

Question 4

Evaluate the following limit:

$$\lim_{x \rightarrow 5} \frac{x^2 - 25}{\sqrt{x+1} - \sqrt{6}}$$

Provide a clear justification for each step.

Question 4 - Solution

We want to evaluate:

$$\lim_{x \rightarrow 5} \frac{x^2 - 25}{\sqrt{x+1} - \sqrt{6}}$$

First, try direct substitution:

$$\frac{5^2 - 25}{\sqrt{5+1} - \sqrt{6}} = \frac{25 - 25}{\sqrt{6} - \sqrt{6}} = \frac{0}{0}$$

This is an indeterminate form, so we need to simplify the expression before evaluating the limit.

Factor the numerator:

$$x^2 - 25 = (x - 5)(x + 5)$$

So the limit becomes:

$$\lim_{x \rightarrow 5} \frac{(x - 5)(x + 5)}{\sqrt{x+1} - \sqrt{6}}$$

The denominator contains square roots, so we multiply by the conjugate:

$$\frac{(x - 5)(x + 5)}{\sqrt{x+1} - \sqrt{6}} \cdot \frac{\sqrt{x+1} + \sqrt{6}}{\sqrt{x+1} + \sqrt{6}}$$

This does not change the value of the expression because we are multiplying by 1.

Now simplify the denominator using the difference of squares formula:

$$(a - b)(a + b) = a^2 - b^2$$

Here,

$$a = \sqrt{x+1} \quad \text{and} \quad b = \sqrt{6}$$

Therefore,

$$(\sqrt{x+1} - \sqrt{6})(\sqrt{x+1} + \sqrt{6}) = (\sqrt{x+1})^2 - (\sqrt{6})^2$$

$$= x + 1 - 6$$

$$= x - 5$$

So the expression becomes:

$$\lim_{x \rightarrow 5} \frac{(x - 5)(x + 5)(\sqrt{x+1} + \sqrt{6})}{x - 5}$$

Since we are evaluating the limit as x approaches 5, we may cancel the common factor $x - 5$ for $x \neq 5$:

$$\lim_{x \rightarrow 5} (x + 5)(\sqrt{x+1} + \sqrt{6})$$

Now substitute $x = 5$:

$$(5 + 5)(\sqrt{5+1} + \sqrt{6})$$

$$= 10(\sqrt{6} + \sqrt{6})$$

$$= 10(2\sqrt{6})$$

$$= 20\sqrt{6}$$

Therefore,

$$\boxed{20\sqrt{6}}$$