

Question 2

Let $f(x) = \ln(x - 1)$, where $x > 1$.

- (a) Find the inverse function $f^{-1}(x)$.
- (b) State the domain and range of $f(x)$ and $f^{-1}(x)$.
- (c) Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Question 2 - Solution

(a) Find the inverse of $f(x) = \ln(x - 1)$:

Let $y = \ln(x - 1)$

Exponentiate both sides:

$$e^y = x - 1 \Rightarrow x = e^y + 1$$

Now swap x and y to get the inverse:

$$f^{-1}(x) = e^x + 1$$

Answer: $f^{-1}(x) = e^x + 1$

(b) **Domain and Range:**

For $f(x) = \ln(x - 1)$:

Domain: $x > 1 \Rightarrow (1, \infty)$, Range of $\ln(x - 1)$ is all real numbers: $(-\infty, \infty)$

So for the inverse:

Domain of $f^{-1}(x)$: $(-\infty, \infty)$, Range of $f^{-1}(x) = e^x + 1$: since $e^x > 0$, we get $f^{-1}(x) > 1 \Rightarrow (1, \infty)$

Summary:

Domain of $f(x) = (1, \infty)$

Range of $f(x) = (-\infty, \infty)$

Domain of $f^{-1}(x) = (-\infty, \infty)$

Range of $f^{-1}(x) = (1, \infty)$

(c) **Verify compositions:**

$$f(f^{-1}(x)) = \ln((e^x + 1) - 1) = \ln(e^x) = x$$

$$f^{-1}(f(x)) = e^{\ln(x-1)} + 1 = x - 1 + 1 = x$$

Both identities verified.