

Question 5

A function f is defined as:

$$f(x) = \begin{cases} \frac{\ln(1+ax)}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

- (a) Find the value of a for which $f(x)$ is continuous at $x = 0$.
- (b) For that value of a , determine whether f is differentiable at $x = 0$.

Question 5 - Solution

(a) For f to be continuous at $x = 0$, we need

$$\lim_{x \rightarrow 0} f(x) = f(0) = 1.$$

For $x \neq 0$,

$$f(x) = \frac{\ln(1 + ax)}{x}.$$

So compute

$$\lim_{x \rightarrow 0} \frac{\ln(1 + ax)}{x}.$$

As $x \rightarrow 0$, the numerator and denominator both approach 0, so we use L'Hospital's Rule:

$$\lim_{x \rightarrow 0} \frac{\ln(1 + ax)}{x} = \lim_{x \rightarrow 0} \frac{\frac{a}{1+ax}}{1} = a.$$

For continuity, this limit must equal $f(0) = 1$. Therefore,

$$a = 1.$$

Thus,

$$\boxed{a = 1}.$$

(b) Now let $a = 1$. Then

$$f(x) = \frac{\ln(1 + x)}{x}, \quad x \neq 0,$$

and $f(0) = 1$.

To check differentiability at $x = 0$, use the definition:

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}.$$

Substitute $f(h) = \frac{\ln(1 + h)}{h}$ and $f(0) = 1$:

$$f'(0) = \lim_{h \rightarrow 0} \frac{\frac{\ln(1+h)}{h} - 1}{h} = \lim_{h \rightarrow 0} \frac{\ln(1 + h) - h}{h^2}.$$

This is also a 0/0 form, so apply L'Hospital's Rule:

$$f'(0) = \lim_{h \rightarrow 0} \frac{\frac{1}{1+h} - 1}{2h} = \lim_{h \rightarrow 0} \frac{-h}{2h(1 + h)} = \lim_{h \rightarrow 0} \frac{-1}{2(1 + h)} = -\frac{1}{2}.$$

Since this limit exists, f is differentiable at $x = 0$, and

$$\boxed{f'(0) = -\frac{1}{2}}.$$