

**Question 5**

Let  $f(x) = \sin(x^2)$ .

- (a) Find  $f'(x)$ ,  $f''(x)$ , and  $f^{(3)}(x)$ .
- (b) Compute  $f''(0)$ .
- (c) Use your results to write the second-degree Taylor polynomial of  $f(x)$  centered at  $x = 0$ .

**Question 5 - Solution**

We are given:

$$f(x) = \sin(x^2)$$

**(a) First three derivatives:**

**First derivative:**

$$f'(x) = \frac{d}{dx} \sin(x^2) = \cos(x^2) \cdot 2x = 2x \cos(x^2)$$

**Second derivative:** Use product rule:

$$f''(x) = \frac{d}{dx} (2x \cos(x^2)) = 2 \cos(x^2) + 2x \cdot (-\sin(x^2)) \cdot 2x = 2 \cos(x^2) - 4x^2 \sin(x^2)$$

**Third derivative:** Differentiate  $f''(x)$ :

$$f^{(3)}(x) = \frac{d}{dx} (2 \cos(x^2) - 4x^2 \sin(x^2))$$

First term:

$$\frac{d}{dx} [2 \cos(x^2)] = -2 \sin(x^2) \cdot 2x = -4x \sin(x^2)$$

Second term (product rule):

$$\frac{d}{dx} [4x^2 \sin(x^2)] = 8x \sin(x^2) + 4x^2 \cdot \cos(x^2) \cdot 2x = 8x \sin(x^2) + 8x^3 \cos(x^2)$$

So:

$$f^{(3)}(x) = -4x \sin(x^2) - (8x \sin(x^2) + 8x^3 \cos(x^2)) = -12x \sin(x^2) - 8x^3 \cos(x^2)$$

**(b) Compute  $f''(0)$ :**

$$f''(x) = 2 \cos(x^2) - 4x^2 \sin(x^2) \Rightarrow f''(0) = 2 \cos(0) - 0 = 2$$

**(c) Taylor Polynomial  $T_2(x)$ :**

We use:

$$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

$$f(0) = \sin(0) = 0, \quad f'(0) = 2(0) \cos(0) = 0, \quad f''(0) = 2$$

So:

$$T_2(x) = 0 + 0 + \frac{2}{2}x^2 = x^2$$

$$\boxed{T_2(x) = x^2}$$