

Question 9

Let

$$f(x) = \sqrt{x^2 + 4x + 5}$$

Task: Use the *definition of the derivative* to find $f'(x)$.

- Use first principles only.
- Do not apply derivative shortcuts.
- Show all algebraic steps.

Question 9 - Solution

We are given:

$$f(x) = \sqrt{x^2 + 4x + 5}$$

Apply the definition of the derivative:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)^2 + 4(x+h) + 5} - \sqrt{x^2 + 4x + 5}}{h}$$

Expand the expression inside the first square root:

$$(x+h)^2 + 4(x+h) + 5 = x^2 + 2xh + h^2 + 4x + 4h + 5$$

So we have:

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} - \sqrt{x^2 + 4x + 5}}{h}$$

Multiply numerator and denominator by the conjugate:

$$= \lim_{h \rightarrow 0} \frac{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} - \sqrt{x^2 + 4x + 5}}{h} \cdot \frac{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5}}{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5}}$$

Use the identity $(a-b)(a+b) = a^2 - b^2$:

$$= \lim_{h \rightarrow 0} \frac{(x^2 + 2xh + h^2 + 4x + 4h + 5) - (x^2 + 4x + 5)}{h (\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5})}$$

Simplify the numerator:

$$= \lim_{h \rightarrow 0} \frac{2xh + h^2 + 4h}{h (\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5})}$$

Factor h from the numerator:

$$= \lim_{h \rightarrow 0} \frac{h(2x + h + 4)}{h (\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5})}$$

Cancel h :

$$= \lim_{h \rightarrow 0} \frac{2x + h + 4}{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5}}$$

Now take the limit as $h \rightarrow 0$:

$$f'(x) = \frac{2x + 4}{2\sqrt{x^2 + 4x + 5}}$$

Final Answer:

$$f'(x) = \boxed{\frac{2x + 4}{2\sqrt{x^2 + 4x + 5}}}$$