

Question 6

Evaluate the limit:

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^3 - 8}$$

Provide detailed steps and justification for each transformation you make.

Question 6 - Solution

We want to evaluate:

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^3 - 8}$$

First, try direct substitution:

$$\frac{2^2 - 4}{2^3 - 8} = \frac{4 - 4}{8 - 8} = \frac{0}{0}$$

This is an indeterminate form, so we need to simplify the expression before evaluating the limit.

Now factor the numerator:

$$x^2 - 4 = (x - 2)(x + 2)$$

This is a difference of squares because:

$$x^2 - 4 = x^2 - 2^2$$

Next, factor the denominator:

$$x^3 - 8 = x^3 - 2^3$$

This is a difference of cubes. Recall the formula:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Here,

$$a = x \quad \text{and} \quad b = 2$$

So:

$$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$$

Now rewrite the original limit using these factorizations:

$$\lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{(x - 2)(x^2 + 2x + 4)}$$

Both the numerator and denominator contain the factor $x - 2$. Since we are finding the limit as x approaches 2, we only need to consider values of x near 2, but not equal to 2. Therefore, for $x \neq 2$, we may cancel the common factor $x - 2$:

$$\lim_{x \rightarrow 2} \frac{x + 2}{x^2 + 2x + 4}$$

Now the expression is no longer indeterminate, so we can substitute $x = 2$:

$$\begin{aligned} & \frac{2 + 2}{2^2 + 2(2) + 4} \\ &= \frac{4}{4 + 4 + 4} \\ &= \frac{4}{12} \\ &= \frac{1}{3} \end{aligned}$$

Therefore,

$$\boxed{\frac{1}{3}}$$