

Question 4

Let $f(x) = e^x \cos(x)$.

- (a) Compute the first three derivatives: $f'(x)$, $f''(x)$, and $f^{(3)}(x)$.
- (b) Find the value of $f''(0)$.
- (c) Determine the third-degree Taylor polynomial of $f(x)$ centered at $x = 0$.

Question 4 - Solution

We are given:

$$f(x) = e^x \cos(x)$$

(a) First three derivatives:

Use product rule: $f(x) = u(x)v(x)$, where $u = e^x$, $v = \cos(x)$

First derivative:

$$f'(x) = u'v + uv' = e^x \cos(x) - e^x \sin(x) = e^x(\cos(x) - \sin(x))$$

Second derivative:

$$\begin{aligned} f''(x) &= \frac{d}{dx}[e^x(\cos(x) - \sin(x))] = e^x(\cos(x) - \sin(x)) + e^x(-\sin(x) - \cos(x)) \\ &= e^x[(\cos(x) - \sin(x)) + (-\sin(x) - \cos(x))] = e^x(-2\sin(x)) \end{aligned}$$

Third derivative:

$$f^{(3)}(x) = \frac{d}{dx}[e^x(-2\sin(x))] = e^x(-2\sin(x)) + e^x(-2\cos(x)) = -2e^x(\sin(x) + \cos(x))$$

(b) Evaluate $f''(0)$:

$$f''(0) = e^0(-2\sin(0)) = 1 \cdot 0 = \boxed{0}$$

(c) Taylor polynomial of degree 3 at $x = 0$:

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3$$

Compute values:

$$f(0) = e^0 \cos(0) = 1, \quad f'(0) = e^0(\cos(0) - \sin(0)) = 1, \quad f''(0) = 0$$

$$f^{(3)}(0) = -2e^0(\sin(0) + \cos(0)) = -2(0 + 1) = -2$$

So:

$$T_3(x) = 1 + x + 0 + \frac{-2}{6}x^3 = 1 + x - \frac{1}{3}x^3$$

$$\boxed{T_3(x) = 1 + x - \frac{1}{3}x^3}$$