

Question 8

Let $f(x) = \frac{x+3}{x-1}$, where $x \neq 1$.

- (a) Show that f is a one-to-one function.
- (b) Find the inverse function $f^{-1}(x)$.
- (c) State the domain and range of both $f(x)$ and $f^{-1}(x)$.
- (d) Verify that $f(f^{-1}(x)) = x$.

Question 8 - Solution

(a) Show that f is one-to-one:Let $f(x_1) = f(x_2)$:

$$\frac{x_1 + 3}{x_1 - 1} = \frac{x_2 + 3}{x_2 - 1} \Rightarrow (x_1 + 3)(x_2 - 1) = (x_2 + 3)(x_1 - 1)$$

Expanding both sides:

$$x_1x_2 - x_1 + 3x_2 - 3 = x_1x_2 - x_2 + 3x_1 - 3$$

Subtract x_1x_2 and -3 from both sides:

$$-x_1 + 3x_2 = -x_2 + 3x_1 \Rightarrow 3x_2 + x_2 = 3x_1 + x_1 \Rightarrow 4x_2 = 4x_1 \Rightarrow x_1 = x_2$$

Therefore, f is one-to-one.

(b) Find the inverse:

$$\text{Let } y = \frac{x + 3}{x - 1}$$

Multiply both sides:

$$y(x - 1) = x + 3 \Rightarrow yx - y = x + 3 \Rightarrow yx - x = y + 3 \Rightarrow x(y - 1) = y + 3 \Rightarrow x = \frac{y + 3}{y - 1}$$

Switch x and y :

$$f^{-1}(x) = \frac{x + 3}{x - 1}$$

So:

$$\boxed{f^{-1}(x) = \frac{x + 3}{x - 1}}$$

Interestingly, $f^{-1}(x) = f(x)$. The function is its own inverse.

(c) Domain and Range:

From $f(x) = \frac{x + 3}{x - 1}$, we see that: - Domain: $x \neq 1 \Rightarrow \boxed{(-\infty, 1) \cup (1, \infty)}$ - As $x \rightarrow \infty$, $f(x) \rightarrow 1$, and $f(x) \neq 1$ - So range: $\boxed{(-\infty, 1) \cup (1, \infty)}$

The same applies to $f^{-1}(x)$, so: - Domain of $f^{-1}(x) : (-\infty, 1) \cup (1, \infty)$ - Range of $f^{-1}(x) : (-\infty, 1) \cup (1, \infty)$

(d) Verify:

$$f(f^{-1}(x)) = f\left(\frac{x + 3}{x - 1}\right) = \frac{\left(\frac{x + 3}{x - 1}\right) + 3}{\left(\frac{x + 3}{x - 1}\right) - 1} = \frac{\frac{x + 3 + 3(x - 1)}{x - 1}}{\frac{x + 3 - (x - 1)}{x - 1}} = \frac{\frac{x + 3 + 3x - 3}{x - 1}}{\frac{x + 3 - x + 1}{x - 1}} = \frac{\frac{4x}{x - 1}}{\frac{4}{x - 1}} = x$$

Verified: $f(f^{-1}(x)) = x$