

Question 1

Let the function $f(x) = \frac{2x + 5}{x - 3}$.

- (a) Show that f is one-to-one.
- (b) Find the inverse function $f^{-1}(x)$.
- (c) State the domain and range of both $f(x)$ and $f^{-1}(x)$.

Question 1 - Solution

(a) **Show that f is one-to-one:**

A function is one-to-one if it passes the Horizontal Line Test, or algebraically if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Suppose:

$$\frac{2x_1 + 5}{x_1 - 3} = \frac{2x_2 + 5}{x_2 - 3}$$

Cross-multiplying:

$$(2x_1 + 5)(x_2 - 3) = (2x_2 + 5)(x_1 - 3)$$

Expanding both sides:

$$2x_1x_2 - 6x_1 + 5x_2 - 15 = 2x_1x_2 - 6x_2 + 5x_1 - 15$$

Subtract $2x_1x_2$ and -15 from both sides:

$$-6x_1 + 5x_2 = -6x_2 + 5x_1 \Rightarrow 5x_2 + 6x_2 = 5x_1 + 6x_1 \Rightarrow 11x_2 = 11x_1 \Rightarrow x_1 = x_2$$

Thus, f is one-to-one.

(b) **Find the inverse function:**

Let $y = \frac{2x + 5}{x - 3}$, and solve for x :

$$y(x - 3) = 2x + 5 \Rightarrow yx - 3y = 2x + 5 \Rightarrow yx - 2x = 3y + 5 \Rightarrow x(y - 2) = 3y + 5 \Rightarrow x = \frac{3y + 5}{y - 2}$$

Now swap x and y :

$$f^{-1}(x) = \frac{3x + 5}{x - 2}$$

(c) **Domain and Range:**

For $f(x) = \frac{2x + 5}{x - 3}$: - Denominator cannot be 0 $x \neq 3$ - So domain of f is: $(-\infty, 3) \cup (3, \infty)$

Range: Find y values that cannot occur. From inverse function: $f^{-1}(x) = \frac{3x + 5}{x - 2} \Rightarrow x \neq 2$

So range of f is $(-\infty, 2) \cup (2, \infty)$

Domain and Range:

Domain of $f(x) : (-\infty, 3) \cup (3, \infty)$

Range of $f(x) : (-\infty, 2) \cup (2, \infty)$

Domain of $f^{-1}(x) : (-\infty, 2) \cup (2, \infty)$

Range of $f^{-1}(x) : (-\infty, 3) \cup (3, \infty)$