

Question 2

Suppose that

$$\lim_{x \rightarrow 3} f(x) = 5 \quad \text{and} \quad \lim_{x \rightarrow 3} g(x) = 0.$$

Define

$$h(x) = \frac{f(x)^2 - 25}{g(x)}.$$

- (a) Explain why direct substitution does not determine $\lim_{x \rightarrow 3} h(x)$.
- (b) Use algebraic manipulation to rewrite $h(x)$.
- (c) State precisely what additional information would be required to evaluate $\lim_{x \rightarrow 3} h(x)$.

Question 2 - Solution

We are given:

$$\lim_{x \rightarrow 3} f(x) = 5, \quad \lim_{x \rightarrow 3} g(x) = 0, \quad h(x) = \frac{f(x)^2 - 25}{g(x)}.$$

(a) Indeterminate form

Substituting the known limits gives:

$$\lim_{x \rightarrow 3} h(x) = \frac{5^2 - 25}{0} = \frac{0}{0},$$

which is an indeterminate form. Therefore, the limit cannot be evaluated by direct substitution.

(b) Algebraic simplification

Factor the numerator:

$$f(x)^2 - 25 = (f(x) - 5)(f(x) + 5).$$

Then,

$$h(x) = \frac{(f(x) - 5)(f(x) + 5)}{g(x)} = \left(\frac{f(x) - 5}{g(x)} \right) (f(x) + 5).$$

Taking limits where possible,

$$\lim_{x \rightarrow 3} (f(x) + 5) = 10.$$

(c) Required additional information

The limit

$$\lim_{x \rightarrow 3} h(x)$$

exists if and only if the limit

$$\lim_{x \rightarrow 3} \frac{f(x) - 5}{g(x)}$$

exists.

If we let

$$L = \lim_{x \rightarrow 3} \frac{f(x) - 5}{g(x)},$$

then

$$\lim_{x \rightarrow 3} h(x) = 10L.$$

Without information about how fast $f(x) \rightarrow 5$ relative to $g(x) \rightarrow 0$, the value of the limit cannot be determined.

The limit cannot be evaluated without additional information.