

Question 10

Let

$$f(x) = \frac{\sqrt{x+4}}{x-1}$$

- (a) Determine the domain of $f(x)$.
- (b) Is f one-to-one on its domain? Justify your answer.
- (c) If possible, find $f^{-1}(x)$. If not, restrict the domain appropriately and find the inverse on that domain.

Question 10 - Solution

Domain. The square root and denominator require

$$D = [-4, 1) \cup (1, \infty).$$

One-to-one. For $x > -4$, $x \neq 1$,

$$f'(x) = \frac{-(x+9)}{2\sqrt{x+4}(x-1)^2} < 0.$$

The left branch decreases from 0 to $-\infty$ and has range $(-\infty, 0]$.

The right branch decreases from $+\infty$ to 0 and has range $(0, \infty)$.

These ranges are disjoint, so f is one-to-one on its entire domain and its range is $\mathbb{R}$.

Inverse. Write $y = \sqrt{x+4}/(x-1)$. Squaring gives

$$y^2 x^2 - (2y^2 + 1)x + y^2 - 4 = 0.$$

For $y \neq 0$, the quadratic formula gives

$$x = \frac{2y^2 + 1 \pm \sqrt{1 + 20y^2}}{2y^2}.$$

The original, unsquared equation requires $x < 1$ when $y < 0$ and $x > 1$ when $y > 0$.

Also $y = 0$ gives $x = -4$. Thus

$$f^{-1}(y) = \begin{cases} \frac{2y^2 + 1 - \sqrt{1 + 20y^2}}{2y^2}, & y < 0, \\ -4, & y = 0, \\ \frac{2y^2 + 1 + \sqrt{1 + 20y^2}}{2y^2}, & y > 0. \end{cases}$$

The sign choice restores the original equation after squaring. The inverse has domain $\mathbb{R}$ and range D .