

Question 10

Let

$$f(x) = \frac{x^2 - 1}{x - 1}$$

Use the ε - δ definition of a limit to prove:

$$\lim_{x \rightarrow 1} f(x) = 2$$

Question 10 - Solution

We are given:

$$f(x) = \frac{x^2 - 1}{x - 1}$$

Note that:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x+1 \quad \text{for } x \neq 1$$

So for $x \neq 1$, $f(x) = x+1$, and:

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (x+1) = 2$$

We now prove this using the ε - δ definition.

We want to show:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - 1| < \delta \Rightarrow |f(x) - 2| < \varepsilon$$

Since $f(x) = x+1$ for $x \neq 1$, we compute:

$$|f(x) - 2| = |x+1-2| = |x-1|$$

So:

$$|f(x) - 2| < \varepsilon \quad \text{whenever} \quad |x - 1| < \varepsilon$$

Step: Choose $\delta = \varepsilon$

Then if $0 < |x - 1| < \delta$, we get:

$$|f(x) - 2| = |x - 1| < \varepsilon$$

Conclusion: By the ε - δ definition, we have:

$$\boxed{\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2}$$