

Question 3

Let $f(x) = \ln(x^2 + 1)$.

- (a) Compute the first three derivatives of $f(x)$.
- (b) Determine the value of $f^{(3)}(0)$.
- (c) Based on the derivatives at $x = 0$, write the third-degree Taylor polynomial centered at $x = 0$.

Question 3 - Solution

We are given:

$$f(x) = \ln(x^2 + 1)$$

(a) First three derivatives:

First derivative:

$$f'(x) = \frac{d}{dx} \ln(x^2 + 1) = \frac{2x}{x^2 + 1}$$

Second derivative:

$$f''(x) = \frac{d}{dx} \left(\frac{2x}{x^2 + 1} \right) = \frac{2(x^2 + 1) - 2x(2x)}{(x^2 + 1)^2} = \frac{2(x^2 + 1 - 2x^2)}{(x^2 + 1)^2} = \frac{2(1 - x^2)}{(x^2 + 1)^2}$$

Third derivative:

We apply the quotient rule to:

$$f''(x) = \frac{2(1 - x^2)}{(x^2 + 1)^2}$$

Let's compute:

$$f^{(3)}(x) = \frac{d}{dx} \left(\frac{2(1 - x^2)}{(x^2 + 1)^2} \right)$$

Numerator derivative:

$$\frac{d}{dx} [2(1 - x^2)] = -4x$$

Denominator derivative:

$$\frac{d}{dx} [(x^2 + 1)^2] = 2(x^2 + 1)(2x) = 4x(x^2 + 1)$$

Now apply quotient rule:

$$f^{(3)}(x) = \frac{(-4x)(x^2 + 1)^2 - 2(1 - x^2)(4x)(x^2 + 1)}{(x^2 + 1)^4}$$

Factor out $4x(x^2 + 1)$:

$$f^{(3)}(x) = \frac{-4x(x^2 + 1) [(x^2 + 1) + 2(1 - x^2)]}{(x^2 + 1)^4}$$

Simplify:

$$(x^2 + 1) + 2(1 - x^2) = x^2 + 1 + 2 - 2x^2 = 3 - x^2$$

So:

$$f^{(3)}(x) = \frac{-4x(x^2 + 1)(3 - x^2)}{(x^2 + 1)^4} = \frac{-4x(3 - x^2)}{(x^2 + 1)^3}$$

(b) Evaluate at $x = 0$:

$$f(0) = \ln(1) = 0, \quad f'(0) = 0, \quad f''(0) = 2, \quad f^{(3)}(0) = \frac{-4(0)(3 - 0)}{1^3} = 0$$

(c) Taylor polynomial of degree 3 at $x = 0$:

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 = 0 + 0 + \frac{2}{2}x^2 + 0 = x^2$$

$$\boxed{T_3(x) = x^2}$$