

Question 4

$$\text{Let } f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x \neq 3 \\ c, & x = 3 \end{cases}$$

- (a) Simplify $f(x)$ for $x \neq 3$.
- (b) Find the value of c that makes $f(x)$ continuous at $x = 3$.
- (c) Is $f(x)$ differentiable at $x = 3$? Justify your answer.

Question 4 - Solution

(a) Simplifying for $x \neq 3$:

$$f(x) = \frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3, \quad x \neq 3$$

So the simplified function is:

$$f(x) = x + 3 \quad \text{for } x \neq 3$$

(b) **Continuity at $x = 3$:**

To ensure continuity at $x = 3$, we must have:

$$\lim_{x \rightarrow 3} f(x) = f(3) = c$$

From (a), since $f(x) = x + 3$ for $x \neq 3$, then:

$$\lim_{x \rightarrow 3} f(x) = 3 + 3 = 6 \Rightarrow c = \boxed{6}$$

(c) **Differentiability at $x = 3$:**

We now check if $f(x)$ is differentiable at $x = 3$. The simplified form $f(x) = x + 3$ is linear and differentiable for all $x \neq 3$.

If we define $f(3) = 6$, then the function becomes:

$$f(x) = \begin{cases} x + 3 & x \neq 3 \\ 6 & x = 3 \end{cases}$$

This function is: Continuous at $x = 3$, Equal to $x + 3$ everywhere except possibly at 3, $f(x)$ approaches $x + 3$ smoothly at 3

So,

$$f'(3) = \lim_{h \rightarrow 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \rightarrow 0} \frac{(3 + h + 3) - 6}{h} = \lim_{h \rightarrow 0} \frac{6 + h - 6}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1$$

Conclusion: $f(x)$ is differentiable at $x = 3$, and $f'(3) = \boxed{1}$