

Question 2

Let

$$f(x) = x^2 + 3x$$

Use the ε - δ definition of a limit to prove that:

$$\lim_{x \rightarrow 1} f(x) = 4$$

Question 2 - Solution

We are given:

$$f(x) = x^2 + 3x, \quad \lim_{x \rightarrow 1} f(x) = 4$$

We want to prove using the ε - δ definition:

Goal: For every $\varepsilon > 0$, we must find $\delta > 0$ such that:

$$0 < |x - 1| < \delta \quad \Rightarrow \quad |f(x) - 4| < \varepsilon$$

Step 1: Write the expression $|f(x) - 4|$:

$$|f(x) - 4| = |x^2 + 3x - 4|$$

Factor the expression:

$$x^2 + 3x - 4 = (x - 1)(x + 4) \Rightarrow |f(x) - 4| = |x - 1||x + 4|$$

We want:

$$|x - 1||x + 4| < \varepsilon$$

Step 2: Estimate $|x + 4|$:

Assume $|x - 1| < 1 \Rightarrow x \in (0, 2) \Rightarrow x + 4 \in (4, 6) \Rightarrow |x + 4| < 6$

Then:

$$|f(x) - 4| = |x - 1||x + 4| < 6|x - 1|$$

To make $|f(x) - 4| < \varepsilon$, it is enough to make:

$$6|x - 1| < \varepsilon \Rightarrow |x - 1| < \frac{\varepsilon}{6}$$

Step 3: Let $\delta = \min \left\{ 1, \frac{\varepsilon}{6} \right\}$

Then if $0 < |x - 1| < \delta$, we have:

$$|f(x) - 4| < \varepsilon$$

Conclusion:

Therefore, $\lim_{x \rightarrow 1} (x^2 + 3x) = 4$ by the ε - δ definition.