

Question 4

Suppose $f(x) = \frac{1}{x^2 + 1}$.

- (a) Determine whether $f(x)$ has an inverse function. Justify.
- (b) If not, restrict the domain of f so that it becomes one-to-one and then find the inverse on that restricted domain.
- (c) State the domain and range of the inverse function you found.

Question 4 - Solution

(a) Does $f(x) = \frac{1}{x^2 + 1}$ have an inverse?

No. Its natural domain is $(-\infty, \infty)$, since $x^2 + 1$ is never zero. To have an inverse function, f must be **one-to-one**: different inputs must give different outputs.

Horizontal line test: Imagine drawing horizontal lines across the graph. Each horizontal line represents a fixed output y . If even one such line crosses the graph more than once, then different inputs give the same output, so the function is not one-to-one.

Testing points: We can show this without drawing the graph by choosing two different inputs and calculating their outputs. Try $x = 1$ and $x = -1$, since squaring either gives 1:

$$f(1) = \frac{1}{1^2 + 1} = \frac{1}{2}, \quad f(-1) = \frac{1}{(-1)^2 + 1} = \frac{1}{2}.$$

Thus, the horizontal line $y = \frac{1}{2}$ meets the graph at both $(1, \frac{1}{2})$ and $(-1, \frac{1}{2})$. An inverse would have to send $\frac{1}{2}$ back to both 1 and -1 , but a function can give only one output for each input.

So, f fails the horizontal line test and has no inverse function on $(-\infty, \infty)$. One pair of different inputs with the same output is enough to prove this; testing a few points with different outputs would not prove that a function is one-to-one everywhere.

(b) **Restrict domain to make f one-to-one:**

Choose only nonnegative inputs. As x increases on $[0, \infty)$, the denominator $x^2 + 1$ increases, so $f(x)$ strictly decreases. Therefore, no output repeats, and the restricted function is one-to-one. We use:

$$x \in [0, \infty)$$

Let $y = \frac{1}{x^2 + 1}$, with $x \geq 0$.
Solve for x :

$$\begin{aligned} y(x^2 + 1) &= 1 \\ x^2 + 1 &= \frac{1}{y} \\ x^2 &= \frac{1}{y} - 1 = \frac{1 - y}{y} \\ x &= \sqrt{\frac{1 - y}{y}}. \end{aligned}$$

We take only the nonnegative square root because our restricted domain requires $x \geq 0$. Now switch x and y to express the inverse using x as its input:

$$f^{-1}(x) = \sqrt{\frac{1 - x}{x}}, \quad \text{for } 0 < x \leq 1$$

(c) **Domain and range of the inverse function:**

The restricted function has domain $[0, \infty)$. Its largest output is $f(0) = 1$. As x increases, its outputs approach 0 but never equal 0, so its range is $(0, 1]$.

An inverse reverses inputs and outputs. Therefore, the original range becomes the inverse's domain, and the original domain becomes the inverse's range:

$$\text{Domain of } f^{-1}(x) = (0, 1], \quad \text{Range of } f^{-1}(x) = [0, \infty)$$

Answer:

$$f^{-1}(x) = \sqrt{\frac{1 - x}{x}}, \quad x \in (0, 1]$$