the denominator by 1:

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$$

(c) Evaluate at $x = 0$:

$$f^{(n)}(0) = \frac{n!}{1^{n+1}} = n!$$

Thus the Taylor series centered at $x = 0$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} x^n$$

$$f(x) = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$