

Question 4

Let

$$f(x) = \frac{1}{x}$$

Prove, using the ε - δ definition of a limit, that:

$$\lim_{x \rightarrow 1} \frac{1}{x} = 1$$

Question 4 - Solution

We are given:

$$f(x) = \frac{1}{x}, \quad \lim_{x \rightarrow 1} f(x) = 1$$

We will prove this using the ε - δ definition of the limit.

Goal: For every $\varepsilon > 0$, find $\delta > 0$ such that:

$$0 < |x - 1| < \delta \quad \Rightarrow \quad \left| \frac{1}{x} - 1 \right| < \varepsilon$$

Step 1: Manipulate the expression

We begin by simplifying:

$$\left| \frac{1}{x} - 1 \right| = \left| \frac{1 - x}{x} \right| = \frac{|x - 1|}{|x|}$$

So we want:

$$\frac{|x - 1|}{|x|} < \varepsilon$$

Step 2: Bound $|x|$ away from zero

Choose $\delta \leq \frac{1}{2}$, so $|x - 1| < \frac{1}{2} \Rightarrow \frac{1}{2} < x < \frac{3}{2}$

This gives us a lower bound: $|x| > \frac{1}{2}$

So:

$$\frac{|x - 1|}{|x|} < \frac{|x - 1|}{1/2} = 2|x - 1|$$

Step 3: Ensure $2|x - 1| < \varepsilon$

To make this less than ε , we require:

$$2|x - 1| < \varepsilon \quad \Rightarrow \quad |x - 1| < \frac{\varepsilon}{2}$$

Step 4: Choose δ

Let:

$$\delta = \min\left(\frac{1}{2}, \frac{\varepsilon}{2}\right)$$

Then whenever $0 < |x - 1| < \delta$, we get:

$$\left| \frac{1}{x} - 1 \right| = \frac{|x - 1|}{|x|} < 2|x - 1| < 2\delta \leq \varepsilon$$

Conclusion:

Thus, by the ε - δ definition of a limit,

$$\boxed{\lim_{x \rightarrow 1} \frac{1}{x} = 1}$$