

Question 1

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} kx^2 + 3x + 1, & \text{if } x < 2 \\ x^3 - 3kx + c, & \text{if } x \geq 2 \end{cases}$$

Find the values of constants k and c such that $f(x)$ is continuous and differentiable at $x = 2$.

Question 1 - Solution

We are given:

$$f(x) = \begin{cases} kx^2 + 3x + 1, & \text{if } x < 2 \\ x^3 - 3kx + c, & \text{if } x \geq 2 \end{cases}$$

We want $f(x)$ to be both:

1. Continuous at $x = 2$
2. Differentiable at $x = 2$

Step 1: Continuity at $x = 2$

For continuity, we need:

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

Left-hand limit:

$$\lim_{x \rightarrow 2^-} f(x) = k(2)^2 + 3(2) + 1 = 4k + 6 + 1 = 4k + 7$$

Right-hand limit:

$$\lim_{x \rightarrow 2^+} f(x) = (2)^3 - 3k(2) + c = 8 - 6k + c$$

Set them equal:

$$4k + 7 = 8 - 6k + c \quad (1)$$

Step 2: Differentiability at $x = 2$

Derivatives:

Left: $f'(x) = 2kx + 3 \Rightarrow f'(2^-) = 4k + 3$

Right: $f'(x) = 3x^2 - 3k \Rightarrow f'(2^+) = 12 - 3k$

Set equal:

$$4k + 3 = 12 - 3k \quad (2)$$

Step 3: Solve the system

From (2):

$$4k + 3 = 12 - 3k \Rightarrow 7k = 9 \Rightarrow k = \frac{9}{7}$$

Substitute into (1):

$$4\left(\frac{9}{7}\right) + 7 = 8 - 6\left(\frac{9}{7}\right) + c \Rightarrow \frac{36}{7} + 7 = 8 - \frac{54}{7} + c \Rightarrow \frac{85}{7} = \frac{2}{7} + c \Rightarrow c = \frac{83}{7}$$

Final Answer:

$k = \frac{9}{7}, \quad c = \frac{83}{7}$
