

Question 2

Let $f(x) = x^2e^{-x}$.

- (a) Compute the first four derivatives of $f(x)$.
- (b) Identify a pattern in the derivatives.
- (c) Use the pattern to write a general formula for $f^{(n)}(x)$ for $n \geq 0$.

Question 2 - Solution

We are given $f(x) = x^2e^{-x}$. Let's compute derivatives step-by-step.

(a) First, compute the derivatives:

First derivative:

$$f'(x) = \frac{d}{dx}(x^2e^{-x}) = 2xe^{-x} + x^2(-e^{-x}) = (2x - x^2)e^{-x}$$

Second derivative:

$$f''(x) = \frac{d}{dx}[(2x - x^2)e^{-x}] = [2 - 2x]e^{-x} + (2x - x^2)(-e^{-x}) = [2 - 2x - 2x + x^2]e^{-x} = (x^2 - 4x + 2)e^{-x}$$

Third derivative:

$$f^{(3)}(x) = \frac{d}{dx}[(x^2 - 4x + 2)e^{-x}] = (2x - 4)e^{-x} + (x^2 - 4x + 2)(-e^{-x}) = (-x^2 + 6x - 6)e^{-x}$$

Fourth derivative:

$$f^{(4)}(x) = \frac{d}{dx}[(-x^2 + 6x - 6)e^{-x}] = (-2x + 6)e^{-x} + (-x^2 + 6x - 6)(-e^{-x}) = (x^2 - 8x + 12)e^{-x}$$

(b) Pattern:

Each derivative is a polynomial times e^{-x} . The polynomials alternate in sign and follow a recursive-like behavior:

$$\begin{aligned}f(x) &= x^2e^{-x} \\f'(x) &= (2x - x^2)e^{-x} \\f''(x) &= (x^2 - 4x + 2)e^{-x} \\f^{(3)}(x) &= (-x^2 + 6x - 6)e^{-x} \\f^{(4)}(x) &= (x^2 - 8x + 12)e^{-x}\end{aligned}$$

(c) General formula (structure):

Let:

$$f^{(n)}(x) = P_n(x)e^{-x}$$

Where $P_n(x)$ is a polynomial of degree 2, alternating in sign:

$$P_n(x) = (-1)^n(x^2 - 2nx + n(n-1))$$

Final boxed result:

$$\boxed{f^{(n)}(x) = (-1)^n(x^2 - 2nx + n(n-1))e^{-x}}$$