

Question 7

Evaluate the limit:

$$\lim_{x \rightarrow \infty} x \left(\sqrt{x^2 + 5x} - x \right).$$

Question 7 - Solution

We evaluate

$$\lim_{x \rightarrow \infty} x \left(\sqrt{x^2 + 5x} - x \right).$$

At first glance, as $x \rightarrow \infty$,

$$\sqrt{x^2 + 5x} \rightarrow \infty \quad \text{and} \quad x \rightarrow \infty,$$

so the expression inside the parentheses has the indeterminate form $\infty - \infty$.

Step 1: Rationalize the Expression

Multiply the expression inside the parentheses by its conjugate:

$$\sqrt{x^2 + 5x} - x = \frac{(\sqrt{x^2 + 5x} - x)(\sqrt{x^2 + 5x} + x)}{\sqrt{x^2 + 5x} + x}.$$

Using the identity $(a - b)(a + b) = a^2 - b^2$, we obtain

$$= \frac{(x^2 + 5x) - x^2}{\sqrt{x^2 + 5x} + x} = \frac{5x}{\sqrt{x^2 + 5x} + x}.$$

Thus the original limit becomes

$$\lim_{x \rightarrow \infty} x \cdot \frac{5x}{\sqrt{x^2 + 5x} + x}.$$

Step 2: Simplify

Rewrite the expression as

$$\lim_{x \rightarrow \infty} \frac{5x^2}{\sqrt{x^2 + 5x} + x}.$$

Factor x out of the square root:

$$\sqrt{x^2 + 5x} = x\sqrt{1 + \frac{5}{x}} \quad (x > 0).$$

Substitute:

$$\frac{5x^2}{x \left(\sqrt{1 + \frac{5}{x}} + 1 \right)} = \frac{5x}{\sqrt{1 + \frac{5}{x}} + 1}.$$

Step 3: Take the Limit

As $x \rightarrow \infty$, $\frac{5}{x} \rightarrow 0$, so

$$\sqrt{1 + \frac{5}{x}} \rightarrow 1.$$

Therefore,

$$\lim_{x \rightarrow \infty} \frac{5x}{\sqrt{1 + \frac{5}{x}} + 1} = \lim_{x \rightarrow \infty} \frac{5x}{2} = \infty.$$

Final Answer

The limit diverges to ∞ .