

Question 10

Let

$$f(x) = \begin{cases} \frac{\sin(3x)}{x}, & x \neq 0 \\ a, & x = 0 \end{cases}$$

- (a) Find the value of a such that $f(x)$ is continuous at $x = 0$.
- (b) Is $f(x)$ differentiable at $x = 0$? Justify your answer.

Question 10 - Solution

(a) For continuity at $x = 0$, we must have

$$\lim_{x \rightarrow 0} f(x) = f(0) = a.$$

We compute the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = \lim_{x \rightarrow 0} 3 \cdot \frac{\sin(3x)}{3x} = 3 \cdot 1 = 3.$$

Thus, to make $f(x)$ continuous at $x = 0$, we must set

$$\boxed{a = 3}.$$

(b) Differentiability at $x = 0$:

We use the definition of the derivative:

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{\sin(3h)}{h} - 3}{h}.$$

Rewrite the expression:

$$\frac{\frac{\sin(3h)}{h} - 3}{h} = \frac{\sin(3h) - 3h}{h^2}.$$

Rescale by $u = 3h$ (so $u^2 = 9h^2$):

$$\frac{\sin(3h) - 3h}{h^2} = 9 \cdot \frac{\sin(3h) - 3h}{(3h)^2}.$$

Let $u = 3h$. Then as $h \rightarrow 0$, $u \rightarrow 0$, and the limit becomes

$$f'(0) = 9 \lim_{u \rightarrow 0} \frac{\sin u - u}{u^2}.$$

Using the standard limit

$$\lim_{u \rightarrow 0} \frac{\sin u - u}{u^2} = 0,$$

we conclude that

$$f'(0) = 0.$$

Conclusion: The limit exists, so $f(x)$ is differentiable at $x = 0$, and

$$\boxed{f'(0) = 0}.$$