

Question 5

Evaluate the following limit:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{\tan(5x)}$$

Give a full explanation of your steps and reasoning.

Question 5 - Solution

We want to evaluate:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{\tan(5x)}$$

First, try direct substitution:

$$\frac{\sin(3 \cdot 0)}{\tan(5 \cdot 0)} = \frac{\sin(0)}{\tan(0)} = \frac{0}{0}$$

This is an indeterminate form, so we need to rewrite the expression before evaluating the limit.

We will use the standard trigonometric limits:

$$\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$$

and

$$\lim_{u \rightarrow 0} \frac{\tan u}{u} = 1$$

Since $3x \rightarrow 0$ and $5x \rightarrow 0$ as $x \rightarrow 0$, we can apply these limits to $\sin(3x)$ and $\tan(5x)$.

Start with the original expression:

$$\frac{\sin(3x)}{\tan(5x)}$$

We want to create the forms

$$\frac{\sin(3x)}{3x} \quad \text{and} \quad \frac{\tan(5x)}{5x}$$

To do this, multiply and divide by the needed factors:

$$\frac{\sin(3x)}{\tan(5x)} = \frac{\sin(3x)}{3x} \cdot \frac{5x}{\tan(5x)} \cdot \frac{3x}{5x}$$

This rewriting is valid because the factors we introduced balance each other out.

Now take the limit of each factor:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} = 1$$

Also,

$$\lim_{x \rightarrow 0} \frac{5x}{\tan(5x)} = 1$$

This is because

$$\lim_{x \rightarrow 0} \frac{\tan(5x)}{5x} = 1$$

so its reciprocal also approaches 1.

Finally,

$$\frac{3x}{5x} = \frac{3}{5}$$

for $x \neq 0$. Since limits only depend on values near 0, not necessarily at 0, this cancellation is allowed.

Therefore,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin(3x)}{\tan(5x)} &= 1 \cdot 1 \cdot \frac{3}{5} \\ &= \boxed{\frac{3}{5}} \end{aligned}$$