

Question 1

Assume $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$. Prove that

$$\lim_{x \rightarrow a} (f(x)g(x)) = LM.$$

Question 1 - Solution

Let $\varepsilon > 0$ be given.

We must show there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then

$$|f(x)g(x) - LM| < \varepsilon.$$

Start by adding and subtracting $Lg(x)$:

$$f(x)g(x) - LM = f(x)g(x) - Lg(x) + Lg(x) - LM = g(x)(f(x) - L) + L(g(x) - M).$$

Take absolute values and use the triangle inequality:

$$|f(x)g(x) - LM| \leq |g(x)| |f(x) - L| + |L| |g(x) - M|.$$

To control $|g(x)|$, use the fact that $g(x) \rightarrow M$.

Since $\lim_{x \rightarrow a} g(x) = M$, there exists $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$, then

$$|g(x) - M| < 1.$$

Then, for $0 < |x - a| < \delta_1$,

$$|g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M| < 1 + |M|.$$

Now choose $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then

$$|f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}.$$

This is possible because $\lim_{x \rightarrow a} f(x) = L$.

Next choose $\delta_3 > 0$ such that if $0 < |x - a| < \delta_3$, then

$$|g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}.$$

This is possible because $\lim_{x \rightarrow a} g(x) = M$.

Let

$$\delta = \min(\delta_1, \delta_2, \delta_3).$$

Then whenever $0 < |x - a| < \delta$, we have simultaneously:

$$|g(x)| < 1 + |M|, \quad |f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}, \quad |g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}.$$

Substitute into the estimate:

$$|f(x)g(x) - LM| \leq |g(x)| |f(x) - L| + |L| |g(x) - M|.$$

For the first term:

$$|g(x)| |f(x) - L| < (1 + |M|) \cdot \frac{\varepsilon}{2(1 + |M|)} = \frac{\varepsilon}{2}.$$

For the second term:

$$|L| |g(x) - M| \leq \frac{\varepsilon |L|}{2(1 + |L|)} < \frac{\varepsilon}{2}.$$

Therefore,

$$|f(x)g(x) - LM| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\lim_{x \rightarrow a} (f(x)g(x)) = LM.$$

Question 2

Assume that $\lim_{x \rightarrow a} f(x) = L$ and that c is a constant. Prove that

$$\lim_{x \rightarrow a} (cf(x)) = cL.$$

Question 2 - Solution

Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that if

$$0 < |x - a| < \delta,$$

then

$$|cf(x) - cL| < \varepsilon.$$

Start by factoring out the constant:

$$|cf(x) - cL| = |c| \cdot |f(x) - L|.$$

Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| < \frac{\varepsilon}{|c|},$$

provided $c \neq 0$.

Substitute this into the inequality:

$$|cf(x) - cL| = |c| \cdot |f(x) - L| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

Thus, whenever $0 < |x - a| < \delta$, we have

$$|cf(x) - cL| < \varepsilon.$$

Therefore,

$$\lim_{x \rightarrow a} (cf(x)) = cL.$$

If $c = 0$, then $cf(x) = 0$ for all x , and the result holds trivially.

Question 3

Assume that $\lim_{x \rightarrow a} f(x) = L$. Prove that

$$\lim_{x \rightarrow a} |f(x)| = |L|.$$

Question 3 - Solution

Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that if

$$0 < |x - a| < \delta,$$

then

$$||f(x)| - |L|| < \varepsilon.$$

Recall the inequality

$$||u| - |v|| \leq |u - v|$$

for all real numbers u and v .

Apply this inequality with $u = f(x)$ and $v = L$:

$$||f(x)| - |L|| \leq |f(x) - L|.$$

Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| < \varepsilon.$$

Combining the inequalities gives

$$||f(x)| - |L|| \leq |f(x) - L| < \varepsilon.$$

Thus, whenever $0 < |x - a| < \delta$,

$$||f(x)| - |L|| < \varepsilon.$$

Therefore,

$$\lim_{x \rightarrow a} |f(x)| = |L|.$$

Question 4

Assume that $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, with $M \neq 0$. Prove that

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Question 4 - Solution

Let $\varepsilon > 0$. Since $g(x) \rightarrow M \neq 0$, choose $\delta_1 > 0$ such that $|g(x) - M| < |M|/2$ when $0 < |x - a| < \delta_1$. Then $|g(x)| > |M|/2$.

By the two assumed limits, choose $\delta_2, \delta_3 > 0$ to ensure respectively

$$|f(x) - L| < \frac{\varepsilon|M|}{4}, \quad |g(x) - M| < \frac{\varepsilon|M|^2}{4(1+|L|)}.$$

Take $\delta = \min(\delta_1, \delta_2, \delta_3)$. For $0 < |x - a| < \delta$,

$$\begin{aligned} \left| \frac{f(x)}{g(x)} - \frac{L}{M} \right| &\leq \frac{|f(x) - L|}{|g(x)|} + \frac{|L| |g(x) - M|}{|M| |g(x)|} \\ &\leq \frac{2|f(x) - L|}{|M|} + \frac{2|L| |g(x) - M|}{|M|^2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon|L|}{2(1+|L|)} < \varepsilon. \end{aligned}$$

(The first term makes the penultimate inequality strict even if $L = 0$.) This proves $\lim_{x \rightarrow a} f(x)/g(x) = L/M$.

Question 5

Assume that $\lim_{x \rightarrow a} f(x) = L$. Prove that

$$\lim_{x \rightarrow a} (f(x) + c) = L + c,$$

where c is a constant.

Question 5 - Solution

Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|(f(x) + c) - (L + c)| < \varepsilon.$$

Simplify the expression inside the absolute value:

$$|(f(x) + c) - (L + c)| = |f(x) - L|.$$

Since $\lim_{x \rightarrow a} f(x) = L$, by definition of the limit there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| < \varepsilon.$$

Therefore, for the same δ ,

$$|(f(x) + c) - (L + c)| < \varepsilon.$$

This proves that

$$\lim_{x \rightarrow a} (f(x) + c) = L + c.$$

Question 6

Assume that $\lim_{x \rightarrow a} f(x) = L$. Prove that

$$\lim_{x \rightarrow a} (f(x) - c) = L - c,$$

where c is a constant.

Question 6 - Solution

Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that if

$$0 < |x - a| < \delta,$$

then

$$|(f(x) - c) - (L - c)| < \varepsilon.$$

Simplify the expression:

$$|(f(x) - c) - (L - c)| = |f(x) - L|.$$

Since $\lim_{x \rightarrow a} f(x) = L$, by the definition of the limit there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| < \varepsilon.$$

Thus, for this same choice of δ ,

$$|(f(x) - c) - (L - c)| < \varepsilon.$$

Therefore,

$$\lim_{x \rightarrow a} (f(x) - c) = L - c.$$

Question 7

Assume that

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = M,$$

and that

$$f(x) \leq g(x) \quad \text{for all } x \text{ near } a.$$

Prove that

$$L \leq M.$$

Question 7 - Solution

We argue by contradiction.

Suppose, to the contrary, that

$$L > M.$$

Then the positive number

$$\varepsilon = \frac{L - M}{2}$$

satisfies $\varepsilon > 0$.

Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta_1 > 0$ such that whenever

$$0 < |x - a| < \delta_1,$$

we have

$$|f(x) - L| < \varepsilon.$$

This implies

$$f(x) > L - \varepsilon = \frac{L + M}{2}.$$

Similarly, since $\lim_{x \rightarrow a} g(x) = M$, there exists $\delta_2 > 0$ such that whenever

$$0 < |x - a| < \delta_2,$$

we have

$$|g(x) - M| < \varepsilon.$$

This implies

$$g(x) < M + \varepsilon = \frac{L + M}{2}.$$

Let

$$\delta = \min(\delta_1, \delta_2).$$

Then for all x satisfying $0 < |x - a| < \delta$, we have

$$f(x) > \frac{L + M}{2} \quad \text{and} \quad g(x) < \frac{L + M}{2}.$$

Thus,

$$f(x) > g(x),$$

which contradicts the assumption that $f(x) \leq g(x)$ near a .

Therefore, our assumption that $L > M$ must be false, and we conclude that

$$L \leq M.$$

Question 8

Assume that

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad f(x) \geq 0 \quad \text{for all } x \text{ near } a.$$

Prove that

$$L \geq 0.$$

Question 8 - Solution

We prove this by contradiction.

Assume, to the contrary, that

$$L < 0.$$

Then the positive number

$$\varepsilon = \frac{-L}{2}$$

satisfies $\varepsilon > 0$.

Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| < \varepsilon.$$

This inequality implies

$$f(x) < L + \varepsilon = L + \frac{-L}{2} = \frac{L}{2}.$$

Because $L < 0$, we have

$$\frac{L}{2} < 0.$$

Thus, for all x with $0 < |x - a| < \delta$,

$$f(x) < 0.$$

This contradicts the assumption that $f(x) \geq 0$ for all x near a .

Therefore, our assumption that $L < 0$ must be false. We conclude that

$$L \geq 0.$$

Question 9

Assume that

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = L.$$

Prove that

$$\lim_{x \rightarrow a} (f(x) - g(x)) = 0.$$

Question 9 - Solution

Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - g(x)| < \varepsilon.$$

Begin by rewriting the difference:

$$f(x) - g(x) = (f(x) - L) - (g(x) - L).$$

Take absolute values and apply the triangle inequality:

$$|f(x) - g(x)| \leq |f(x) - L| + |g(x) - L|.$$

Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta_1 > 0$ such that

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{whenever } 0 < |x - a| < \delta_1.$$

Similarly, since $\lim_{x \rightarrow a} g(x) = L$, there exists $\delta_2 > 0$ such that

$$|g(x) - L| < \frac{\varepsilon}{2} \quad \text{whenever } 0 < |x - a| < \delta_2.$$

Let

$$\delta = \min(\delta_1, \delta_2).$$

Then for all x satisfying $0 < |x - a| < \delta$,

$$|f(x) - g(x)| \leq |f(x) - L| + |g(x) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore,

$$\lim_{x \rightarrow a} (f(x) - g(x)) = 0.$$

Question 10

Assume that

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a} f(x) = M.$$

Prove that

$$L = M.$$

Question 10 - Solution

We prove this by contradiction.

Assume, to the contrary, that

$$L \neq M.$$

Then the positive number

$$\varepsilon = \frac{|L - M|}{2}$$

satisfies $\varepsilon > 0$.

Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta_1 > 0$ such that whenever

$$0 < |x - a| < \delta_1,$$

we have

$$|f(x) - L| < \varepsilon.$$

Similarly, since $\lim_{x \rightarrow a} f(x) = M$, there exists $\delta_2 > 0$ such that whenever

$$0 < |x - a| < \delta_2,$$

we have

$$|f(x) - M| < \varepsilon.$$

Let

$$\delta = \min(\delta_1, \delta_2).$$

Then for all x such that $0 < |x - a| < \delta$, both inequalities hold.

Using the triangle inequality,

$$|L - M| = |L - f(x) + f(x) - M| \leq |f(x) - L| + |f(x) - M|.$$

Substitute the bounds:

$$|L - M| < \varepsilon + \varepsilon = 2\varepsilon = |L - M|.$$

This is a contradiction.

Therefore, our assumption that $L \neq M$ is false. We conclude that

$$L = M.$$

Question 1

Assume that f and g are differentiable at $x = a$. Prove that

$$(f + g)'(a) = f'(a) + g'(a).$$

Question 1 - Solution

By definition of the derivative,

$$(f + g)'(a) = \lim_{h \rightarrow 0} \frac{(f + g)(a + h) - (f + g)(a)}{h}.$$

Expand the numerator:

$$(f + g)(a + h) - (f + g)(a) = (f(a + h) + g(a + h)) - (f(a) + g(a)).$$

Group terms:

$$= (f(a + h) - f(a)) + (g(a + h) - g(a)).$$

Substitute back into the limit:

$$(f + g)'(a) = \lim_{h \rightarrow 0} \left[\frac{f(a + h) - f(a)}{h} + \frac{g(a + h) - g(a)}{h} \right].$$

Use the limit law for sums:

$$(f + g)'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} + \lim_{h \rightarrow 0} \frac{g(a + h) - g(a)}{h}.$$

Since f and g are differentiable at a , both limits exist and equal $f'(a)$ and $g'(a)$, respectively. Therefore,

$$(f + g)'(a) = f'(a) + g'(a).$$

Question 2

Assume that f is differentiable at $x = a$ and that c is a constant. Prove that

$$(cf)'(a) = c f'(a).$$

Question 2 - Solution

By definition of the derivative,

$$(cf)'(a) = \lim_{h \rightarrow 0} \frac{cf(a+h) - cf(a)}{h}.$$

Factor out the constant c from the numerator:

$$(cf)'(a) = \lim_{h \rightarrow 0} \frac{c(f(a+h) - f(a))}{h}.$$

Rewrite the fraction:

$$(cf)'(a) = c \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Since f is differentiable at a , the limit exists and equals $f'(a)$. Therefore,

$$(cf)'(a) = c f'(a).$$

Question 3

Assume that f and g are differentiable at $x = a$. Prove the product rule:

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Question 3 - Solution

By definition of the derivative,

$$(fg)'(a) = \lim_{h \rightarrow 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}.$$

Add and subtract $f(a+h)g(a)$ in the numerator:

$$= \lim_{h \rightarrow 0} \frac{f(a+h)g(a+h) - f(a+h)g(a) + f(a+h)g(a) - f(a)g(a)}{h}.$$

Group the terms:

$$= \lim_{h \rightarrow 0} \left[\frac{f(a+h)(g(a+h) - g(a))}{h} + \frac{g(a)(f(a+h) - f(a))}{h} \right].$$

Separate the limit:

$$= \lim_{h \rightarrow 0} f(a+h) \lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h} + g(a) \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Since f is differentiable at a , it is continuous at a , so

$$\lim_{h \rightarrow 0} f(a+h) = f(a).$$

Also, by differentiability,

$$\lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h} = g'(a), \quad \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

Substitute these limits:

$$(fg)'(a) = f(a)g'(a) + g(a)f'(a).$$

Thus,

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Question 4

Assume that f and g are differentiable at $x = a$ and that $g(a) \neq 0$. Prove the quotient rule:

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.$$

Question 4 - Solution

By definition of the derivative,

$$\left(\frac{f}{g}\right)'(a) = \lim_{h \rightarrow 0} \frac{\frac{f(a+h)}{g(a+h)} - \frac{f(a)}{g(a)}}{h}.$$

Combine the fractions in the numerator:

$$= \lim_{h \rightarrow 0} \frac{f(a+h)g(a) - f(a)g(a+h)}{h g(a+h)g(a)}.$$

Rewrite the numerator by adding and subtracting $f(a)g(a)$:

$$f(a+h)g(a) - f(a)g(a+h) = g(a)(f(a+h) - f(a)) - f(a)(g(a+h) - g(a)).$$

Substitute this expression:

$$= \lim_{h \rightarrow 0} \frac{g(a)(f(a+h) - f(a)) - f(a)(g(a+h) - g(a))}{h g(a+h)g(a)}.$$

Separate the fraction:

$$= \lim_{h \rightarrow 0} \left[\frac{g(a)}{g(a+h)g(a)} \cdot \frac{f(a+h) - f(a)}{h} - \frac{f(a)}{g(a+h)g(a)} \cdot \frac{g(a+h) - g(a)}{h} \right].$$

Simplify coefficients:

$$= \lim_{h \rightarrow 0} \left[\frac{1}{g(a+h)} \cdot \frac{f(a+h) - f(a)}{h} - \frac{f(a)}{g(a)g(a+h)} \cdot \frac{g(a+h) - g(a)}{h} \right].$$

Since g is differentiable at a , it is continuous at a , so

$$\lim_{h \rightarrow 0} g(a+h) = g(a).$$

Taking limits term by term,

$$\left(\frac{f}{g}\right)'(a) = \frac{1}{g(a)}f'(a) - \frac{f(a)}{[g(a)]^2}g'(a).$$

Combine the terms:

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.$$

Question 4

Assume that f is differentiable at $x = a$. Prove that f is continuous at $x = a$.

Question 4 - Solution

To show that f is continuous at $x = a$, we must prove that

$$\lim_{x \rightarrow a} f(x) = f(a).$$

Since f is differentiable at a , by definition,

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Because this limit exists, the expression

$$\frac{f(a+h) - f(a)}{h}$$

remains bounded and approaches a finite value as $h \rightarrow 0$.

Rewrite $f(a+h) - f(a)$ as

$$f(a+h) - f(a) = \left(\frac{f(a+h) - f(a)}{h} \right) h.$$

Now take limits as $h \rightarrow 0$:

$$\lim_{h \rightarrow 0} (f(a+h) - f(a)) = \left(\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \right) \left(\lim_{h \rightarrow 0} h \right).$$

Since

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = f'(a) \quad \text{and} \quad \lim_{h \rightarrow 0} h = 0,$$

we obtain

$$\lim_{h \rightarrow 0} (f(a+h) - f(a)) = 0.$$

Therefore,

$$\lim_{h \rightarrow 0} f(a+h) = f(a).$$

This shows that

$$\lim_{x \rightarrow a} f(x) = f(a),$$

so f is continuous at $x = a$.

Question 6

Assume that f is differentiable at $x = a$ and that $f'(a) = 0$. Prove that

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = 0.$$

Question 6 - Solution

By the definition of the derivative,

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

We are given that

$$f'(a) = 0.$$

Thus,

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = 0.$$

To make this explicit using ε - δ language, let $\varepsilon > 0$ be given.

Since the limit defining $f'(a)$ exists and equals 0, there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$\left| \frac{f(x) - f(a)}{x - a} - 0 \right| < \varepsilon.$$

That is,

$$\left| \frac{f(x) - f(a)}{x - a} \right| < \varepsilon.$$

This proves that

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = 0.$$

Question 7

Prove that the derivative of a constant function is zero. That is, if

$$f(x) = c$$

for all x , where c is a constant, then

$$f'(a) = 0$$

for every real number a .

Question 7 - Solution

By definition of the derivative,

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Since $f(x) = c$ for all x , we have

$$f(a+h) = c \quad \text{and} \quad f(a) = c.$$

Substitute into the difference quotient:

$$f'(a) = \lim_{h \rightarrow 0} \frac{c - c}{h}.$$

Simplify the numerator:

$$f'(a) = \lim_{h \rightarrow 0} \frac{0}{h}.$$

Since $\frac{0}{h} = 0$ for all $h \neq 0$, the limit is

$$f'(a) = 0.$$

Therefore, the derivative of a constant function is zero:

$$\boxed{f'(a) = 0}.$$

Question 8

Prove from first principles that if

$$f(x) = x^2,$$

then

$$f'(a) = 2a.$$

Question 8 - Solution

By definition of the derivative,

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Substitute $f(x) = x^2$:

$$f'(a) = \lim_{h \rightarrow 0} \frac{(a+h)^2 - a^2}{h}.$$

Expand the square:

$$(a+h)^2 = a^2 + 2ah + h^2.$$

Substitute back:

$$f'(a) = \lim_{h \rightarrow 0} \frac{a^2 + 2ah + h^2 - a^2}{h}.$$

Simplify the numerator:

$$f'(a) = \lim_{h \rightarrow 0} \frac{2ah + h^2}{h}.$$

Factor out h :

$$f'(a) = \lim_{h \rightarrow 0} \frac{h(2a + h)}{h}.$$

Cancel h (for $h \neq 0$):

$$f'(a) = \lim_{h \rightarrow 0} (2a + h).$$

Now take the limit:

$$f'(a) = 2a.$$

Thus, using first principles,

$$\boxed{f'(a) = 2a}.$$

Question 9

Prove from first principles that if

$$f(x) = x^3,$$

then

$$f'(a) = 3a^2.$$

Question 9 - Solution

By definition of the derivative,

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Substitute $f(x) = x^3$:

$$f'(a) = \lim_{h \rightarrow 0} \frac{(a+h)^3 - a^3}{h}.$$

Expand the cube:

$$(a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3.$$

Substitute into the difference quotient:

$$f'(a) = \lim_{h \rightarrow 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h}.$$

Simplify the numerator:

$$f'(a) = \lim_{h \rightarrow 0} \frac{3a^2h + 3ah^2 + h^3}{h}.$$

Factor out h :

$$f'(a) = \lim_{h \rightarrow 0} \frac{h(3a^2 + 3ah + h^2)}{h}.$$

Cancel h (for $h \neq 0$):

$$f'(a) = \lim_{h \rightarrow 0} (3a^2 + 3ah + h^2).$$

Now take the limit:

$$f'(a) = 3a^2.$$

Thus, using first principles,

$$\boxed{f'(a) = 3a^2}.$$

Question 10

Assume that f is differentiable at $x = a$ and that g is differentiable at $x = f(a)$. Prove the Chain Rule:

$$(g \circ f)'(a) = g'(f(a)) f'(a).$$

Question 10 - Solution

Put $b = f(a)$. Differentiability of g at b means

$$g(b + u) - g(b) = g'(b)u + u\eta(u),$$

where $\eta(u) \rightarrow 0$ as $u \rightarrow 0$; define $\eta(0) = 0$ so this identity also holds at $u = 0$.

For $h \neq 0$, set $u = f(a + h) - f(a)$. Then

$$\frac{g(f(a + h)) - g(f(a))}{h} = (g'(b) + \eta(f(a + h) - f(a))) \frac{f(a + h) - f(a)}{h}.$$

Differentiability implies continuity of f at a , so the first factor tends to $g'(b)$ and the second to $f'(a)$.

$$\boxed{(g \circ f)'(a) = g'(f(a))f'(a).}$$

This argument remains valid when $f(a + h) = f(a)$ for some or all nearby h .

Question 1

Prove that

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Question 1 - Solution

We prove this limit using geometric inequalities.

Consider a unit circle centered at the origin.

Let $0 < x < \pi/2$ be measured in radians, and consider the angle x .

The area of the sector with angle x is

$$\text{Area of sector} = \frac{1}{2}x.$$

The area of the triangle formed by the radius and the angle x is

$$\text{Area of triangle} = \frac{1}{2} \sin x.$$

The area of the triangle formed by the tangent line is

$$\text{Area of outer triangle} = \frac{1}{2} \tan x.$$

Thus, the areas satisfy

$$\frac{1}{2} \sin x < \frac{1}{2}x < \frac{1}{2} \tan x.$$

Multiply through by 2:

$$\sin x < x < \tan x.$$

Divide all parts by $\sin x$ (which is positive for $0 < x < \pi/2$):

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

Taking reciprocals reverses the inequalities:

$$\cos x < \frac{\sin x}{x} < 1.$$

Now take the limit as $x \rightarrow 0^+$.

Since

$$\lim_{x \rightarrow 0} \cos x = 1,$$

the Squeeze Theorem gives

$$\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1.$$

A similar argument applies for $x \rightarrow 0^-$, so the two-sided limit exists.

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Question 2

Prove that

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0.$$

Question 2 - Solution

We begin by rewriting the expression using a trigonometric identity.

Recall the identity

$$1 - \cos x = 2 \sin^2\left(\frac{x}{2}\right).$$

Substitute this into the limit:

$$\frac{1 - \cos x}{x} = \frac{2 \sin^2\left(\frac{x}{2}\right)}{x}.$$

Rewrite the fraction:

$$\frac{2 \sin^2\left(\frac{x}{2}\right)}{x} = \sin\left(\frac{x}{2}\right) \cdot \frac{2 \sin\left(\frac{x}{2}\right)}{x}.$$

Now rewrite the second factor:

$$\frac{2 \sin\left(\frac{x}{2}\right)}{x} = \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}.$$

Thus,

$$\frac{1 - \cos x}{x} = \sin\left(\frac{x}{2}\right) \cdot \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}.$$

Now take limits of each factor as $x \rightarrow 0$.

Since $\lim_{u \rightarrow 0} \sin u = 0$, we have

$$\lim_{x \rightarrow 0} \sin\left(\frac{x}{2}\right) = 0.$$

Also, using the fundamental trig limit

$$\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1,$$

we obtain

$$\lim_{x \rightarrow 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} = 1.$$

Therefore, by the product law for limits,

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0 \cdot 1 = 0.$$

0

Question 3

Prove that

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = 3.$$

Question 3 - Solution

We begin by rewriting the expression so that the fundamental trigonometric limit appears.

Write

$$\frac{\sin(3x)}{x} = \frac{\sin(3x)}{3x} \cdot 3.$$

Now take limits as $x \rightarrow 0$.

Since $3x \rightarrow 0$ as $x \rightarrow 0$, the fundamental trigonometric limit gives

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} = 1.$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = \left(\lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} \right) \cdot 3 = 1 \cdot 3 = 3.$$

□ 3

Question 4

Prove that

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1.$$

Question 4 - Solution

We rewrite the expression in terms of sine and cosine.

Recall that

$$\tan x = \frac{\sin x}{\cos x}.$$

Substitute this into the limit:

$$\frac{\tan x}{x} = \frac{\sin x}{x \cos x} = \left(\frac{\sin x}{x} \right) \left(\frac{1}{\cos x} \right).$$

Now take limits of each factor as $x \rightarrow 0$.

From the fundamental trigonometric limit,

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Also, since cosine is continuous at 0,

$$\lim_{x \rightarrow 0} \cos x = \cos 0 = 1,$$

and therefore

$$\lim_{x \rightarrow 0} \frac{1}{\cos x} = 1.$$

By the product law for limits,

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \cdot 1 = 1.$$

□

Question 5

Prove that

$$\lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{x^2} = 2.$$

Question 5 - Solution

We begin by using a trigonometric identity.

Recall that

$$1 - \cos(2x) = 2 \sin^2 x.$$

Substitute this into the expression:

$$\frac{1 - \cos(2x)}{x^2} = \frac{2 \sin^2 x}{x^2}.$$

Rewrite the fraction:

$$\frac{2 \sin^2 x}{x^2} = 2 \left(\frac{\sin x}{x} \right)^2.$$

Now take limits as $x \rightarrow 0$.

Using the fundamental trigonometric limit,

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Therefore,

$$\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 = 1^2 = 1.$$

Multiply by 2:

$$\lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{x^2} = 2 \cdot 1 = 2.$$

$\boxed{2}$

Question 6

Prove that

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x} = 0.$$

Question 6 - Solution

We begin by factoring the expression.

Rewrite the fraction as

$$\frac{\sin x - x}{x} = \frac{\sin x}{x} - 1.$$

Now take the limit as $x \rightarrow 0$.

Using the fundamental trigonometric limit,

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Therefore,

$$\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} - 1 \right) = 1 - 1 = 0.$$

Hence,

$$\boxed{\lim_{x \rightarrow 0} \frac{\sin x - x}{x} = 0}.$$

Question 7

Prove that

$$\lim_{x \rightarrow 0} \frac{\sin x}{\tan x} = 1.$$

Question 7 - Solution

Rewrite the expression using the identity

$$\tan x = \frac{\sin x}{\cos x}.$$

Substitute into the fraction:

$$\frac{\sin x}{\tan x} = \frac{\sin x}{\frac{\sin x}{\cos x}} = \cos x,$$

for all x where $\tan x$ is defined and $\sin x \neq 0$.

Now take the limit as $x \rightarrow 0$.

Since cosine is continuous at 0,

$$\lim_{x \rightarrow 0} \cos x = \cos 0 = 1.$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{\sin x}{\tan x} = 1.$$

□

Question 8

Prove that

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0.$$

Question 8 - Solution

We begin by rewriting the expression using a trigonometric identity.

Recall that

$$\cos x - 1 = -(1 - \cos x).$$

Thus,

$$\frac{\cos x - 1}{x} = -\frac{1 - \cos x}{x}.$$

From a previously established trigonometric limit,

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0.$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = -\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = -0 = 0.$$

$$\boxed{0}$$

Question 9

Prove that

$$\lim_{x \rightarrow 0} \frac{\sin(2x)}{\tan x} = 2.$$

Question 9 - Solution

We begin by rewriting the expression using trigonometric identities.

Recall that

$$\tan x = \frac{\sin x}{\cos x}.$$

Substitute this into the fraction:

$$\frac{\sin(2x)}{\tan x} = \sin(2x) \cdot \frac{\cos x}{\sin x}.$$

Next, use the identity

$$\sin(2x) = 2 \sin x \cos x.$$

Substitute:

$$\frac{\sin(2x)}{\tan x} = (2 \sin x \cos x) \cdot \frac{\cos x}{\sin x}.$$

Cancel $\sin x$ (for $x \neq 0$):

$$\frac{\sin(2x)}{\tan x} = 2 \cos^2 x.$$

Now take the limit as $x \rightarrow 0$.

Since cosine is continuous,

$$\lim_{x \rightarrow 0} \cos x = \cos 0 = 1.$$

Therefore,

$$\lim_{x \rightarrow 0} 2 \cos^2 x = 2(1)^2 = 2.$$

2

Question 10

Prove that

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x} = 0.$$

Question 10 - Solution

We begin by rewriting the expression to isolate a known limit.

Factor out x from the numerator:

$$\frac{\tan x - x}{x} = \frac{\tan x}{x} - 1.$$

Now rewrite $\tan x$ in terms of sine and cosine:

$$\frac{\tan x}{x} = \frac{\sin x}{x \cos x} = \left(\frac{\sin x}{x} \right) \left(\frac{1}{\cos x} \right).$$

Take limits of each factor as $x \rightarrow 0$.

Using the fundamental trigonometric limit,

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Since cosine is continuous at 0,

$$\lim_{x \rightarrow 0} \cos x = \cos 0 = 1, \quad \text{so} \quad \lim_{x \rightarrow 0} \frac{1}{\cos x} = 1.$$

Thus,

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1.$$

Finally,

$$\lim_{x \rightarrow 0} \left(\frac{\tan x}{x} - 1 \right) = 1 - 1 = 0.$$

0

Question 1

Assume that f has a local maximum at $x = a$ and that f is differentiable at $x = a$. Prove that

$$f'(a) = 0.$$

Question 1 - Solution

Since f has a local maximum at $x = a$, there exists $\delta > 0$ such that

$$f(x) \leq f(a) \quad \text{for all } x \text{ with } |x - a| < \delta.$$

Consider the difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

for small values of h .

Case 1: $h > 0$

If $0 < h < \delta$, then $a + h$ lies within the interval where $f(a)$ is a local maximum. Hence,

$$f(a+h) - f(a) \leq 0.$$

Since $h > 0$, dividing by h preserves the inequality:

$$\frac{f(a+h) - f(a)}{h} \leq 0.$$

Case 2: $h < 0$

If $-\delta < h < 0$, then again $a + h$ lies within the interval of the local maximum, so

$$f(a+h) - f(a) \leq 0.$$

Since $h < 0$, dividing by h reverses the inequality:

$$\frac{f(a+h) - f(a)}{h} \geq 0.$$

Step 3: Take the limit

For $h > 0$, the difference quotient is less than or equal to 0. For $h < 0$, the difference quotient is greater than or equal to 0.

Since f is differentiable at a , the limit

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

exists and must be the same from both sides.

The only number that is both ≤ 0 and ≥ 0 is 0.

Therefore,

$$f'(a) = 0.$$

$$\boxed{f'(a) = 0}$$

Question 2

Assume that f is continuous on $[a, b]$, differentiable on (a, b) , and

$$f(a) = f(b).$$

Prove that there exists a number $c \in (a, b)$ such that

$$f'(c) = 0.$$

Question 2 - Solution

Since f is continuous on the closed interval $[a, b]$, the Extreme Value Theorem guarantees that f attains both a maximum and a minimum on $[a, b]$.

Let

$$M = \max_{x \in [a, b]} f(x), \quad m = \min_{x \in [a, b]} f(x).$$

If $M = m$, then f is constant on $[a, b]$, and hence

$$f'(x) = 0 \quad \text{for all } x \in (a, b).$$

In particular, there exists $c \in (a, b)$ such that $f'(c) = 0$.

Now assume that $M \neq m$. Since $f(a) = f(b)$, the maximum or minimum cannot occur at both endpoints.

Thus, at least one of the extreme values M or m must occur at some point $c \in (a, b)$.

At such a point c , the function f has either a local maximum or a local minimum. Because f is differentiable at c , the derivative must vanish there.

Therefore,

$$f'(c) = 0.$$

There exists $c \in (a, b)$ such that $f'(c) = 0$

Question 3

Assume that f is continuous on $[a, b]$ and differentiable on (a, b) . Prove that there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Question 3 - Solution

Define a new function

$$g(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a} \right) (x - a).$$

The function g is continuous on $[a, b]$ and differentiable on (a, b) because it is formed from functions with these properties.

Evaluate g at the endpoints:

$$g(a) = f(a) - \left(\frac{f(b) - f(a)}{b - a} \right) (a - a) = f(a),$$

$$g(b) = f(b) - \left(\frac{f(b) - f(a)}{b - a} \right) (b - a) = f(b) - (f(b) - f(a)) = f(a).$$

Thus,

$$g(a) = g(b).$$

By Rolle's Theorem, there exists a number $c \in (a, b)$ such that

$$g'(c) = 0.$$

Differentiate $g(x)$:

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Substitute $x = c$:

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Solve for $f'(c)$:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Question 4

Assume that f is differentiable on an interval (a, b) and that

$$f'(x) > 0 \quad \text{for all } x \in (a, b).$$

Prove that f is increasing on (a, b) .

Question 4 - Solution

To show that f is increasing on (a, b) , we must prove that for any $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, we have

$$f(x_1) < f(x_2).$$

Let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. Since f is differentiable on (a, b) , it is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) .

By the Mean Value Theorem, there exists a number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By assumption, $f'(c) > 0$, and since $x_2 - x_1 > 0$, it follows that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0.$$

Multiply both sides by $x_2 - x_1$:

$$f(x_2) - f(x_1) > 0.$$

Thus,

$$f(x_2) > f(x_1).$$

Since this holds for all $x_1 < x_2$ in (a, b) , the function f is increasing on (a, b) .

f is increasing on (a, b)

Question 5

Assume that f is differentiable on an interval (a, b) and that

$$f'(x) < 0 \quad \text{for all } x \in (a, b).$$

Prove that f is decreasing on (a, b) .

Question 5 - Solution

To prove that f is decreasing on (a, b) , we must show that for any $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, the inequality

$$f(x_1) > f(x_2)$$

holds.

Let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. Because f is differentiable on (a, b) , it is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) .

By the Mean Value Theorem, there exists a number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By assumption, $f'(c) < 0$, and since $x_2 - x_1 > 0$, it follows that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0.$$

Multiply both sides by $x_2 - x_1$:

$$f(x_2) - f(x_1) < 0.$$

Thus,

$$f(x_2) < f(x_1).$$

Since this holds for all $x_1 < x_2$ in (a, b) , the function f is decreasing on (a, b) .

$f \text{ is decreasing on } (a, b)$

Question 6

Assume that f is twice differentiable on an interval (a, b) and that

$$f''(x) > 0 \quad \text{for all } x \in (a, b).$$

Prove that f is concave up on (a, b) .

Question 6 - Solution

To show that f is concave up on (a, b) , we must prove that the derivative $f'(x)$ is increasing on (a, b) .

Let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. Since f is twice differentiable on (a, b) , the function f' is differentiable on (a, b) and therefore continuous on $[x_1, x_2]$.

Apply the Mean Value Theorem to f' on the interval $[x_1, x_2]$. There exists a number $c \in (x_1, x_2)$ such that

$$f''(c) = \frac{f'(x_2) - f'(x_1)}{x_2 - x_1}.$$

By assumption, $f''(c) > 0$ and since $x_2 - x_1 > 0$, it follows that

$$\frac{f'(x_2) - f'(x_1)}{x_2 - x_1} > 0.$$

Multiply both sides by $x_2 - x_1$:

$$f'(x_2) - f'(x_1) > 0.$$

Thus,

$$f'(x_2) > f'(x_1).$$

Since this holds for all $x_1 < x_2$ in (a, b) , the derivative $f'(x)$ is increasing on (a, b) .

Therefore, the function f is concave up on (a, b) .

$f \text{ is concave up on } (a, b)$

Question 7

Assume that f is twice differentiable on an interval (a, b) and that

$$f''(x) < 0 \quad \text{for all } x \in (a, b).$$

Prove that f is concave down on (a, b) .

Question 7 - Solution

To show that f is concave down on (a, b) , we must show that the derivative $f'(x)$ is decreasing on (a, b) .

Let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. Since f is twice differentiable on (a, b) , the function f' is differentiable on (a, b) and therefore continuous on $[x_1, x_2]$.

Apply the Mean Value Theorem to f' on the interval $[x_1, x_2]$. There exists a number $c \in (x_1, x_2)$ such that

$$f''(c) = \frac{f'(x_2) - f'(x_1)}{x_2 - x_1}.$$

By assumption, $f''(c) < 0$, and since $x_2 - x_1 > 0$, it follows that

$$\frac{f'(x_2) - f'(x_1)}{x_2 - x_1} < 0.$$

Multiply both sides by $x_2 - x_1$:

$$f'(x_2) - f'(x_1) < 0.$$

Thus,

$$f'(x_2) < f'(x_1).$$

Since this holds for all $x_1 < x_2$ in (a, b) , the derivative $f'(x)$ is decreasing on (a, b) .

Therefore, the function f is concave down on (a, b) .

$f \text{ is concave down on } (a, b)$

Question 8

Assume that f is differentiable on an open interval I containing a , and that

$$f'(a) = 0 \quad \text{and} \quad f''(a) > 0.$$

Prove that f has a strict local minimum at $x = a$.

Question 8 - Solution

By the definition of the second derivative and $f'(a) = 0$,

$$\lim_{h \rightarrow 0} \frac{f'(a+h) - f'(a)}{h} = f''(a).$$

Because this limit is nonzero, the quotient has the same sign as $f''(a)$ for all sufficiently small nonzero h .

Consequently, $f'(a+h) < 0$ for negative h and $f'(a+h) > 0$ for positive h . Thus f is strictly decreasing just left of a and strictly increasing just right of a , by the Mean Value Theorem.

Therefore f has a strict local minimum at a . No continuity assumption on f'' near a is needed.

Question 9

Assume that f is continuous on $[a, b]$, differentiable on (a, b) , and that

$$f'(x) = 0 \quad \text{for all } x \in (a, b).$$

Prove that f is constant on $[a, b]$.

Question 9 - Solution

Let $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. We will show that

$$f(x_1) = f(x_2).$$

Since f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) , the Mean Value Theorem applies. Thus, there exists a number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By assumption, $f'(c) = 0$, so

$$0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $x_2 - x_1 \neq 0$, it follows that

$$f(x_2) - f(x_1) = 0.$$

Therefore,

$$f(x_2) = f(x_1).$$

Because x_1 and x_2 were arbitrary points in $[a, b]$, we conclude that $f(x)$ has the same value at every point in the interval.

f is constant on $[a, b]$

Question 10

Assume that f is differentiable on an open interval I containing a , and that

$$f'(a) = 0 \quad \text{and} \quad f''(a) < 0.$$

Prove that f has a strict local maximum at $x = a$.

Question 10 - Solution

By the definition of the second derivative and $f'(a) = 0$,

$$\lim_{h \rightarrow 0} \frac{f'(a+h)}{h} = f''(a).$$

Because this limit is nonzero, the quotient has the same sign as $f''(a)$ for all sufficiently small nonzero h .

Consequently, $f'(a+h) > 0$ for negative h and $f'(a+h) < 0$ for positive h . Thus f is strictly increasing just left of a and strictly decreasing just right of a , by the Mean Value Theorem.

Therefore f has a strict local maximum at a . No continuity assumption on f'' near a is needed.

Question 1

Assume that f is integrable on the interval $[a, b]$ and that c is a constant. Prove that

$$\int_a^b c f(x) dx = c \int_a^b f(x) dx.$$

Question 1 - Solution

We prove this property using the definition of the definite integral in terms of Riemann sums.

Let P be a partition of $[a, b]$:

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let $\Delta x_i = x_i - x_{i-1}$. Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

A Riemann sum for f on $[a, b]$ is

$$\sum_{i=1}^n f(x_i^*) \Delta x_i.$$

A Riemann sum for cf on $[a, b]$ is

$$\sum_{i=1}^n cf(x_i^*) \Delta x_i.$$

Factor out the constant c :

$$\sum_{i=1}^n cf(x_i^*) \Delta x_i = c \sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Now take the limit as the norm of the partition $\|P\| \rightarrow 0$. Since f is integrable on $[a, b]$, the limit of its Riemann sums exists:

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i = \int_a^b f(x) dx.$$

Therefore,

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n cf(x_i^*) \Delta x_i = c \int_a^b f(x) dx.$$

By definition of the definite integral, this limit equals

$$\int_a^b cf(x) dx.$$

Hence,

$$\int_a^b cf(x) dx = c \int_a^b f(x) dx.$$

$$\int_a^b cf(x) dx = c \int_a^b f(x) dx$$

Question 2

Assume that f and g are integrable on the interval $[a, b]$. Prove that

$$\int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx.$$

Question 2 - Solution

We prove this property using Riemann sums.

Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of $[a, b]$, and let $\Delta x_i = x_i - x_{i-1}$. Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

A Riemann sum for $f + g$ is

$$\sum_{i=1}^n (f(x_i^*) + g(x_i^*)) \Delta x_i.$$

Using algebra, split the sum:

$$\sum_{i=1}^n (f(x_i^*) + g(x_i^*)) \Delta x_i = \sum_{i=1}^n f(x_i^*) \Delta x_i + \sum_{i=1}^n g(x_i^*) \Delta x_i.$$

Now take the limit as the norm of the partition $\|P\| \rightarrow 0$.

Since f and g are integrable on $[a, b]$, the limits of both sums exist:

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i = \int_a^b f(x) dx,$$

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n g(x_i^*) \Delta x_i = \int_a^b g(x) dx.$$

Therefore,

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

$$\boxed{\int_a^b (f + g) = \int_a^b f + \int_a^b g}$$

Question 3

Assume that f is integrable on $[a, b]$ and that $a < b$. Prove that

$$\int_a^a f(x) \, dx = 0.$$

Question 3 - Solution

We use the definition of the definite integral in terms of Riemann sums.

Consider the interval $[a, a]$. Any partition of this interval consists of a single point:

$$a = x_0 = x_1.$$

The width of the subinterval is

$$\Delta x = x_1 - x_0 = a - a = 0.$$

Choose any sample point x^* in the interval $[a, a]$.

A Riemann sum for f over $[a, a]$ is

$$\sum f(x^*) \Delta x = f(x^*) \cdot 0 = 0.$$

Since every Riemann sum over $[a, a]$ equals 0, the limit of the Riemann sums exists and equals 0.

By definition of the definite integral,

$$\int_a^a f(x) dx = 0.$$

$$\int_a^a f(x) dx = 0$$

Question 4

Assume that f is integrable on $[a, b]$. Prove that

$$\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx.$$

Question 4 - Solution

We use the definition of the definite integral in terms of Riemann sums.

Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of the interval $[a, b]$, with subinterval widths

$$\Delta x_i = x_i - x_{i-1}.$$

A Riemann sum for f over $[a, b]$ is

$$\sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Now consider the interval $[b, a]$. Reverse the order of the partition:

$$b = x_n > x_{n-1} > \cdots > x_0 = a.$$

The corresponding subinterval widths are

$$\Delta x'_i = x_{i-1} - x_i = -\Delta x_i.$$

A Riemann sum for f over $[b, a]$ is

$$\sum_{i=1}^n f(x_i^*) \Delta x'_i = \sum_{i=1}^n f(x_i^*) (-\Delta x_i) = - \sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Take the limit as the norm of the partition goes to zero. Since f is integrable on $[a, b]$, the limit exists:

$$\int_b^a f(x) dx = - \int_a^b f(x) dx.$$

Rewriting,

$$\int_a^b f(x) dx = - \int_b^a f(x) dx.$$

$$\boxed{\int_a^b f(x) dx = - \int_b^a f(x) dx}$$

Question 5

Assume that f is integrable on $[a, b]$ and that

$$f(x) \geq 0 \quad \text{for all } x \in [a, b].$$

Prove that

$$\int_a^b f(x) \, dx \geq 0.$$

Question 5 - Solution

We use the definition of the definite integral in terms of Riemann sums.

Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of the interval $[a, b]$, and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

A Riemann sum for f on $[a, b]$ is

$$\sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Since $f(x) \geq 0$ for all $x \in [a, b]$ and $\Delta x_i > 0$, each term in the sum satisfies

$$f(x_i^*) \Delta x_i \geq 0.$$

Thus, every Riemann sum for f over $[a, b]$ is nonnegative:

$$\sum_{i=1}^n f(x_i^*) \Delta x_i \geq 0.$$

Now take the limit as the norm of the partition $\|P\| \rightarrow 0$. Since f is integrable, the limit of the Riemann sums exists and equals the definite integral:

$$\int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Because every sum in the limit is nonnegative, the limit itself must be nonnegative.

Therefore,

$$\int_a^b f(x) dx \geq 0.$$

$$\int_a^b f(x) dx \geq 0$$

Question 6

Assume that f and g are integrable on $[a, b]$ and that

$$f(x) \leq g(x) \quad \text{for all } x \in [a, b].$$

Prove that

$$\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx.$$

Question 6 - Solution

Define a new function

$$h(x) = g(x) - f(x).$$

Since f and g are integrable on $[a, b]$, the function h is also integrable on $[a, b]$.

From the assumption $f(x) \leq g(x)$ for all $x \in [a, b]$, we have

$$h(x) = g(x) - f(x) \geq 0 \quad \text{for all } x \in [a, b].$$

By the positivity property of integrals,

$$\int_a^b h(x) \, dx \geq 0.$$

Substitute back for $h(x)$:

$$\int_a^b (g(x) - f(x)) \, dx \geq 0.$$

Use linearity of the integral:

$$\int_a^b g(x) \, dx - \int_a^b f(x) \, dx \geq 0.$$

Rearranging gives

$$\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx.$$

$$\boxed{\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx}$$

Question 7

Assume that f is integrable on $[a, b]$. Prove that

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$$

Question 7 - Solution

We begin with a basic inequality involving absolute values.

For all real numbers u , we have

$$-|u| \leq u \leq |u|.$$

Apply this inequality to $f(x)$:

$$-|f(x)| \leq f(x) \leq |f(x)| \quad \text{for all } x \in [a, b].$$

Integrate each part of the inequality over $[a, b]$:

$$\int_a^b -|f(x)| \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx.$$

Simplify the left-hand integral:

$$-\int_a^b |f(x)| \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx.$$

This shows that the value of $\int_a^b f(x) \, dx$ lies between $-\int_a^b |f(x)| \, dx$ and $\int_a^b |f(x)| \, dx$.

By definition of absolute value, this implies

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$$

$$\boxed{\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx}$$

Question 8

Assume that f is integrable on $[a, b]$ and that c satisfies

$$a < c < b.$$

Prove that

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$$

Question 8 - Solution

We prove this property using Riemann sums.

Let

$$a = x_0 < x_1 < \cdots < x_k = c < \cdots < x_n = b$$

be a partition of $[a, b]$ that includes the point c .

Let

$$\Delta x_i = x_i - x_{i-1}$$

and choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

A Riemann sum for f over $[a, b]$ is

$$\sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Split this sum at c :

$$\sum_{i=1}^n f(x_i^*) \Delta x_i = \sum_{i=1}^k f(x_i^*) \Delta x_i + \sum_{i=k+1}^n f(x_i^*) \Delta x_i.$$

The first sum is a Riemann sum for f over $[a, c]$, and the second sum is a Riemann sum for f over $[c, b]$.

Now take the limit as the norm of the partition $\|P\| \rightarrow 0$.

Since f is integrable on $[a, b]$, it is integrable on both subintervals $[a, c]$ and $[c, b]$.

Thus,

$$\begin{aligned} \lim_{\|P\| \rightarrow 0} \sum_{i=1}^k f(x_i^*) \Delta x_i &= \int_a^c f(x) dx, \\ \lim_{\|P\| \rightarrow 0} \sum_{i=k+1}^n f(x_i^*) \Delta x_i &= \int_c^b f(x) dx. \end{aligned}$$

Therefore,

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

Question 9

Assume that f is integrable on $[a, b]$ and that there exists a real number m such that

$$f(x) \geq m \quad \text{for all } x \in [a, b].$$

Prove that

$$\int_a^b f(x) \, dx \geq m(b - a).$$

Question 9 - Solution

Define a new function

$$h(x) = f(x) - m.$$

Since f is integrable on $[a, b]$ and m is a constant, the function h is also integrable on $[a, b]$.

From the assumption $f(x) \geq m$ for all $x \in [a, b]$, we have

$$h(x) = f(x) - m \geq 0 \quad \text{for all } x \in [a, b].$$

By the positivity property of integrals,

$$\int_a^b h(x) \, dx \geq 0.$$

Substitute back for $h(x)$:

$$\int_a^b (f(x) - m) \, dx \geq 0.$$

Use linearity of the integral:

$$\int_a^b f(x) \, dx - \int_a^b m \, dx \geq 0.$$

Since $\int_a^b m \, dx = m(b - a)$, we obtain

$$\int_a^b f(x) \, dx - m(b - a) \geq 0.$$

Rearranging gives

$$\int_a^b f(x) \, dx \geq m(b - a).$$

$$\int_a^b f(x) \, dx \geq m(b - a)$$

Question 10

Assume that f is integrable on $[a, b]$ and that there exists a real number M such that

$$f(x) \leq M \quad \text{for all } x \in [a, b].$$

Prove that

$$\int_a^b f(x) \, dx \leq M(b - a).$$

Question 10 - Solution

Define the function

$$h(x) = M - f(x).$$

Since f is integrable on $[a, b]$ and M is a constant, the function h is also integrable on $[a, b]$.

From the assumption $f(x) \leq M$ for all $x \in [a, b]$, we have

$$h(x) = M - f(x) \geq 0 \quad \text{for all } x \in [a, b].$$

By the positivity property of integrals,

$$\int_a^b h(x) \, dx \geq 0.$$

Substitute back for $h(x)$:

$$\int_a^b (M - f(x)) \, dx \geq 0.$$

Use linearity of the integral:

$$\int_a^b M \, dx - \int_a^b f(x) \, dx \geq 0.$$

Since

$$\int_a^b M \, dx = M(b - a),$$

we obtain

$$M(b - a) - \int_a^b f(x) \, dx \geq 0.$$

Rearranging gives

$$\int_a^b f(x) \, dx \leq M(b - a).$$

$$\int_a^b f(x) \, dx \leq M(b - a)$$

Question 1

Assume that f is a continuous, nonnegative function on the interval $[a, b]$. Prove that the area of the region bounded by the graph of $y = f(x)$, the x -axis, and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b f(x) \, dx.$$

Question 1 - Solution

We interpret area using approximating rectangles.

Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of $[a, b]$, and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

Since $f(x) \geq 0$, the area of the rectangle over $[x_{i-1}, x_i]$ with height $f(x_i^*)$ is

$$\text{Area}_i = f(x_i^*) \Delta x_i.$$

The total area of all rectangles is

$$\sum_{i=1}^n f(x_i^*) \Delta x_i.$$

As the partition is refined (that is, as $\|P\| \rightarrow 0$), the rectangles better approximate the region under the curve. Because f is continuous on $[a, b]$, it is integrable, and the limit of these sums exists.

Taking the limit gives

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i = \int_a^b f(x) dx.$$

By definition, this limit equals the exact area of the region under the curve.

Therefore, the area of the region bounded by $y = f(x)$, the x -axis, and $x = a$, $x = b$ is

$$A = \int_a^b f(x) dx.$$

$$A = \int_a^b f(x) dx$$

Question 2

Assume that f is a continuous, nonnegative function on $[a, b]$. Prove that the volume of the solid obtained by revolving the region bounded by $y = f(x)$, the x -axis, and the lines $x = a$ and $x = b$ about the x -axis is

$$V = \pi \int_a^b [f(x)]^2 dx.$$

Question 2 - Solution

We approximate the solid of revolution using thin disks.

Partition the interval $[a, b]$ as

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let $\Delta x_i = x_i - x_{i-1}$. Choose a sample point x_i^* in each subinterval.

Over the interval $[x_{i-1}, x_i]$, the graph of $y = f(x)$ generates a disk when revolved about the x -axis.

The radius of this disk is

$$r_i = f(x_i^*).$$

Thus, the volume of the disk is

$$\Delta V_i = \pi r_i^2 \Delta x_i = \pi [f(x_i^*)]^2 \Delta x_i.$$

The total volume of all disks is approximated by the sum

$$\sum_{i=1}^n \pi [f(x_i^*)]^2 \Delta x_i.$$

As the partition is refined and $\|P\| \rightarrow 0$, this sum approaches the exact volume of the solid. Since f is continuous, the limit exists.

Taking the limit yields

$$V = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n \pi [f(x_i^*)]^2 \Delta x_i = \pi \int_a^b [f(x)]^2 dx.$$

$$V = \pi \int_a^b [f(x)]^2 dx$$

Question 3

Assume that f and g are continuous functions on $[a, b]$ with

$$f(x) \geq g(x) \quad \text{for all } x \in [a, b].$$

Prove that the area of the region bounded by the graphs of $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is

$$A = \int_a^b (f(x) - g(x)) \, dx.$$

Question 3 - Solution

We approximate the region using vertical rectangles.

Partition the interval $[a, b]$ as

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

At the point x_i^* , the height of the region between the curves is

$$f(x_i^*) - g(x_i^*).$$

Thus, the area of the corresponding rectangle is

$$\Delta A_i = (f(x_i^*) - g(x_i^*)) \Delta x_i.$$

The total area of all rectangles is approximated by the sum

$$\sum_{i=1}^n (f(x_i^*) - g(x_i^*)) \Delta x_i.$$

As the partition is refined and $\|P\| \rightarrow 0$, these rectangles more accurately approximate the region between the curves. Since f and g are continuous, the limit of the sums exists.

Taking the limit gives

$$A = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n (f(x_i^*) - g(x_i^*)) \Delta x_i = \int_a^b (f(x) - g(x)) \, dx.$$

$$A = \int_a^b (f(x) - g(x)) \, dx$$

Question 4

Assume $0 \leq a < b$ and that f is continuous and nonnegative on $[a, b]$. Prove that the volume of the solid obtained by revolving the region bounded by $y = f(x)$, the x -axis, and the lines $x = a$ and $x = b$ about the y -axis is

$$V = 2\pi \int_a^b x f(x) dx.$$

Question 4 - Solution

Take a partition $a = x_0 < \cdots < x_n = b$ and let m_i, M_i be the minimum and maximum of f on $[x_{i-1}, x_i]$.

Since $a \geq 0$, the annular cylinders occupy disjoint radial intervals. Their volumes bound the desired volume:

$$\sum_i \pi(x_i^2 - x_{i-1}^2)m_i \leq V \leq \sum_i \pi(x_i^2 - x_{i-1}^2)M_i.$$

Let $c_i = (x_i + x_{i-1})/2$. Then $\pi(x_i^2 - x_{i-1}^2) = 2\pi c_i \Delta x_i$.

Uniform continuity of f makes $\max_i (M_i - m_i) \rightarrow 0$ as the mesh tends to zero, so the difference between the upper and lower bounds tends to zero.

Both bounds approach the Riemann integral of $2\pi x f(x)$. Hence

$$V = 2\pi \int_a^b x f(x) dx.$$

Question 5

Assume that f is a continuous, nonnegative function on $[a, b]$. Prove that the volume of the solid obtained by revolving the region bounded by $y = f(x)$, the x -axis, and the lines $x = a$ and $x = b$ about the x -axis is

$$V = \pi \int_a^b [f(x)]^2 dx.$$

Question 5 - Solution

We derive the formula using the idea of approximating the solid with thin disks.

Partition the interval $[a, b]$ into subintervals

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

Over a small interval $[x_{i-1}, x_i]$, the graph of $y = f(x)$ is approximately constant at height $f(x_i^*)$. When this vertical strip is revolved about the x -axis, it forms a thin circular disk.

The radius of the disk is

$$r_i = f(x_i^*).$$

The area of the circular face of the disk is

$$A_i = \pi r_i^2 = \pi [f(x_i^*)]^2.$$

Since the thickness of the disk is Δx_i , its volume is approximately

$$\Delta V_i = \pi [f(x_i^*)]^2 \Delta x_i.$$

Adding the volumes of all disks gives an approximation to the total volume:

$$\sum_{i=1}^n \pi [f(x_i^*)]^2 \Delta x_i.$$

As the partition is refined and the maximum subinterval length approaches zero, these disks more accurately fill the solid. Because f is continuous on $[a, b]$, the limit of the sum exists.

Taking the limit yields

$$V = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n \pi [f(x_i^*)]^2 \Delta x_i = \pi \int_a^b [f(x)]^2 dx.$$

$$V = \pi \int_a^b [f(x)]^2 dx$$

Question 6

Assume that f and g are continuous functions on $[a, b]$ with

$$f(x) \geq g(x) \geq 0 \quad \text{for all } x \in [a, b].$$

Prove that the volume of the solid obtained by revolving the region bounded by $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$ about the x -axis is

$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx.$$

Question 6 - Solution

We derive the formula using washers (disks with holes).

Partition the interval $[a, b]$ as

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

Over a small interval near x_i^* , the region between the curves has an outer radius

$$R_i = f(x_i^*)$$

and an inner radius

$$r_i = g(x_i^*).$$

When this vertical strip is revolved about the x -axis, it forms a thin washer.

The area of the outer disk is

$$\pi R_i^2 = \pi [f(x_i^*)]^2,$$

and the area of the inner disk (the hole) is

$$\pi r_i^2 = \pi [g(x_i^*)]^2.$$

Thus, the area of the washer is

$$\pi ([f(x_i^*)]^2 - [g(x_i^*)]^2).$$

Multiplying by the thickness Δx_i , the volume of the washer is

$$\Delta V_i = \pi ([f(x_i^*)]^2 - [g(x_i^*)]^2) \Delta x_i.$$

Adding the volumes of all washers gives an approximation to the total volume:

$$\sum_{i=1}^n \pi ([f(x_i^*)]^2 - [g(x_i^*)]^2) \Delta x_i.$$

As the partition is refined and $\|P\| \rightarrow 0$, the washers fill the solid more accurately. Since f and g are continuous on $[a, b]$, the limit of the sum exists.

Taking the limit yields

$$V = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n \pi ([f(x_i^*)]^2 - [g(x_i^*)]^2) \Delta x_i = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx.$$

$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx$$

Question 7

Prove that the area of a circle of radius R is

$$A = \pi R^2.$$

Question 7 - Solution

We derive the area formula using integration.

Consider a circle of radius R centered at the origin. Its equation is

$$x^2 + y^2 = R^2.$$

Solving for y gives the upper semicircle:

$$y = \sqrt{R^2 - x^2}.$$

The area of the entire circle is twice the area of the upper semicircle:

$$A = 2 \int_{-R}^R \sqrt{R^2 - x^2} dx.$$

To evaluate the integral, use the substitution

$$x = R \sin \theta, \quad dx = R \cos \theta d\theta.$$

When $x = -R$, $\theta = -\frac{\pi}{2}$, and when $x = R$, $\theta = \frac{\pi}{2}$.

Substitute into the integral:

$$\int_{-R}^R \sqrt{R^2 - x^2} dx = \int_{-\pi/2}^{\pi/2} \sqrt{R^2 - R^2 \sin^2 \theta} (R \cos \theta) d\theta.$$

Simplify:

$$\sqrt{R^2(1 - \sin^2 \theta)} = R \cos \theta.$$

Thus,

$$\int_{-R}^R \sqrt{R^2 - x^2} dx = \int_{-\pi/2}^{\pi/2} R^2 \cos^2 \theta d\theta.$$

Use the identity

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}.$$

Then

$$\int_{-\pi/2}^{\pi/2} R^2 \cos^2 \theta d\theta = \frac{R^2}{2} \int_{-\pi/2}^{\pi/2} (1 + \cos(2\theta)) d\theta.$$

Evaluate:

$$\int_{-\pi/2}^{\pi/2} 1 d\theta = \pi, \quad \int_{-\pi/2}^{\pi/2} \cos(2\theta) d\theta = 0.$$

So,

$$\int_{-R}^R \sqrt{R^2 - x^2} dx = \frac{R^2}{2} \pi.$$

Finally, multiply by 2 to get the area of the full circle:

$$A = 2 \cdot \frac{R^2}{2} \pi = \pi R^2.$$

$$\boxed{A = \pi R^2}$$

Question 8

Prove that the volume of a right circular cylinder with radius R and height h is

$$V = \pi R^2 h.$$

Question 8 - Solution

We derive the volume formula using cross-sectional areas and integration.

Place the cylinder so that its base lies in the xy -plane and its axis is along the x -axis. Then the cylinder extends from $x = 0$ to $x = h$.

At any position x with $0 \leq x \leq h$, a cross-section perpendicular to the x -axis is a circle of radius R .

The area of this circular cross-section is constant and equal to

$$A(x) = \pi R^2.$$

The volume of the cylinder can be approximated by slicing it into thin slabs of thickness dx . Each slab has volume approximately

$$dV = A(x) dx = \pi R^2 dx.$$

Adding the volumes of all such slabs gives

$$V = \int_0^h \pi R^2 dx.$$

Since πR^2 is constant, evaluate the integral:

$$V = \pi R^2 \int_0^h dx = \pi R^2 [x]_0^h = \pi R^2 h.$$

$$\boxed{V = \pi R^2 h}$$

Question 9

Prove that the volume of a right circular cone with base radius R and height h is

$$V = \frac{1}{3}\pi R^2 h.$$

Question 9 - Solution

We derive the formula using cross-sectional disks and integration.

Place the cone so that its tip is at the origin and its base lies in the plane $x = h$. The axis of the cone lies along the x -axis.

At a distance x from the tip, where $0 \leq x \leq h$, the radius of the cone is determined by similar triangles.

Since the radius grows linearly from 0 to R over the height h , the radius at position x is

$$r(x) = \frac{R}{h}x.$$

A cross-section perpendicular to the x -axis at position x is a circle of radius $r(x)$. Its area is

$$A(x) = \pi[r(x)]^2 = \pi \left(\frac{R}{h}x \right)^2 = \pi \frac{R^2}{h^2}x^2.$$

An infinitesimal slice of thickness dx has volume

$$dV = A(x) dx = \pi \frac{R^2}{h^2}x^2 dx.$$

Integrate from $x = 0$ to $x = h$ to obtain the total volume:

$$V = \int_0^h \pi \frac{R^2}{h^2}x^2 dx.$$

Factor out the constants:

$$V = \pi \frac{R^2}{h^2} \int_0^h x^2 dx.$$

Evaluate the integral:

$$\int_0^h x^2 dx = \left[\frac{x^3}{3} \right]_0^h = \frac{h^3}{3}.$$

Substitute back:

$$V = \pi \frac{R^2}{h^2} \cdot \frac{h^3}{3} = \frac{1}{3}\pi R^2 h.$$

$$V = \frac{1}{3}\pi R^2 h$$

Question 10

Prove that the volume of a sphere of radius R is

$$V = \frac{4}{3}\pi R^3.$$

Question 10 - Solution

We derive the formula using the method of cross-sectional disks.

Place the sphere of radius R so that it is centered at the origin. Its equation is

$$x^2 + y^2 + z^2 = R^2.$$

Fix a value of x with $-R \leq x \leq R$. A cross-section perpendicular to the x -axis is a circle whose radius depends on x .

Solving the equation of the sphere for $y^2 + z^2$ gives

$$y^2 + z^2 = R^2 - x^2.$$

Thus, the radius of the cross-sectional disk at position x is

$$r(x) = \sqrt{R^2 - x^2}.$$

The area of this circular cross-section is

$$A(x) = \pi[r(x)]^2 = \pi(R^2 - x^2).$$

An infinitesimally thin slice of thickness dx has volume

$$dV = A(x) dx = \pi(R^2 - x^2) dx.$$

Integrate from $x = -R$ to $x = R$ to obtain the total volume:

$$V = \int_{-R}^R \pi(R^2 - x^2) dx.$$

Factor out π :

$$V = \pi \int_{-R}^R (R^2 - x^2) dx.$$

Evaluate the integral:

$$\int_{-R}^R R^2 dx = 2R^3, \quad \int_{-R}^R x^2 dx = \frac{2R^3}{3}.$$

Thus,

$$\int_{-R}^R (R^2 - x^2) dx = 2R^3 - \frac{2R^3}{3} = \frac{4R^3}{3}.$$

Substitute back:

$$V = \pi \cdot \frac{4R^3}{3} = \frac{4}{3}\pi R^3.$$

$$\boxed{V = \frac{4}{3}\pi R^3}$$

Question 1

Prove that

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0.$$

Question 1 - Solution

We compare the growth rates of the numerator and denominator.

For $x > 0$, both $\ln x$ and x are differentiable. Apply L'Hôpital's Rule to the limit

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x}.$$

Differentiate the numerator and denominator:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad \frac{d}{dx}(x) = 1.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = \lim_{x \rightarrow \infty} \frac{1}{x}.$$

From basic limits,

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

Therefore,

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0.$$

$$\boxed{0}$$

Question 2

Prove that

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0 \quad \text{for any positive integer } n.$$

Question 2 - Solution

We compare the growth rates of the polynomial x^n and the exponential function e^x .

Both x^n and e^x are differentiable for all real x . Apply L'Hôpital's Rule repeatedly to the limit

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^x}.$$

Differentiate the numerator and denominator once:

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad \frac{d}{dx}(e^x) = e^x.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = \lim_{x \rightarrow \infty} \frac{nx^{n-1}}{e^x}.$$

Apply L'Hôpital's Rule again. After n applications, the numerator becomes a constant:

$$\lim_{x \rightarrow \infty} \frac{n!}{e^x}.$$

Since e^x grows without bound as $x \rightarrow \infty$,

$$\lim_{x \rightarrow \infty} \frac{n!}{e^x} = 0.$$

Therefore,

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0.$$

$$\boxed{0}$$

Question 3

Prove that

$$\lim_{x \rightarrow \infty} \sqrt{x^2 + x} - x = \frac{1}{2}.$$

Question 3 - Solution

We begin by algebraically simplifying the expression.

Consider

$$\sqrt{x^2 + x} - x.$$

Multiply and divide by the conjugate:

$$\sqrt{x^2 + x} - x = \frac{(\sqrt{x^2 + x} - x)(\sqrt{x^2 + x} + x)}{\sqrt{x^2 + x} + x}.$$

Simplify the numerator:

$$(x^2 + x) - x^2 = x.$$

Thus,

$$\sqrt{x^2 + x} - x = \frac{x}{\sqrt{x^2 + x} + x}.$$

Factor x out of the square root in the denominator:

$$\sqrt{x^2 + x} = x\sqrt{1 + \frac{1}{x}}.$$

Substitute:

$$\frac{x}{x\sqrt{1 + \frac{1}{x}} + x} = \frac{x}{x\left(\sqrt{1 + \frac{1}{x}} + 1\right)}.$$

Cancel x :

$$\frac{1}{\sqrt{1 + \frac{1}{x}} + 1}.$$

Now take the limit as $x \rightarrow \infty$:

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{x}} + 1} = \frac{1}{\sqrt{1 + 0} + 1} = \frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$

Question 4

Prove that

$$\lim_{x \rightarrow \infty} (x - \sqrt{x^2 - 3x}) = \frac{3}{2}.$$

Question 4 - Solution

We simplify the expression using algebraic manipulation.

Consider

$$x - \sqrt{x^2 - 3x}.$$

Multiply and divide by the conjugate:

$$x - \sqrt{x^2 - 3x} = \frac{(x - \sqrt{x^2 - 3x})(x + \sqrt{x^2 - 3x})}{x + \sqrt{x^2 - 3x}}.$$

Simplify the numerator:

$$x^2 - (x^2 - 3x) = 3x.$$

Thus,

$$x - \sqrt{x^2 - 3x} = \frac{3x}{x + \sqrt{x^2 - 3x}}.$$

Factor x out of the square root in the denominator:

$$\sqrt{x^2 - 3x} = x\sqrt{1 - \frac{3}{x}}.$$

Substitute into the expression:

$$\frac{3x}{x + x\sqrt{1 - \frac{3}{x}}} = \frac{3}{1 + \sqrt{1 - \frac{3}{x}}}.$$

Now take the limit as $x \rightarrow \infty$:

$$\lim_{x \rightarrow \infty} \frac{3}{1 + \sqrt{1 - \frac{3}{x}}} = \frac{3}{1 + \sqrt{1}} = \frac{3}{2}.$$

$$\boxed{\frac{3}{2}}$$

Question 5

Prove that

$$\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 5x} - \sqrt{x^2 + x} \right) = 2.$$

Question 5 - Solution

We simplify the expression by factoring out x from each square root.

Consider

$$\sqrt{x^2 + 5x} - \sqrt{x^2 + x}.$$

Factor x^2 inside each square root:

$$\sqrt{x^2 \left(1 + \frac{5}{x}\right)} - \sqrt{x^2 \left(1 + \frac{1}{x}\right)}.$$

Since $x > 0$ for large x , this becomes

$$x \left(\sqrt{1 + \frac{5}{x}} - \sqrt{1 + \frac{1}{x}} \right).$$

Multiply and divide by the conjugate:

$$x \cdot \frac{\left(\sqrt{1 + \frac{5}{x}} - \sqrt{1 + \frac{1}{x}}\right) \left(\sqrt{1 + \frac{5}{x}} + \sqrt{1 + \frac{1}{x}}\right)}{\sqrt{1 + \frac{5}{x}} + \sqrt{1 + \frac{1}{x}}}.$$

Simplify the numerator:

$$\left(1 + \frac{5}{x}\right) - \left(1 + \frac{1}{x}\right) = \frac{4}{x}.$$

Thus the expression becomes

$$x \cdot \frac{\frac{4}{x}}{\sqrt{1 + \frac{5}{x}} + \sqrt{1 + \frac{1}{x}}} = \frac{4}{\sqrt{1 + \frac{5}{x}} + \sqrt{1 + \frac{1}{x}}}.$$

Now take the limit as $x \rightarrow \infty$:

$$\lim_{x \rightarrow \infty} \frac{4}{\sqrt{1 + \frac{5}{x}} + \sqrt{1 + \frac{1}{x}}} = \frac{4}{1 + 1} = 2.$$

2

Question 6

Prove that

$$\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 4x + 1} - x \right) = 2.$$

Question 6 - Solution

Start with the expression

$$\sqrt{x^2 + 4x + 1} - x.$$

Multiply and divide by the conjugate:

$$\sqrt{x^2 + 4x + 1} - x = \frac{(\sqrt{x^2 + 4x + 1} - x)(\sqrt{x^2 + 4x + 1} + x)}{\sqrt{x^2 + 4x + 1} + x}.$$

Simplify the numerator:

$$(x^2 + 4x + 1) - x^2 = 4x + 1.$$

Thus,

$$\sqrt{x^2 + 4x + 1} - x = \frac{4x + 1}{\sqrt{x^2 + 4x + 1} + x}.$$

Factor x from the square root in the denominator:

$$\sqrt{x^2 + 4x + 1} = x\sqrt{1 + \frac{4}{x} + \frac{1}{x^2}}.$$

Substitute:

$$\frac{4x + 1}{x\sqrt{1 + \frac{4}{x} + \frac{1}{x^2}} + x} = \frac{4 + \frac{1}{x}}{\sqrt{1 + \frac{4}{x} + \frac{1}{x^2}} + 1}.$$

Take the limit as $x \rightarrow \infty$:

$$\lim_{x \rightarrow \infty} \frac{4 + \frac{1}{x}}{\sqrt{1 + \frac{4}{x} + \frac{1}{x^2}} + 1} = \frac{4}{1 + 1} = 2.$$

2

Question 7

Prove that

$$\lim_{x \rightarrow \infty} x(\sqrt{x^2 + 1} - x) = \frac{1}{2}.$$

Question 7 - Solution

Begin with the expression

$$x(\sqrt{x^2 + 1} - x).$$

Factor the expression inside the parentheses using a conjugate:

$$\sqrt{x^2 + 1} - x = \frac{(\sqrt{x^2 + 1} - x)(\sqrt{x^2 + 1} + x)}{\sqrt{x^2 + 1} + x} = \frac{(x^2 + 1) - x^2}{\sqrt{x^2 + 1} + x} = \frac{1}{\sqrt{x^2 + 1} + x}.$$

Substitute this back:

$$x(\sqrt{x^2 + 1} - x) = \frac{x}{\sqrt{x^2 + 1} + x}.$$

Factor x out of the square root in the denominator:

$$\sqrt{x^2 + 1} = x\sqrt{1 + \frac{1}{x^2}}.$$

Thus,

$$\frac{x}{x\sqrt{1 + \frac{1}{x^2}} + x} = \frac{1}{\sqrt{1 + \frac{1}{x^2}} + 1}.$$

Now take the limit as $x \rightarrow \infty$:

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{x^2}} + 1} = \frac{1}{1 + 1} = \frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$

Question 8

Prove that

$$\lim_{x \rightarrow \infty} (\ln(x+1) - \ln x) = 0.$$

Question 8 - Solution

Combine the logarithms using logarithmic properties:

$$\ln(x+1) - \ln x = \ln\left(\frac{x+1}{x}\right).$$

Rewrite the fraction:

$$\frac{x+1}{x} = 1 + \frac{1}{x}.$$

Thus, the expression becomes

$$\ln\left(1 + \frac{1}{x}\right).$$

Now take the limit as $x \rightarrow \infty$. Since

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0,$$

and the natural logarithm function is continuous at 1, we obtain

$$\lim_{x \rightarrow \infty} \ln\left(1 + \frac{1}{x}\right) = \ln(1) = 0.$$

0

Question 9

Prove that

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = 0.$$

Question 9 - Solution

We compare the growth rates of the logarithmic function and the power function.

Both $\ln x$ and $\sqrt{x}$ are positive and differentiable for $x > 0$. Apply L'Hôpital's Rule to the limit

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}}.$$

Differentiate the numerator and denominator:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{2\sqrt{x}}{x}.$$

Simplify:

$$\frac{2\sqrt{x}}{x} = \frac{2}{\sqrt{x}}.$$

Since

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} = 0,$$

it follows that

$$\lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0.$$

Therefore,

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = 0.$$

0

Question 10

Prove that

$$\lim_{x \rightarrow \infty} \left(\frac{x}{x+1} \right)^x = \frac{1}{e}.$$

Question 10 - Solution

Let

$$L = \lim_{x \rightarrow \infty} \left(\frac{x}{x+1} \right)^x.$$

Take the natural logarithm:

$$\ln L = \lim_{x \rightarrow \infty} x \ln \left(\frac{x}{x+1} \right).$$

Rewrite the logarithm:

$$\ln \left(\frac{x}{x+1} \right) = \ln \left(\frac{1}{1 + \frac{1}{x}} \right) = -\ln \left(1 + \frac{1}{x} \right).$$

Thus,

$$\ln L = - \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x} \right).$$

Consider the limit

$$\lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x} \right).$$

Rewrite it as a quotient:

$$\lim_{x \rightarrow \infty} \frac{\ln(1 + \frac{1}{x})}{\frac{1}{x}}.$$

This is an indeterminate form $\frac{0}{0}$, so apply L'Hôpital's Rule.

Differentiate numerator and denominator:

$$\frac{d}{dx} \ln \left(1 + \frac{1}{x} \right) = \frac{-1}{x(x+1)}, \quad \frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2}.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{\ln(1 + \frac{1}{x})}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{-1}{x(x+1)}}{\frac{-1}{x^2}} = \lim_{x \rightarrow \infty} \frac{x^2}{x(x+1)} = \lim_{x \rightarrow \infty} \frac{x}{x+1} = 1.$$

Therefore,

$$\ln L = -1.$$

Exponentiating both sides:

$$L = e^{-1} = \frac{1}{e}.$$

$$\boxed{\frac{1}{e}}$$

Question 1

Prove that for any positive integer n ,

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}.$$

Question 1 - Solution

We prove the formula using mathematical induction.

Base Case:

For $n = 1$,

$$\sum_{k=1}^1 k = 1, \quad \frac{1(1+1)}{2} = 1.$$

Thus, the formula holds for $n = 1$.

Inductive Hypothesis:

Assume that for some positive integer n ,

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}.$$

Inductive Step:

Consider the sum up to $n + 1$:

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^n k + (n+1).$$

Using the inductive hypothesis,

$$\sum_{k=1}^{n+1} k = \frac{n(n+1)}{2} + (n+1).$$

Factor out $(n+1)$:

$$\sum_{k=1}^{n+1} k = (n+1) \left(\frac{n}{2} + 1 \right) = (n+1) \frac{n+2}{2}.$$

Thus,

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}.$$

This matches the given formula with n replaced by $n + 1$.

By mathematical induction, the formula holds for all positive integers n .

$$\boxed{\sum_{k=1}^n k = \frac{n(n+1)}{2}}$$

Question 2

Prove that for any positive integer n ,

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Question 2 - Solution

We prove the formula using mathematical induction.

Base Case:

For $n = 1$,

$$\sum_{k=1}^1 k^2 = 1^2 = 1,$$

and

$$\frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{6}{6} = 1.$$

Thus, the formula holds for $n = 1$.

Inductive Hypothesis:

Assume that for some positive integer n ,

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Inductive Step:

Consider the sum up to $n + 1$:

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^n k^2 + (n+1)^2.$$

Using the inductive hypothesis,

$$\sum_{k=1}^{n+1} k^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2.$$

Factor out $(n+1)$:

$$\sum_{k=1}^{n+1} k^2 = (n+1) \left(\frac{n(2n+1)}{6} + (n+1) \right).$$

Combine terms inside the parentheses:

$$\frac{n(2n+1) + 6(n+1)}{6} = \frac{2n^2 + 7n + 6}{6} = \frac{(2n+3)(n+2)}{6}.$$

Thus,

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}.$$

This matches the given formula with n replaced by $n + 1$.

By mathematical induction, the formula holds for all positive integers n .

$$\boxed{\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}}$$

Question 3

Prove that for any positive integer n ,

$$\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2} \right)^2.$$

Question 3 - Solution

We prove the formula using mathematical induction.

Base Case:

For $n = 1$,

$$\sum_{k=1}^1 k^3 = 1^3 = 1, \quad \left(\frac{1(1+1)}{2} \right)^2 = 1.$$

Thus, the formula holds for $n = 1$.

Inductive Hypothesis:

Assume that for some positive integer n ,

$$\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2} \right)^2.$$

Inductive Step:

Consider the sum up to $n + 1$:

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^n k^3 + (n+1)^3.$$

Using the inductive hypothesis,

$$\sum_{k=1}^{n+1} k^3 = \left(\frac{n(n+1)}{2} \right)^2 + (n+1)^3.$$

Factor out $(n+1)^2$:

$$\sum_{k=1}^{n+1} k^3 = (n+1)^2 \left(\frac{n^2}{4} + n + 1 \right).$$

Combine terms:

$$\frac{n^2 + 4n + 4}{4} = \frac{(n+2)^2}{4}.$$

Thus,

$$\sum_{k=1}^{n+1} k^3 = (n+1)^2 \cdot \frac{(n+2)^2}{4} = \left(\frac{(n+1)(n+2)}{2} \right)^2.$$

This matches the given formula with n replaced by $n + 1$.

By mathematical induction, the formula holds for all positive integers n .

$$\boxed{\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2} \right)^2}$$

Question 4

Prove that for any positive integer n ,

$$\sum_{k=1}^n (2k - 1) = n^2.$$

Question 4 - Solution

We prove the identity using mathematical induction.

Base Case:

For $n = 1$,

$$\sum_{k=1}^1 (2k - 1) = 2(1) - 1 = 1,$$

and

$$1^2 = 1.$$

Thus, the statement holds for $n = 1$.

Inductive Hypothesis:

Assume that for some positive integer n ,

$$\sum_{k=1}^n (2k - 1) = n^2.$$

Inductive Step:

Consider the sum up to $n + 1$:

$$\sum_{k=1}^{n+1} (2k - 1) = \sum_{k=1}^n (2k - 1) + (2(n + 1) - 1).$$

Using the inductive hypothesis,

$$\sum_{k=1}^{n+1} (2k - 1) = n^2 + (2n + 1).$$

Simplify:

$$n^2 + 2n + 1 = (n + 1)^2.$$

Thus,

$$\sum_{k=1}^{n+1} (2k - 1) = (n + 1)^2.$$

This matches the given formula with n replaced by $n + 1$.

By mathematical induction, the identity holds for all positive integers n .

$$\boxed{\sum_{k=1}^n (2k - 1) = n^2}$$

Question 5

Prove that for any positive integer n ,

$$\sum_{k=1}^n (3k + 2) = \frac{3n(n+1)}{2} + 2n.$$

Question 5 - Solution

We prove the identity using properties of summations.

Start with the given sum:

$$\sum_{k=1}^n (3k + 2).$$

Split the sum using linearity:

$$\sum_{k=1}^n (3k + 2) = \sum_{k=1}^n 3k + \sum_{k=1}^n 2.$$

Factor out constants:

$$\sum_{k=1}^n 3k = 3 \sum_{k=1}^n k, \quad \sum_{k=1}^n 2 = 2 \sum_{k=1}^n 1.$$

Use known summation formulas:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^n 1 = n.$$

Substitute these results:

$$\sum_{k=1}^n (3k + 2) = 3 \cdot \frac{n(n+1)}{2} + 2n.$$

Thus,

$$\sum_{k=1}^n (3k + 2) = \frac{3n(n+1)}{2} + 2n.$$

$$\sum_{k=1}^n (3k + 2) = \frac{3n(n+1)}{2} + 2n$$

Question 6

Prove that for any positive integer n ,

$$\sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) = \frac{n}{n+1}.$$

Question 6 - Solution

Write out the first few terms of the sum:

$$\sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) = \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right).$$

Notice that consecutive terms cancel:

$$-\frac{1}{2} + \frac{1}{2} = 0, \quad -\frac{1}{3} + \frac{1}{3} = 0, \quad \dots$$

After all cancellations, only the first and last terms remain:

$$1 - \frac{1}{n+1}.$$

Simplify:

$$1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

$$\boxed{\sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) = \frac{n}{n+1}}$$

Question 7

Prove that for any positive integer n ,

$$\sum_{k=1}^n \frac{1}{k(k+1)} = \frac{n}{n+1}.$$

Question 7 - Solution

Begin by rewriting the general term using partial fractions.

For each $k \geq 1$,

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Substitute this into the sum:

$$\sum_{k=1}^n \frac{1}{k(k+1)} = \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right).$$

Write out the first few terms:

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

All intermediate terms cancel, leaving

$$1 - \frac{1}{n+1}.$$

Simplify:

$$1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

$$\boxed{\sum_{k=1}^n \frac{1}{k(k+1)} = \frac{n}{n+1}}$$

Question 8

Prove that for any positive integer n and real number $r \neq 1$,

$$\sum_{k=0}^n r^k = \frac{1 - r^{n+1}}{1 - r}.$$

Question 8 - Solution

Let

$$S = \sum_{k=0}^n r^k = 1 + r + r^2 + \cdots + r^n.$$

Multiply both sides by r :

$$rS = r + r^2 + r^3 + \cdots + r^{n+1}.$$

Subtract the second equation from the first:

$$S - rS = (1 + r + r^2 + \cdots + r^n) - (r + r^2 + \cdots + r^{n+1}).$$

All intermediate terms cancel, leaving

$$S - rS = 1 - r^{n+1}.$$

Factor the left-hand side:

$$(1 - r)S = 1 - r^{n+1}.$$

Since $r \neq 1$, divide both sides by $1 - r$:

$$S = \frac{1 - r^{n+1}}{1 - r}.$$

Thus,

$$\boxed{\sum_{k=0}^n r^k = \frac{1 - r^{n+1}}{1 - r}}$$

Question 9

Prove that for every positive integer n ,

$$\sum_{k=1}^n \frac{k}{k+1} = n - \sum_{k=1}^n \frac{1}{k+1}.$$

Question 9 - Solution

For every $k \geq 1$,

$$\frac{k}{k+1} = 1 - \frac{1}{k+1}.$$

Summing all n terms gives

$$\boxed{\sum_{k=1}^n \frac{k}{k+1} = n - \sum_{k=1}^n \frac{1}{k+1}.$$

Equivalently, the result is $n + 1 - H_{n+1}$, where $H_{n+1} = \sum_{j=1}^{n+1} 1/j$.
For $n = 1$, both sides of the corrected identity equal $1/2$.

Question 10

Prove that for any positive integer n ,

$$\sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^n \frac{1}{k+1} = 1 - \frac{1}{n+1}.$$

Question 10 - Solution

Write both sums explicitly:

$$\sum_{k=1}^n \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$
$$\sum_{k=1}^n \frac{1}{k+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n+1}.$$

Subtract the second sum from the first:

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right).$$

All common terms cancel:

$$\frac{1}{2} - \frac{1}{2} = 0, \quad \frac{1}{3} - \frac{1}{3} = 0, \quad \dots$$

Only the first term of the first sum and the last term of the second sum remain:

$$1 - \frac{1}{n+1}.$$

Thus,

$$\sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^n \frac{1}{k+1} = 1 - \frac{1}{n+1}.$$

$$\boxed{1 - \frac{1}{n+1}}$$

Question 1

Suppose that F and G are antiderivatives of the same function f on an interval I . Prove that there exists a constant C such that

$$F(x) = G(x) + C \quad \text{for all } x \in I.$$

Question 1 - Solution

Since F and G are both antiderivatives of f on I , we have

$$F'(x) = f(x) \quad \text{and} \quad G'(x) = f(x) \quad \text{for all } x \in I.$$

Consider the function

$$H(x) = F(x) - G(x).$$

Differentiate $H(x)$:

$$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0 \quad \text{for all } x \in I.$$

Thus, the derivative of H is zero on the interval I .

From the result that a function with zero derivative on an interval is constant, there exists a real number C such that

$$H(x) = C \quad \text{for all } x \in I.$$

Substituting back for $H(x)$ gives

$$F(x) - G(x) = C.$$

Rewriting,

$$F(x) = G(x) + C.$$

$$\boxed{F(x) = G(x) + C}$$

Question 2

Suppose that $F'(x) = 0$ for all x in an interval I . Prove that F is constant on I .

Question 2 - Solution

Let $x_1, x_2 \in I$ with $x_1 < x_2$. We will show that

$$F(x_1) = F(x_2).$$

Since $F'(x) = 0$ for all $x \in I$, the function F is differentiable on I and continuous on the closed interval $[x_1, x_2]$.

By the Mean Value Theorem, there exists a number $c \in (x_1, x_2)$ such that

$$F'(c) = \frac{F(x_2) - F(x_1)}{x_2 - x_1}.$$

But $F'(c) = 0$ by assumption, so

$$0 = \frac{F(x_2) - F(x_1)}{x_2 - x_1}.$$

Since $x_2 - x_1 \neq 0$, it follows that

$$F(x_2) - F(x_1) = 0.$$

Thus,

$$F(x_2) = F(x_1).$$

Because x_1 and x_2 were arbitrary points in I , the function F has the same value at every point of I .

F is constant on I

Question 3

Let f be a continuous function on an interval I . Suppose that

$$\int f(x) \, dx = F(x) + C_1 \quad \text{and} \quad \int f(x) \, dx = G(x) + C_2$$

are two indefinite integrals of f . Prove that

$$F(x) - G(x) = \text{constant}.$$

Question 3 - Solution

Since both expressions represent antiderivatives of f , we have

$$F'(x) = f(x) \quad \text{and} \quad G'(x) = f(x) \quad \text{for all } x \in I.$$

Consider the difference of the two functions:

$$H(x) = F(x) - G(x).$$

Differentiate $H(x)$:

$$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0 \quad \text{for all } x \in I.$$

Thus, the derivative of H is zero on the interval I .

From the Mean Value Theorem, a function whose derivative is zero everywhere on an interval must be constant. Therefore, there exists a real number K such that

$$H(x) = K \quad \text{for all } x \in I.$$

Substituting back for $H(x)$ gives

$$F(x) - G(x) = K.$$

$F(x) - G(x) = \text{constant}$

Question 4

Suppose that f is continuous on an interval I and that

$$F'(x) = f(x) \quad \text{for all } x \in I.$$

Show that every solution of the differential equation

$$y' = f(x)$$

has the form

$$y = F(x) + C$$

for some constant C .

Question 4 - Solution

Let y be any solution of the differential equation

$$y' = f(x)$$

on the interval I .

Since $F'(x) = f(x)$ and $y'(x) = f(x)$ for all $x \in I$, we have

$$y'(x) - F'(x) = 0 \quad \text{for all } x \in I.$$

Consider the function

$$H(x) = y(x) - F(x).$$

Differentiate $H(x)$:

$$H'(x) = y'(x) - F'(x) = 0 \quad \text{for all } x \in I.$$

Thus, $H'(x) = 0$ on I .

By the result that a function with zero derivative on an interval is constant, there exists a real number C such that

$$H(x) = C \quad \text{for all } x \in I.$$

Substituting back gives

$$y(x) - F(x) = C.$$

Rewriting,

$$y = F(x) + C.$$

$$\boxed{y = F(x) + C}$$

Question 5

Suppose that f and g are continuous on a nondegenerate interval I , that $a \in I$, and that

$$\int_a^x f(t) \, dt = \int_a^x g(t) \, dt \quad \text{for all } x \in I.$$

Prove that

$$f(x) = g(x) \quad \text{for all } x \in I.$$

Question 5 - Solution

Set $F(x) = \int_a^x f(t) dt$ and $G(x) = \int_a^x g(t) dt$.

The hypothesis gives $F = G$ on I . At every interior point x of I , continuity of both integrands allows the Fundamental Theorem of Calculus:

$$f(x) = F'(x) = G'(x) = g(x).$$

At an included endpoint, take limits from the interior and use continuity of f and g . Thus

$$\boxed{f(x) = g(x) \quad \text{for every } x \in I.}$$

Question 6

Let f be continuous on an interval I , and fix one antiderivative F . Write the general antiderivative as

$$y(x) = F(x) + C.$$

Show that the condition

$$y(x_0) = y_0, \quad x_0 \in I,$$

uniquely determines C .

Question 6 - Solution

Fix an antiderivative F of f on I . Every other antiderivative has the form $y(x) = F(x) + C$.

Imposing $y(x_0) = y_0$ yields $F(x_0) + C = y_0$, so

$$\boxed{C = y_0 - F(x_0)}.$$

This value is unique because F , x_0 , and y_0 are fixed.

Question 7

Suppose that f is continuous on an interval I and that

$$\int_a^x f(t) dt = F(x) + C$$

for some constant C . Show that choosing a different lower limit $b \in I$ changes the antiderivative only by a constant.

Question 7 - Solution

Consider the two functions

$$F_1(x) = \int_a^x f(t) dt, \quad F_2(x) = \int_b^x f(t) dt.$$

Both F_1 and F_2 are antiderivatives of f on I by the Fundamental Theorem of Calculus.

Evaluate the difference between them:

$$F_1(x) - F_2(x) = \int_a^x f(t) dt - \int_b^x f(t) dt.$$

Using properties of definite integrals,

$$\int_a^x f(t) dt - \int_b^x f(t) dt = \int_a^b f(t) dt.$$

The right-hand side is a fixed real number that does not depend on x .

Therefore,

$$F_1(x) - F_2(x) = \text{constant}.$$

This shows that changing the lower limit of integration alters the antiderivative only by an additive constant.

$$\int_a^x f(t) dt = \int_b^x f(t) dt + \text{constant}$$

Question 8

Suppose that f is continuous on an interval I and that

$$\int f(x) \, dx = F(x) + C.$$

Prove that if

$$F'(x) = 0 \quad \text{for all } x \in I,$$

then

$$\int f(x) \, dx = C.$$

Question 8 - Solution

Since

$$\int f(x) dx = F(x) + C,$$

the function $F(x)$ is an antiderivative of f .

Given that

$$F'(x) = 0 \quad \text{for all } x \in I,$$

it follows from the Mean Value Theorem that F is constant on the interval I .

Thus, there exists a real number K such that

$$F(x) = K \quad \text{for all } x \in I.$$

Substitute this into the expression for the indefinite integral:

$$\int f(x) dx = K + C.$$

Since $K + C$ is itself a constant, we may write

$$\int f(x) dx = C'.$$

Therefore, if the derivative of an antiderivative is zero on an interval, the indefinite integral reduces to a constant.

$$\boxed{\int f(x) dx = C}$$

Question 9

Suppose that f is continuous on an interval I and that

$$\int f(x) dx = F(x) + C.$$

Show that if two antiderivatives of f agree at a single point, then they are identical on the entire interval.

Question 9 - Solution

Let F and G be two antiderivatives of f on the interval I . Then

$$F'(x) = f(x) \quad \text{and} \quad G'(x) = f(x) \quad \text{for all } x \in I.$$

Assume that there exists a point $x_0 \in I$ such that

$$F(x_0) = G(x_0).$$

Consider the function

$$H(x) = F(x) - G(x).$$

Differentiate $H(x)$:

$$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0 \quad \text{for all } x \in I.$$

Thus, H has zero derivative on I , so H is constant on I . That is,

$$H(x) = K \quad \text{for all } x \in I.$$

Evaluate H at x_0 :

$$H(x_0) = F(x_0) - G(x_0) = 0.$$

Hence, $K = 0$, and therefore

$$F(x) - G(x) = 0 \quad \text{for all } x \in I.$$

This implies

$$F(x) = G(x) \quad \text{for all } x \in I.$$

$$\boxed{F(x) \equiv G(x)}$$

Question 10

Suppose that f is continuous on an interval I and that

$$\int f(x) dx = F(x) + C.$$

Show that the family of curves

$$y = F(x) + C$$

consists of vertical translations of a single curve.

Question 10 - Solution

Consider two members of the family of antiderivatives:

$$y_1 = F(x) + C_1, \quad y_2 = F(x) + C_2,$$

where C_1 and C_2 are constants.

Subtract the equations:

$$y_2 - y_1 = (F(x) + C_2) - (F(x) + C_1) = C_2 - C_1.$$

The difference $C_2 - C_1$ is a constant independent of x . Thus, for every value of x , the vertical distance between the two curves is the same.

This means that changing the constant of integration does not alter the shape of the graph, but only shifts it upward or downward by a fixed amount.

Therefore, the family of functions

$$y = F(x) + C$$

represents all vertical translations of a single curve.

Different values of C produce vertical shifts of the same graph