

Question 4

Let $f(x) = \sin(x)\cos(x) + \cos^2(x)$

- (a) Express $f(x)$ in terms of a single trigonometric function or a simpler expression.
- (b) Find the maximum and minimum values of $f(x)$ on the interval $[0, 2\pi]$.
- (c) Determine all values of $x \in [0, 2\pi]$ where $f(x) = 1$.

Question 4 - Solution

Simplify. Double-angle identities give

$$f(x) = \frac{1}{2} + \frac{\sqrt{2}}{2} \sin(2x + \pi/4).$$

The sine attains both -1 and 1 on the given interval, so

$$f_{\min} = \frac{1 - \sqrt{2}}{2}, \quad f_{\max} = \frac{1 + \sqrt{2}}{2}.$$

Solve without losing roots. Subtracting 1 from the original expression gives

$$f(x) - 1 = \sin x \cos x - \sin^2 x = \sin x(\cos x - \sin x).$$

Thus $\sin x = 0$ or $\cos x = \sin x$. Restricting both families to $[0, 2\pi]$ gives

$$x = 0, \pi/4, \pi, 5\pi/4, 2\pi.$$

Each listed value makes one factor zero, so each satisfies the original equation.