

Question 8

Let

$$f(x) = x^2$$

Use the ε - δ definition of a limit to prove that:

$$\lim_{x \rightarrow 3} f(x) = 9$$

Question 8 - Solution

We are given:

$$f(x) = x^2, \quad \lim_{x \rightarrow 3} x^2 = 9$$

We want to prove that:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - 3| < \delta \Rightarrow |x^2 - 9| < \varepsilon$$

Step 1: Manipulate the expression.

$$|x^2 - 9| = |x - 3||x + 3|$$

We want to bound this expression. Assume $|x - 3| < 1$. Then:

$$2 < x < 4 \Rightarrow 5 < x + 3 < 7 \Rightarrow |x + 3| < 7$$

So:

$$|x^2 - 9| = |x - 3||x + 3| < |x - 3| \cdot 7$$

We want:

$$|x - 3| \cdot 7 < \varepsilon \Rightarrow |x - 3| < \frac{\varepsilon}{7}$$

Step 2: Choose δ .

Let:

$$\delta = \min\left(1, \frac{\varepsilon}{7}\right)$$

Step 3: Verify.

If $0 < |x - 3| < \delta$, then:

$$|x^2 - 9| = |x - 3||x + 3| < \delta \cdot 7 \leq \frac{\varepsilon}{7} \cdot 7 = \varepsilon$$

Conclusion: By the ε - δ definition of the limit:

$$\lim_{x \rightarrow 3} x^2 = 9$$