

Question 3

Let the function f be defined by

$$f(x) = \cos(\sin x)$$

- (a) Determine the period of $f(x)$.
- (b) Prove that $f(x)$ is an even function.
- (c) Find all values of $x \in [0, 2\pi]$ such that $f(x) = 1$.

Question 3 - Solution

(a) Although $\sin x$ has period 2π , the function

$$f(x) = \cos(\sin x)$$

has a smaller period because cosine is even.

Using

$$\sin(x + \pi) = -\sin x$$

and

$$\cos(-u) = \cos(u),$$

we get

$$f(x + \pi) = \cos(\sin(x + \pi)) = \cos(-\sin x) = \cos(\sin x) = f(x).$$

Therefore, $f(x)$ repeats after π . The reason the period changes from 2π to π is that shifting by π changes $\sin x$ to $-\sin x$, but cosine gives the same value for opposite inputs.

Thus, the period is

$$\boxed{\pi}.$$

(b) To prove that $f(x)$ is even, compute $f(-x)$:

$$f(-x) = \cos(\sin(-x)) = \cos(-\sin x) = \cos(\sin x) = f(x).$$

Therefore,

$$\boxed{f(x) \text{ is even.}}$$

(c) Solve

$$\cos(\sin x) = 1.$$

Cosine equals 1 when its input is $2k\pi$. Thus,

$$\sin x = 2k\pi.$$

But since $-1 \leq \sin x \leq 1$, the only possible value is

$$\sin x = 0.$$

On $[0, 2\pi]$, this occurs at

$$x = 0, \pi, 2\pi.$$

Therefore,

$$\boxed{x = 0, \pi, 2\pi}.$$