

Question 3

Let $f(x) = \frac{x}{x+1}$, where $x \neq -1$.

- (a) Find the inverse function $f^{-1}(x)$.
- (b) State the domain and range of both $f(x)$ and $f^{-1}(x)$.
- (c) Verify the identity $f(f^{-1}(x)) = x$.

Question 3 - Solution

(a) Find the inverse of $f(x) = \frac{x}{x+1}$:

Let $y = \frac{x}{x+1}$. Solve for x :

Multiply both sides by $x+1$:

$$y(x+1) = x \Rightarrow yx + y = x \Rightarrow yx - x = -y \Rightarrow x(y-1) = -y \Rightarrow x = \frac{-y}{y-1}$$

Now switch x and y :

$$f^{-1}(x) = \frac{-x}{x-1}$$

Answer: $f^{-1}(x) = \frac{-x}{x-1}$

(b) Domain and Range:

For $f(x) = \frac{x}{x+1}$: Domain: $x \neq -1$, As $x \rightarrow \infty$, $f(x) \rightarrow 1$, and as $x \rightarrow -\infty$, $f(x) \rightarrow 1$, Solve $\frac{x}{x+1} = 1 \Rightarrow x = x+1 \Rightarrow 0 = 1 \rightarrow$ contradiction
So $f(x) \neq 1$

Domain: $(-\infty, -1) \cup (-1, \infty)$ **Range:** $(-\infty, 1) \cup (1, \infty)$

Now for $f^{-1}(x) = \frac{-x}{x-1}$:

Domain excludes $x = 1$, since denominator is zero Range excludes $y = -1$, since solving $\frac{-x}{x-1} = -1 \Rightarrow x = \frac{1}{2}$, which contradicts range of original function

Domain of $f^{-1}(x) = (-\infty, 1) \cup (1, \infty)$ **Range of** $f^{-1}(x) = (-\infty, -1) \cup (-1, \infty)$

(c) Verify:

$$f(f^{-1}(x)) = f\left(\frac{-x}{x-1}\right) = \frac{\frac{-x}{x-1}}{\frac{-x}{x-1} + 1} = \frac{\frac{-x}{x-1}}{\frac{-x + (x-1)}{x-1}} = \frac{\frac{-x}{x-1}}{\frac{-1}{x-1}} = x$$

Verified: $f(f^{-1}(x)) = x$