

Question 6

Let $f(x) = \sqrt{2x - 1}$ with domain $x \geq \frac{1}{2}$.

- (a) Show that f is one-to-one and has an inverse.
- (b) Find the inverse function $f^{-1}(x)$.
- (c) Determine the domain and range of $f^{-1}(x)$.
- (d) Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Question 6 - Solution**(a) One-to-One Check:**

Since $f(x) = \sqrt{2x-1}$ is a square root function with a linear expression inside and domain $x \geq \frac{1}{2}$, it is strictly increasing. Therefore, it passes the horizontal line test and is one-to-one.

(b) Find the inverse:

Let $y = \sqrt{2x-1}$. Solve for x :

$$y^2 = 2x - 1 \Rightarrow 2x = y^2 + 1 \Rightarrow x = \frac{y^2 + 1}{2}$$

Now switch x and y :

$$f^{-1}(x) = \frac{x^2 + 1}{2}$$

Answer: $\boxed{f^{-1}(x) = \frac{x^2 + 1}{2}}$

(c) Domain and Range of Inverse:

Since original domain is $x \geq \frac{1}{2}$, the range is:

$$f(x) = \sqrt{2x-1} \geq 0 \Rightarrow \text{Range of } f = [0, \infty)$$

So: - Domain of $f^{-1}(x) = [0, \infty)$ - Range of $f^{-1}(x) = [\frac{1}{2}, \infty)$

(d) Verifying composition:

$$f(f^{-1}(x)) = f\left(\frac{x^2 + 1}{2}\right) = \sqrt{2 \cdot \frac{x^2 + 1}{2} - 1} = \sqrt{x^2 + 1 - 1} = \sqrt{x^2} = x \quad (\text{since } x \geq 0)$$

$$f^{-1}(f(x)) = f^{-1}(\sqrt{2x-1}) = \frac{(\sqrt{2x-1})^2 + 1}{2} = \frac{2x - 1 + 1}{2} = x$$

Both identities verified.