

Question 5

Let $f(x) = \frac{3x - 4}{5x + 2}$, where $x \neq -\frac{2}{5}$.

- (a) Show that f is a one-to-one function.
- (b) Find the inverse function $f^{-1}(x)$.
- (c) Determine the domain and range of $f(x)$ and $f^{-1}(x)$.

Question 5 - Solution

(a) **Show that f is one-to-one:**

Suppose $f(x_1) = f(x_2)$. Then:

$$\frac{3x_1 - 4}{5x_1 + 2} = \frac{3x_2 - 4}{5x_2 + 2}$$

Cross-multiplying:

$$(3x_1 - 4)(5x_2 + 2) = (3x_2 - 4)(5x_1 + 2)$$

Expanding both sides:

$$15x_1x_2 + 6x_1 - 20x_2 - 8 = 15x_1x_2 + 6x_2 - 20x_1 - 8$$

Cancel $15x_1x_2$ and -8 :

$$6x_1 - 20x_2 = 6x_2 - 20x_1 \Rightarrow 6x_1 + 20x_1 = 6x_2 + 20x_2 \Rightarrow 26x_1 = 26x_2 \Rightarrow x_1 = x_2$$

Therefore, f is one-to-one.

(b) **Find the inverse function:**

Let $y = \frac{3x-4}{5x+2}$

Multiply both sides:

$$y(5x + 2) = 3x - 4 \Rightarrow 5xy + 2y = 3x - 4 \Rightarrow 5xy - 3x = -2y - 4 \Rightarrow x(5y - 3) = -2y - 4 \Rightarrow x = \frac{-2y - 4}{5y - 3}$$

Now swap x and y :

$$f^{-1}(x) = \frac{-2x - 4}{5x - 3}$$

(c) **Domain and Range:**

The only restriction on the domain of f is $5x + 2 \neq 0 \Rightarrow x \neq -\frac{2}{5}$

The inverse is undefined when the denominator is 0:

$$5x - 3 \neq 0 \Rightarrow x \neq \frac{3}{5}$$

So:

$$\text{Domain of } f(x) : \left(-\infty, -\frac{2}{5} \right) \cup \left(-\frac{2}{5}, \infty \right)$$

$$\text{Range of } f(x) : \left(-\infty, \frac{3}{5} \right) \cup \left(\frac{3}{5}, \infty \right)$$

$$\text{Domain of } f^{-1}(x) : \left(-\infty, \frac{3}{5} \right) \cup \left(\frac{3}{5}, \infty \right)$$

$$\text{Range of } f^{-1}(x) : \left(-\infty, -\frac{2}{5} \right) \cup \left(-\frac{2}{5}, \infty \right)$$