

Question 8

Evaluate the limit:

$$\lim_{x \rightarrow \infty} x^2 \left(\sqrt{x^2 + 4x} - x \right).$$

Question 8 - Solution

We evaluate

$$\lim_{x \rightarrow \infty} x^2 \left(\sqrt{x^2 + 4x} - x \right).$$

As $x \rightarrow \infty$, both $\sqrt{x^2 + 4x}$ and x approach infinity, so the expression inside the parentheses has the indeterminate form $\infty - \infty$. Multiplying by x^2 significantly changes the behavior of the limit, so careful analysis is required.

Step 1: Rationalize the Expression

We first rationalize the difference:

$$\sqrt{x^2 + 4x} - x = \frac{(\sqrt{x^2 + 4x} - x)(\sqrt{x^2 + 4x} + x)}{\sqrt{x^2 + 4x} + x}.$$

Using the identity $(a - b)(a + b) = a^2 - b^2$, this becomes

$$= \frac{(x^2 + 4x) - x^2}{\sqrt{x^2 + 4x} + x} = \frac{4x}{\sqrt{x^2 + 4x} + x}.$$

Substitute into the original expression:

$$\lim_{x \rightarrow \infty} x^2 \cdot \frac{4x}{\sqrt{x^2 + 4x} + x} = \lim_{x \rightarrow \infty} \frac{4x^3}{\sqrt{x^2 + 4x} + x}.$$

Step 2: Simplify

Factor x out of the square root:

$$\sqrt{x^2 + 4x} = x\sqrt{1 + \frac{4}{x}} \quad (x > 0).$$

Then

$$\frac{4x^3}{x\left(\sqrt{1 + \frac{4}{x}} + 1\right)} = \frac{4x^2}{\sqrt{1 + \frac{4}{x}} + 1}.$$

Step 3: Take the Limit

As $x \rightarrow \infty$, $\frac{4}{x} \rightarrow 0$, so

$$\sqrt{1 + \frac{4}{x}} \rightarrow 1.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{4x^2}{\sqrt{1 + \frac{4}{x}} + 1} = \lim_{x \rightarrow \infty} \frac{4x^2}{2} = \infty.$$

Final Answer

The limit diverges to ∞ .