

Question 1

Find the average value of the function

$$f(x) = x\sqrt{1+x^2}$$

on the interval $[0, 2]$.

Question 1 - Solution

The average value of a function f on an interval $[a, b]$ is defined by

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here, $a = 0$ and $b = 2$, so

$$f_{\text{avg}} = \frac{1}{2} \int_0^2 x \sqrt{1+x^2} dx.$$

Evaluate the integral using substitution. Let

$$u = 1 + x^2.$$

Then

$$du = 2x dx \quad \Rightarrow \quad x dx = \frac{1}{2} du.$$

Change the limits. When $x = 0$, $u = 1$. When $x = 2$, $u = 5$.

Substitute:

$$\int_0^2 x \sqrt{1+x^2} dx = \frac{1}{2} \int_1^5 u^{1/2} du.$$

Integrate:

$$\frac{1}{2} \int u^{1/2} du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} = \frac{1}{3} u^{3/2}.$$

Apply the limits:

$$\left. \frac{1}{3} u^{3/2} \right|_1^5 = \frac{1}{3} (5^{3/2} - 1).$$

Now divide by the interval length:

$$f_{\text{avg}} = \frac{1}{2} \cdot \frac{1}{3} (5^{3/2} - 1) = \frac{1}{6} (5^{3/2} - 1).$$

$$\boxed{\frac{1}{6} (5^{3/2} - 1)}$$

Question 2

Find the average value of the function

$$f(x) = \ln(1 + x^2)$$

on the interval $[-1, 1]$.

Question 2 - Solution

The average value of a function f on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here $a = -1$ and $b = 1$, so

$$f_{\text{avg}} = \frac{1}{2} \int_{-1}^1 \ln(1+x^2) dx.$$

Notice that $\ln(1+x^2)$ is an even function, since

$$\ln(1+(-x)^2) = \ln(1+x^2).$$

Therefore,

$$\int_{-1}^1 \ln(1+x^2) dx = 2 \int_0^1 \ln(1+x^2) dx.$$

Thus,

$$f_{\text{avg}} = \int_0^1 \ln(1+x^2) dx.$$

Evaluate the integral using integration by parts. Let

$$u = \ln(1+x^2), \quad dv = dx.$$

Then

$$du = \frac{2x}{1+x^2} dx, \quad v = x.$$

Apply integration by parts:

$$\int \ln(1+x^2) dx = x \ln(1+x^2) - \int \frac{2x^2}{1+x^2} dx.$$

Rewrite the integrand:

$$\frac{2x^2}{1+x^2} = 2 - \frac{2}{1+x^2}.$$

Thus,

$$\int \ln(1+x^2) dx = x \ln(1+x^2) - \int \left(2 - \frac{2}{1+x^2}\right) dx.$$

Integrate:

$$x \ln(1+x^2) - 2x + 2 \arctan x.$$

Evaluate from 0 to 1:

$$\left[x \ln(1+x^2) - 2x + 2 \arctan x \right]_0^1 = \ln 2 - 2 + \frac{\pi}{2}.$$

Therefore,

$$\boxed{\ln 2 - 2 + \frac{\pi}{2}}$$

Question 3

Find the average value of the function

$$f(x) = x^2 \cos x$$

on the interval $[0, \pi]$.

Question 3 - Solution

The average value of a function f on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{\pi} \int_0^\pi x^2 \cos x dx.$$

Evaluate the integral using integration by parts.

Let

$$u = x^2, \quad dv = \cos x dx.$$

Then

$$du = 2x dx, \quad v = \sin x.$$

Apply integration by parts:

$$\int x^2 \cos x dx = x^2 \sin x - \int 2x \sin x dx.$$

Apply integration by parts again to the remaining integral. Let

$$u = 2x, \quad dv = \sin x dx.$$

Then

$$du = 2 dx, \quad v = -\cos x.$$

Thus,

$$\int 2x \sin x dx = -2x \cos x + 2 \int \cos x dx = -2x \cos x + 2 \sin x.$$

Substitute back:

$$\int x^2 \cos x dx = x^2 \sin x + 2x \cos x - 2 \sin x.$$

Evaluate from 0 to π :

$$[x^2 \sin x + 2x \cos x - 2 \sin x]_0^\pi.$$

At $x = \pi$:

$$\pi^2 \sin \pi + 2\pi \cos \pi - 2 \sin \pi = -2\pi.$$

At $x = 0$:

$$0 + 0 - 0 = 0.$$

Thus,

$$\int_0^\pi x^2 \cos x dx = -2\pi.$$

Divide by the interval length:

$$f_{\text{avg}} = \frac{1}{\pi}(-2\pi) = -2.$$

$$\boxed{-2}$$

Question 4

Find the average value of the function

$$f(x) = \sqrt{x}(1+x)$$

on the interval $[0, 1]$.

Question 4 - Solution

The average value of a function on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here,

$$f_{\text{avg}} = \int_0^1 \sqrt{x}(1+x) dx.$$

First expand the integrand:

$$\sqrt{x}(1+x) = x^{1/2} + x^{3/2}.$$

Integrate term by term:

$$\int x^{1/2} dx = \frac{2}{3}x^{3/2}, \quad \int x^{3/2} dx = \frac{2}{5}x^{5/2}.$$

Thus,

$$\int_0^1 \sqrt{x}(1+x) dx = \left[\frac{2}{3}x^{3/2} + \frac{2}{5}x^{5/2} \right]_0^1.$$

Evaluate:

$$\frac{2}{3} + \frac{2}{5} = \frac{10+6}{15} = \frac{16}{15}.$$

$$\boxed{\frac{16}{15}}$$

Question 5

Find the average value of the function

$$f(x) = \frac{x}{1+x^2}$$

on the interval $[0, 1]$.

Question 5 - Solution

The average value of a function f on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here $a = 0$ and $b = 1$, so

$$f_{\text{avg}} = \int_0^1 \frac{x}{1+x^2} dx.$$

Evaluate the integral using substitution.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x dx \quad \Rightarrow \quad x dx = \frac{1}{2} du.$$

Change the limits. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

Substitute:

$$\int_0^1 \frac{x}{1+x^2} dx = \frac{1}{2} \int_1^2 \frac{1}{u} du.$$

Integrate:

$$\frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln u.$$

Apply the limits:

$$\frac{1}{2} \ln u \Big|_1^2 = \frac{1}{2} \ln 2.$$

$$\boxed{\frac{1}{2} \ln 2}$$

Question 6

Find the average value of the function

$$f(x) = \sin^2 x$$

on the interval $[0, \pi]$.

Question 6 - Solution

The average value of a function on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{\pi} \int_0^\pi \sin^2 x dx.$$

Use the identity

$$\sin^2 x = \frac{1 - \cos(2x)}{2}.$$

Then

$$\int_0^\pi \sin^2 x dx = \frac{1}{2} \int_0^\pi (1 - \cos 2x) dx.$$

Integrate:

$$\frac{1}{2} \left[x - \frac{1}{2} \sin(2x) \right]_0^\pi = \frac{1}{2} (\pi - 0) = \frac{\pi}{2}.$$

Now divide by the interval length:

$$f_{\text{avg}} = \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$

Question 7

Find the average value of the function

$$f(x) = \sqrt{1 - x^2}$$

on the interval $[-1, 1]$.

Question 7 - Solution

The average value of a function on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{2} \int_{-1}^1 \sqrt{1-x^2} dx.$$

Method 1: Geometry

The graph of

$$y = \sqrt{1-x^2}$$

is the upper half of a circle of radius 1 centered at the origin.

The area under this curve from $x = -1$ to $x = 1$ is the area of a semicircle:

$$\frac{1}{2} \pi (1)^2 = \frac{\pi}{2}.$$

Therefore,

$$f_{\text{avg}} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$$

Method 2: Trigonometric Substitution

Evaluate the integral directly.

Let

$$x = \sin \theta, \quad dx = \cos \theta d\theta.$$

Change the limits. When $x = -1$, $\theta = -\frac{\pi}{2}$. When $x = 1$, $\theta = \frac{\pi}{2}$.

Substitute:

$$\int_{-1}^1 \sqrt{1-x^2} dx = \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta.$$

Use the identity

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}.$$

Then

$$\int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) d\theta.$$

Integrate:

$$\frac{1}{2} \left[\theta + \frac{1}{2} \sin(2\theta) \right]_{-\pi/2}^{\pi/2}.$$

Since $\sin(\pi) = \sin(-\pi) = 0$, this becomes

$$\frac{1}{2} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) = \frac{\pi}{2}.$$

Now compute the average value:

$$f_{\text{avg}} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$$

$$\boxed{\frac{\pi}{4}}$$

Question 8

Find the average value of the function

$$f(x) = |x|$$

on the interval $[-2, 2]$.

Question 8 - Solution

The average value of a function on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

Here $a = -2$ and $b = 2$, so

$$f_{\text{avg}} = \frac{1}{4} \int_{-2}^2 |x| \, dx.$$

Since $|x|$ is an even function,

$$\int_{-2}^2 |x| \, dx = 2 \int_0^2 x \, dx.$$

Evaluate:

$$2 \left[\frac{x^2}{2} \right]_0^2 = 2 \cdot 2 = 4.$$

Compute the average value:

$$f_{\text{avg}} = \frac{1}{4} \cdot 4 = 1.$$

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Question 9

Find the average value of the function

$$f(x) = \sqrt{x^2 + 1}$$

on the interval $[-1, 1]$.

Question 9 - Solution

The average value of a function on an interval $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{2} \int_{-1}^1 \sqrt{x^2 + 1} dx.$$

Notice that $\sqrt{x^2 + 1}$ is an even function, so

$$\int_{-1}^1 \sqrt{x^2 + 1} dx = 2 \int_0^1 \sqrt{x^2 + 1} dx.$$

Thus,

$$f_{\text{avg}} = \int_0^1 \sqrt{x^2 + 1} dx.$$

Evaluate the integral using trigonometric substitution. Let

$$x = \tan \theta, \quad dx = \sec^2 \theta d\theta.$$

When $x = 0$, $\theta = 0$. When $x = 1$, $\theta = \frac{\pi}{4}$.

Substitute:

$$\int_0^1 \sqrt{x^2 + 1} dx = \int_0^{\pi/4} \sec \theta \cdot \sec^2 \theta d\theta = \int_0^{\pi/4} \sec^3 \theta d\theta.$$

Use the standard result

$$\int \sec^3 \theta d\theta = \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|).$$

Apply the limits:

$$\frac{1}{2} (\sec \theta \tan \theta + \ln(\sec \theta + \tan \theta)) \Big|_0^{\pi/4}.$$

Evaluate:

$$\begin{aligned} \sec\left(\frac{\pi}{4}\right) &= \sqrt{2}, \quad \tan\left(\frac{\pi}{4}\right) = 1, \\ \sec(0) &= 1, \quad \tan(0) = 0. \end{aligned}$$

Thus,

$$f_{\text{avg}} = \frac{1}{2} \left(\sqrt{2} + \ln(1 + \sqrt{2}) \right).$$

$$\boxed{\frac{1}{2} \left(\sqrt{2} + \ln(1 + \sqrt{2}) \right)}$$

Question 10

Find the average value of the function

$$f(x) = x e^{-x}$$

on the interval $[0, 2]$.

Question 10 - Solution

The average value of a function on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{2} \int_0^2 x e^{-x} dx.$$

Evaluate the integral using integration by parts.

Let

$$u = x, \quad dv = e^{-x} dx.$$

Then

$$du = dx, \quad v = -e^{-x}.$$

Apply integration by parts:

$$\int x e^{-x} dx = -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x}.$$

Thus,

$$\int x e^{-x} dx = -(x+1)e^{-x}.$$

Evaluate from 0 to 2:

$$\left[-(x+1)e^{-x} \right]_0^2 = -3e^{-2} - (-1) = 1 - 3e^{-2}.$$

Now compute the average value:

$$f_{\text{avg}} = \frac{1}{2} (1 - 3e^{-2}).$$

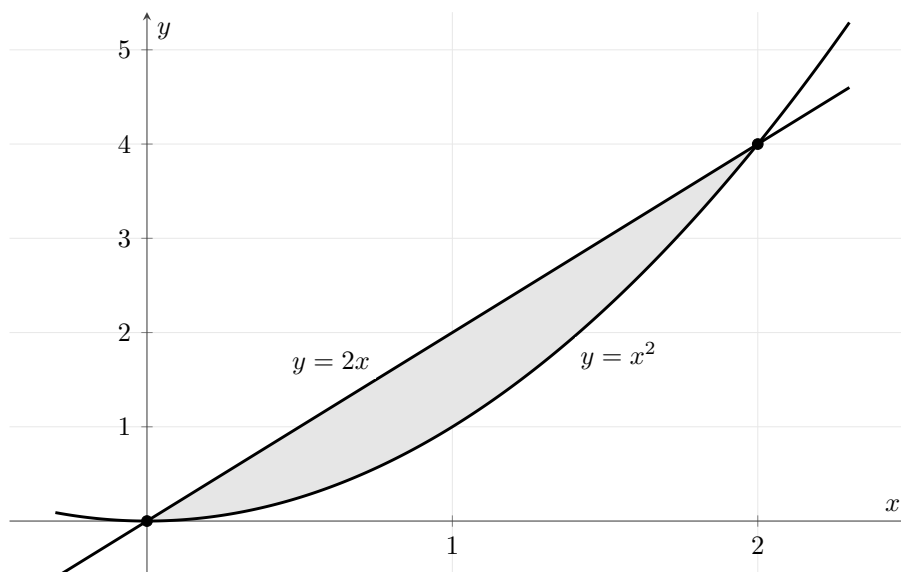
$$\frac{1}{2} (1 - 3e^{-2})$$

Question 1

Find the exact area of the region enclosed by the curves

$$y = x^2 \quad \text{and} \quad y = 2x,$$

between their points of intersection.



The shaded region is the area to be found.

Question 1 – Solution**1. Find the intersections.**

$$x^2 = 2x \implies x(x - 2) = 0 \implies x = 0, 2.$$

Factoring gives $x = 0$ or $x - 2 = 0$. Substituting into $y = 2x$ gives $y = 0$ or $y = 4$. The intersection points are $(0, 0)$ and $(2, 4)$.

2. Determine which curve is above the other.

On $0 \leq x \leq 2$, $2x - x^2 = x(2 - x) \geq 0$, so the line is above the parabola.

3. Set up upper minus lower.

$$A = \int_0^2 (\text{upper} - \text{lower}) \, dx = \int_0^2 (2x - x^2) \, dx.$$

4. Find an antiderivative.

By the power rule, $\int 2x \, dx = x^2$ and $\int x^2 \, dx = x^3/3$.

5. Apply the bounds and simplify.

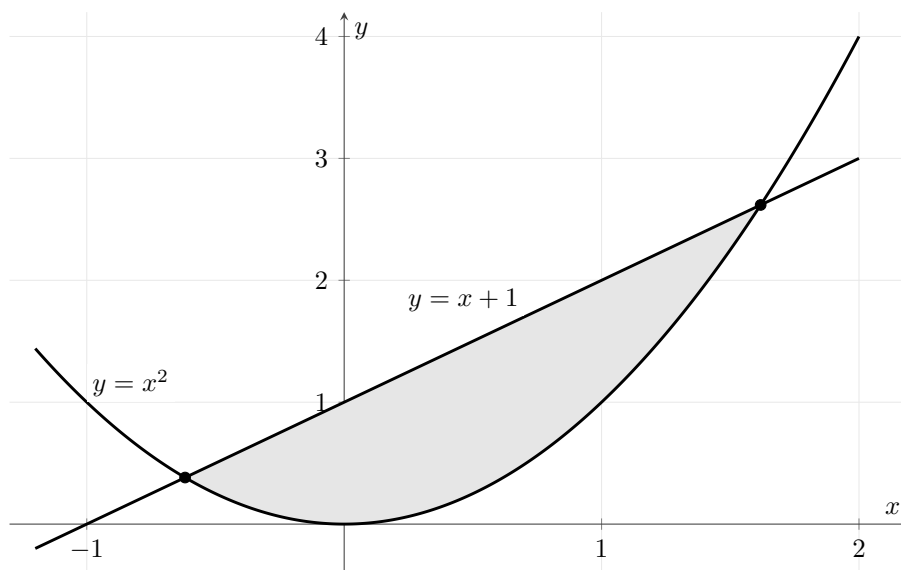
$$\begin{aligned} A &= \left[x^2 - \frac{x^3}{3} \right]_0^2 \\ &= \left(4 - \frac{8}{3} \right) - 0 = \boxed{\frac{4}{3}}. \end{aligned}$$

Question 2

Find the exact area of the region enclosed by the curves

$$y = x + 1 \quad \text{and} \quad y = x^2,$$

between their points of intersection.



The shaded region is the area to be found.

Question 2 – Solution

1. Find the intersections.

$$x^2 = x + 1 \implies x^2 - x - 1 = 0.$$

The quadratic formula gives $x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$. Write the two roots as

$$a = \frac{1 - \sqrt{5}}{2}, \quad b = \frac{1 + \sqrt{5}}{2}.$$

2. Determine which curve is above the other.

Test $x = 0$: the line has height 1 and the parabola has height 0. Between the roots, $(x + 1) - x^2 \geq 0$, so the line is above the parabola.

3. Set up upper minus lower.

$$A = \int_a^b (-x^2 + x + 1) dx.$$

4. Complete the square to simplify the bounds.

Complete the square:

$$-x^2 + x + 1 = \frac{5}{4} - \left(x - \frac{1}{2}\right)^2.$$

Let $u = x - \frac{1}{2}$. The new bounds are $-\sqrt{5}/2$ and $\sqrt{5}/2$. Here $du = dx$. The integrand is even, so the two halves have equal area.

5. Integrate and simplify.

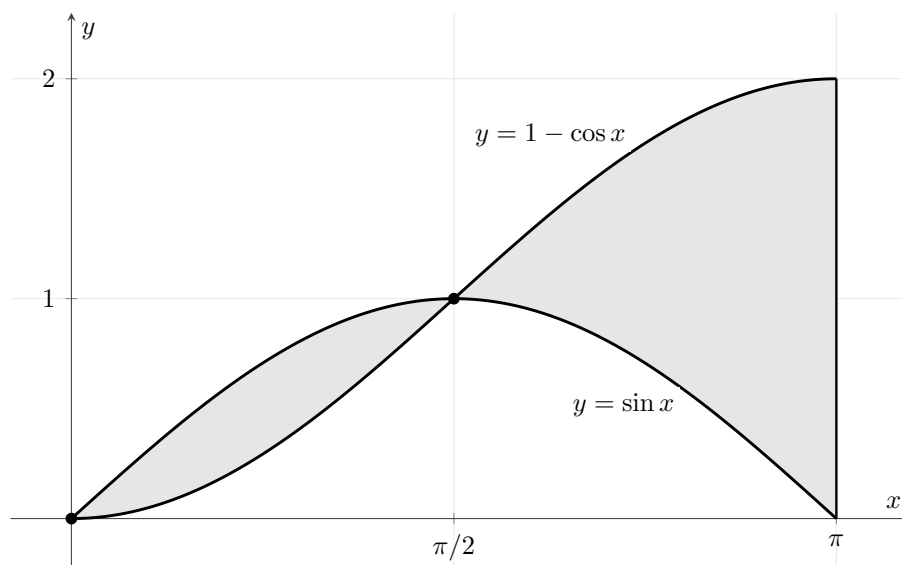
$$\begin{aligned} A &= 2 \int_0^{\sqrt{5}/2} \left(\frac{5}{4} - u^2 \right) du \\ &= 2 \left[\frac{5}{4}u - \frac{u^3}{3} \right]_0^{\sqrt{5}/2} \\ &= 2 \left(\frac{5\sqrt{5}}{8} - \frac{5\sqrt{5}}{24} \right) = \boxed{\frac{5\sqrt{5}}{6}}. \end{aligned}$$

Question 3

Find the exact area of the region between the curves

$$y = \sin x \quad \text{and} \quad y = 1 - \cos x,$$

between $x = 0$ and $x = \pi$.



The shaded region is the area to be found.

Question 3 – Solution

1. Find where the curves cross on $[0, \pi]$.

Using half-angle identities,

$$\begin{aligned}\sin x = 1 - \cos x &\iff 2 \sin(x/2) (\cos(x/2) - \sin(x/2)) = 0 \\ &\iff x = 0 \quad \text{or} \quad x = \frac{\pi}{2}.\end{aligned}$$

2. Determine the upper curve on each interval.

On $(0, \pi/2)$, $\sin x > 1 - \cos x$; on $(\pi/2, \pi)$, the order reverses. Test $x = \pi/4$: $\sin x = \sqrt{2}/2 > 1 - \sqrt{2}/2$. At $x = 3\pi/4$, $1 - \cos x = 1 + \sqrt{2}/2 > \sqrt{2}/2$. The line $x = \pi$ closes the region on the right.

3. Set up the sum of the two areas.

$$A = \int_0^{\pi/2} (\sin x - 1 + \cos x) dx + \int_{\pi/2}^{\pi} (1 - \cos x - \sin x) dx.$$

4. Integrate each piece and apply its bounds.

$$A_1 = [-\cos x - x + \sin x]_0^{\pi/2} = 2 - \frac{\pi}{2},$$

$$A_2 = [x - \sin x + \cos x]_{\pi/2}^{\pi} = (\pi - 1) - \left(\frac{\pi}{2} - 1\right) = \frac{\pi}{2}.$$

5. Add the positive areas.

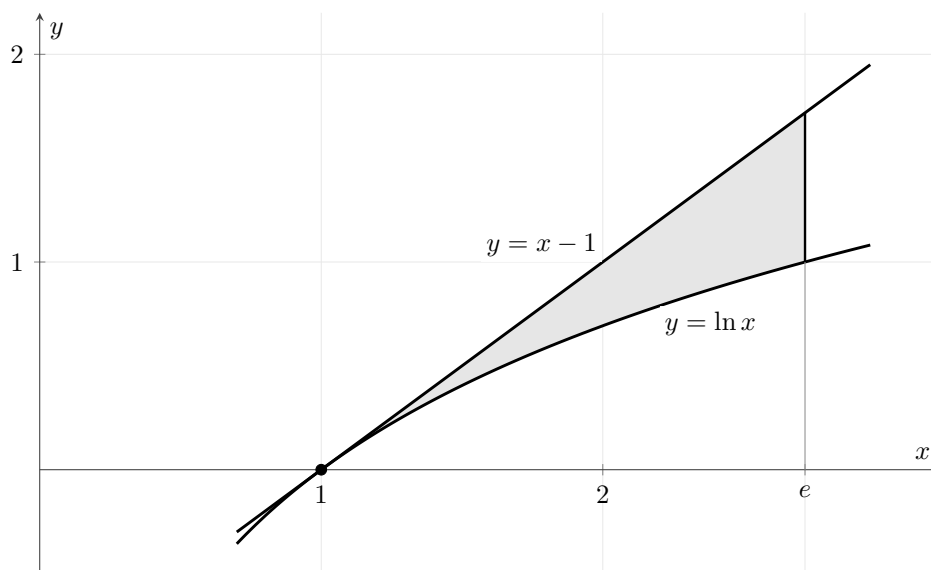
$$A = A_1 + A_2 = \left(2 - \frac{\pi}{2}\right) + \frac{\pi}{2} = \boxed{2}.$$

Question 4

Find the exact area of the region between the curves

$$y = \ln x \quad \text{and} \quad y = x - 1,$$

between $x = 1$ and $x = e$.



The shaded region is the area to be found.

Question 4 – Solution

1. Identify the upper curve on $[1, e]$.

Let $h(x) = x - 1 - \ln x$. Then $h(1) = 0$ and

$$h'(x) = 1 - \frac{1}{x} \geq 0 \quad (1 \leq x \leq e).$$

Thus $x - 1 \geq \ln x$ throughout the interval.

2. Set up the area.

$$A = \int_1^e (x - 1 - \ln x) dx.$$

3. Find the logarithm antiderivative.

Integration by parts, with $u = \ln x$ and $dv = dx$, gives

$$\int \ln x dx = x \ln x - \int 1 dx = x \ln x - x.$$

Here $du = dx/x$ and $v = x$. Combine the terms: $x^2/2 - x - (x \ln x - x) = x^2/2 - x \ln x$.

4. Apply the upper and lower bounds.

$$\begin{aligned} A &= \left[\frac{x^2}{2} - x - (x \ln x - x) \right]_1^e \\ &= \left[\frac{x^2}{2} - x \ln x \right]_1^e \\ &= \left(\frac{e^2}{2} - e \right) - \frac{1}{2} = \boxed{\frac{e^2}{2} - e - \frac{1}{2}}. \end{aligned}$$

5. Check the result.

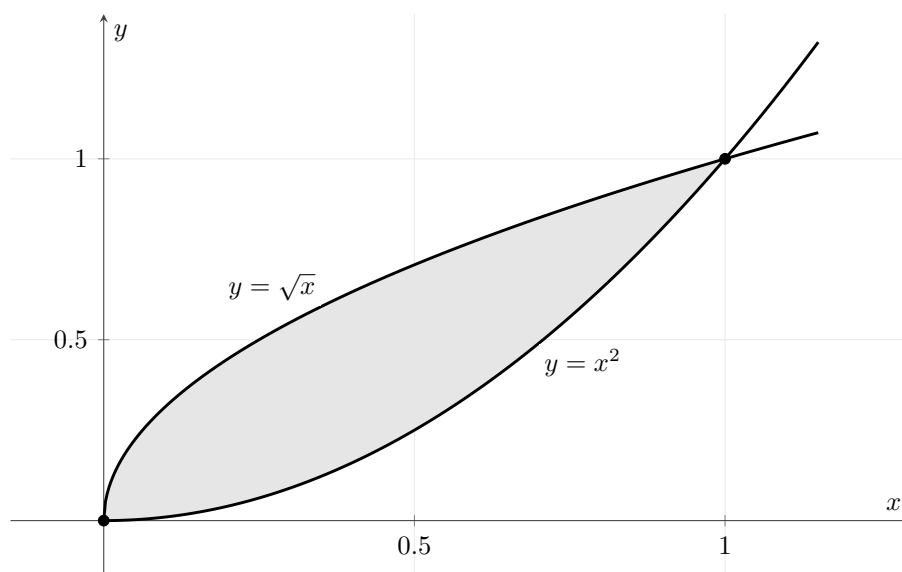
Since $\ln e = 1$ and $\ln 1 = 0$, the endpoint terms above give an area of approximately 0.4762 square units, which is positive.

Question 5

Find the exact area of the region enclosed by the curves

$$y = \sqrt{x} \quad \text{and} \quad y = x^2,$$

between their points of intersection.



The shaded region is the area to be found.

Question 5 – Solution

1. Find the intersections.

The square root requires $x \geq 0$. Both sides are nonnegative, so squaring gives

$$\sqrt{x} = x^2 \implies x = x^4 \implies x(x^3 - 1) = 0.$$

The factor $x^3 - 1 = (x - 1)(x^2 + x + 1)$ has only one real zero, $x = 1$. The real solutions $x = 0$ and $x = 1$ both satisfy the original equation. The intersection points are $(0, 0)$ and $(1, 1)$.

2. Determine which curve is above the other.

For $0 < x < 1$, $x^{1/2} > x^2$. For example, at $x = 1/4$, $\sqrt{x} = 1/2$ while $x^2 = 1/16$.

3. Set up upper minus lower.

$$A = \int_0^1 (\sqrt{x} - x^2) dx.$$

4. Integrate each power.

Use $\int x^n dx = x^{n+1}/(n+1)$ for $n \neq -1$:

$$\int x^{1/2} dx = \frac{x^{3/2}}{3/2} = \frac{2}{3}x^{3/2}, \quad \int x^2 dx = \frac{x^3}{3}.$$

5. Apply the bounds and subtract.

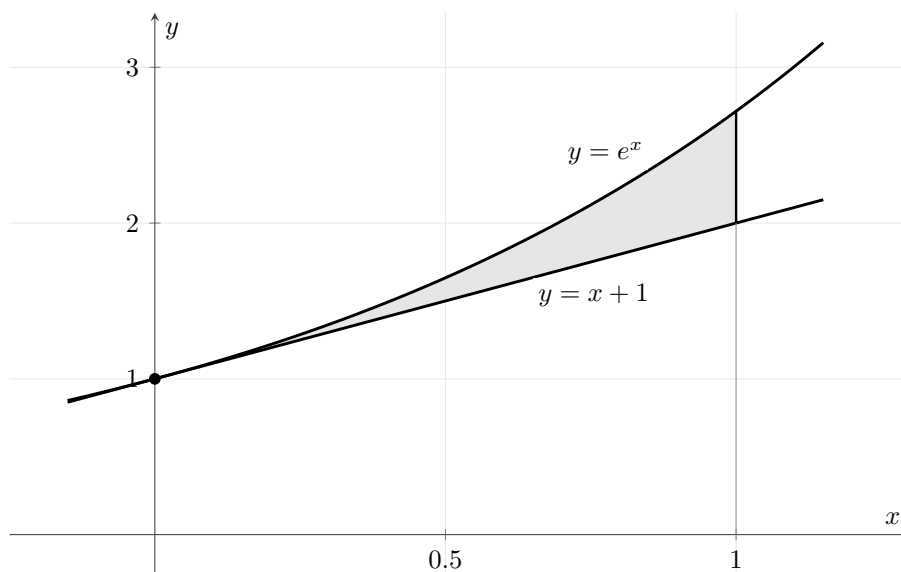
$$\begin{aligned} A &= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3} \right]_0^1 \\ &= \left(\frac{2}{3} - \frac{1}{3} \right) - 0 = \boxed{\frac{1}{3}}. \end{aligned}$$

Question 6

Find the exact area of the region bounded by the curves

$$y = e^x \quad \text{and} \quad y = x + 1,$$

and the vertical lines $x = 0$ and $x = 1$.



The shaded region is the area to be found.

Question 6 – Solution

1. Identify the upper curve on $[0, 1]$.

Let $h(x) = e^x - x - 1$. Then $h(0) = 0$ and

$$h'(x) = e^x - 1 \geq 0 \quad (0 \leq x \leq 1).$$

Thus $e^x \geq x + 1$, with equality only at $x = 0$.

2. Use the given vertical boundaries.

$$A = \int_0^1 (e^x - (x + 1)) \, dx.$$

3. Integrate term by term.

$$\int e^x \, dx = e^x, \quad \int x \, dx = \frac{x^2}{2}, \quad \int 1 \, dx = x.$$

4. Evaluate at $x = 1$ and subtract the value at $x = 0$.

$$\begin{aligned} A &= \left[e^x - \frac{x^2}{2} - x \right]_0^1 \\ &= \left(e - \frac{1}{2} - 1 \right) - (1 - 0 - 0) \\ &= \boxed{e - \frac{5}{2}}. \end{aligned}$$

5. Check the sign and units.

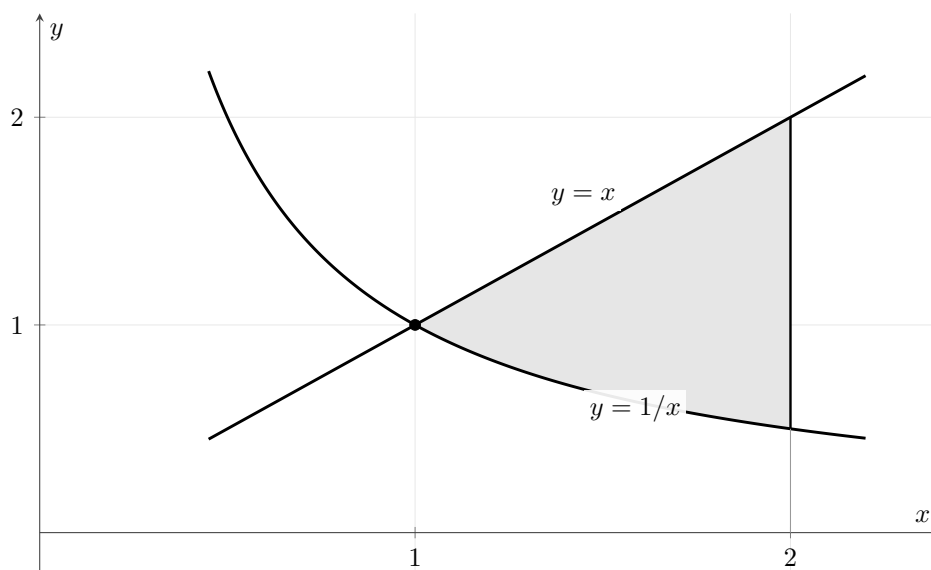
The area is approximately $2.71828 - 2.5 = 0.21828$ square units. It is positive because e^x lies above the line on $(0, 1]$.

Question 7

Find the exact area of the region between the curves

$$y = \frac{1}{x} \quad \text{and} \quad y = x,$$

between $x = 1$ and $x = 2$.



The shaded region is the area to be found.

Question 7 – Solution

1. Identify the upper curve on $[1, 2]$.

For $x \geq 1$,

$$x - \frac{1}{x} = \frac{x^2 - 1}{x} \geq 0.$$

Thus the line $y = x$ lies above $y = 1/x$. They meet at $(1, 1)$, and the boundary $x = 2$ closes the region.

2. Set up the area.

$$A = \int_1^2 \left(x - \frac{1}{x} \right) dx.$$

3. Integrate each term.

Since $x > 0$ on this interval, an antiderivative of $1/x$ is $\ln x$.

$$\int x \, dx = \frac{x^2}{2}, \quad \int \frac{1}{x} \, dx = \ln x.$$

4. Substitute the bounds and simplify.

$$\begin{aligned} A &= \left[\frac{x^2}{2} - \ln x \right]_1^2 \\ &= (2 - \ln 2) - \left(\frac{1}{2} - \ln 1 \right) \\ &= \boxed{\frac{3}{2} - \ln 2}. \end{aligned}$$

5. Check the result.

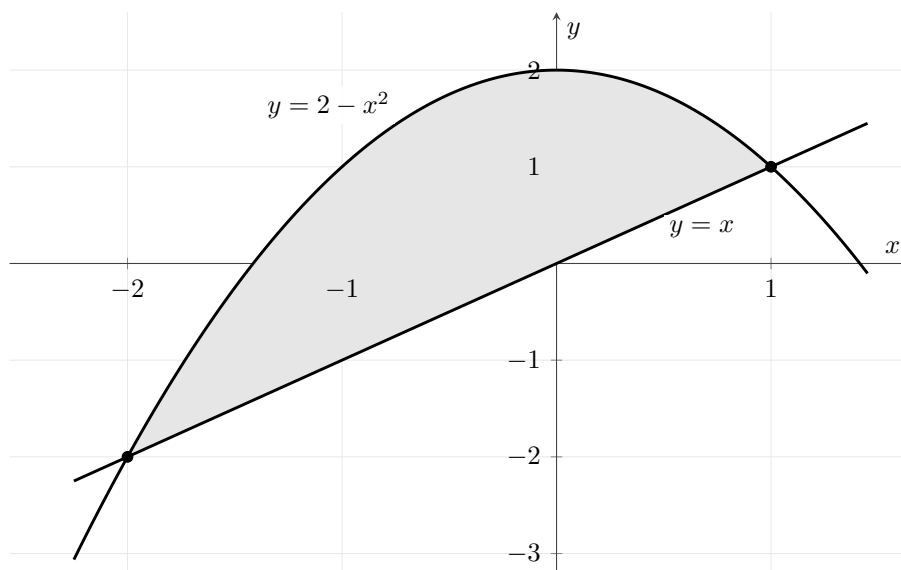
Using $\ln 2 \approx 0.69315$, the area is $1.5 - 0.69315 \approx 0.80685$ square units. The result is positive, as required.

Question 8

Find the exact area of the region enclosed by the curves

$$y = 2 - x^2 \quad \text{and} \quad y = x,$$

between their points of intersection.



The shaded region is the area to be found.

Question 8 – Solution

1. Find the intersections.

$$2 - x^2 = x \implies x^2 + x - 2 = 0 \implies (x + 2)(x - 1) = 0.$$

The zero-product property gives $x + 2 = 0$ or $x - 1 = 0$. Thus $x = -2$ and $x = 1$, giving points $(-2, -2)$ and $(1, 1)$.

2. Determine which curve is above the other.

On $[-2, 1]$, $(2 - x^2) - x = (1 - x)(x + 2) \geq 0$. The parabola is above the line, even where both curves are below the x -axis.

3. Set up upper minus lower.

$$A = \int_{-2}^1 (2 - x - x^2) dx.$$

4. Find an antiderivative.

$$\int 2 dx = 2x, \quad \int x dx = \frac{x^2}{2}, \quad \int x^2 dx = \frac{x^3}{3}.$$

5. Apply the bounds, taking care with negative signs.

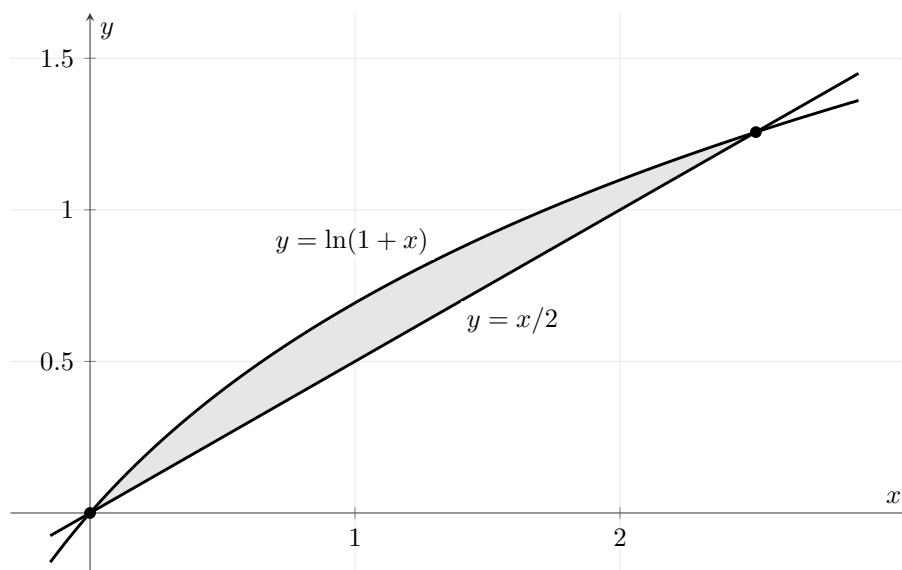
$$\begin{aligned} A &= \left[2x - \frac{x^2}{2} - \frac{x^3}{3} \right]_{-2}^1 \\ &= \left(2 - \frac{1}{2} - \frac{1}{3} \right) - \left(-4 - 2 + \frac{8}{3} \right) \\ &= \frac{7}{6} - \left(-\frac{10}{3} \right) = \boxed{\frac{9}{2}}. \end{aligned}$$

Question 9

Find the exact area of the region enclosed by the curves

$$y = \ln(1 + x) \quad \text{and} \quad y = \frac{x}{2},$$

between their points of intersection.



The shaded region is the area to be found.

Question 9 – Solution

1. Describe the intersections.

Let $h(x) = \ln(1+x) - x/2$, with domain $x > -1$. Then

$$h'(x) = \frac{1-x}{2(1+x)}.$$

So h increases up to $x = 1$ and decreases thereafter. Since $h(0) = 0$, $h(1) > 0$, and $h(x) \rightarrow -\infty$ as $x \rightarrow \infty$, the only roots are 0 and a unique number $a > 1$ satisfying

$$\ln(1+a) = \frac{a}{2}, \quad a \approx 2.512862417.$$

The nonzero root has no elementary closed form; this equation defines a exactly.

2. Set up the area.

On $(0, a)$, $h(x) > 0$, so the logarithm is above the line.

$$A = \int_0^a \left(\ln(1+x) - \frac{x}{2} \right) dx.$$

3. Find an antiderivative.

With $u = 1+x$, integration by parts gives

$$\int \ln(1+x) dx = (1+x) \ln(1+x) - x + C.$$

Also, $\int (x/2) dx = x^2/4$. At $x = 0$, all terms below are zero.

4. Evaluate, then substitute $\ln(1+a) = a/2$.

$$\begin{aligned} A &= \left[(1+x) \ln(1+x) - x - \frac{x^2}{4} \right]_0^a \\ &= (1+a) \frac{a}{2} - a - \frac{a^2}{4} \\ &= \boxed{\frac{a(a-2)}{4}} \approx 0.322188173. \end{aligned}$$

5. Interpret the exact answer.

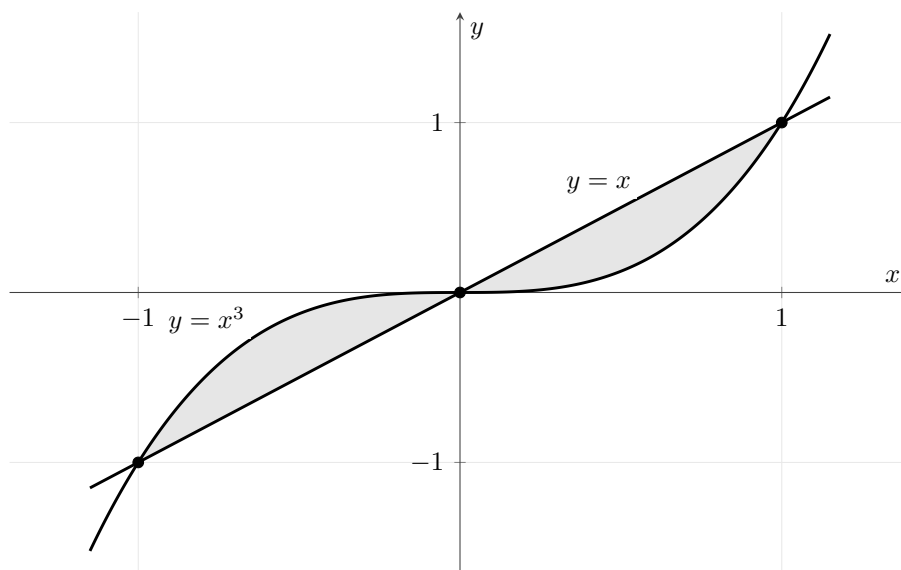
The boxed expression is exact with a defined above; using its numerical approximation gives 0.322188173 square units.

Question 10

Find the exact total area of the regions enclosed by the curves

$$y = x^3 \quad \text{and} \quad y = x,$$

between their points of intersection.



The shaded region is the area to be found.

Question 10 – Solution

1. Find all intersections.

$$x^3 = x \implies x(x-1)(x+1) = 0 \implies x = -1, 0, 1.$$

2. Split the area where the upper curve changes.

On $(-1, 0)$, $x^3 > x$; on $(0, 1)$, $x > x^3$. For example, at $x = -1/2$, $x^3 = -1/8 > -1/2$; at $x = 1/2$, $x = 1/2 > 1/8 = x^3$. Both enclosed regions contribute positive area:

3. Set up upper minus lower on each interval.

$$A = \int_{-1}^0 (x^3 - x) dx + \int_0^1 (x - x^3) dx.$$

4. Integrate and evaluate each region.

The power rule gives $\int x^3 dx = x^4/4$ and $\int x dx = x^2/2$.

$$A_1 = \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^0 = 0 - \left(\frac{1}{4} - \frac{1}{2} \right) = \frac{1}{4},$$

$$A_2 = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

5. Add both areas.

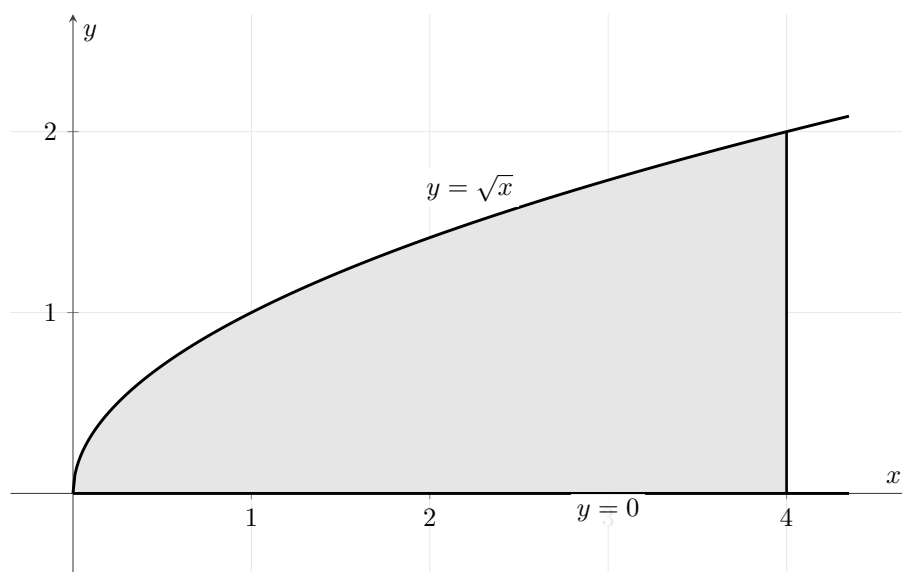
$$A = A_1 + A_2 = \frac{1}{4} + \frac{1}{4} = \boxed{\frac{1}{2}}.$$

Question 1

Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x} \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = 4$, about the x -axis.



Rotate the shaded region about the x -axis.

Question 1 – Solution**1. Choose slices and bounds.**

The rotation axis is horizontal, so use vertical slices of thickness dx . The stated interval is $0 \leq x \leq 4$. Each slice reaches the axis, producing a disk.

2. Measure the radii from the axis.

The distance from $y = 0$ to $y = \sqrt{x}$ is the outer radius. There is no central hole.

$$R(x) = \sqrt{x} - 0 = \sqrt{x}, \quad r(x) = 0.$$

3. Write the cross-sectional area and volume.

$$A(x) = \pi R(x)^2 - \pi r(x)^2 = \pi((\sqrt{x})^2 - 0^2) = \pi x.$$

$$V = \int_0^4 A(x) dx = \pi \int_0^4 x dx.$$

4. Integrate using the power rule.

$$\int x dx = \frac{x^{1+1}}{1+1} = \frac{x^2}{2}, \quad V = \pi \left[\frac{x^2}{2} \right]_0^4.$$

5. Apply the bounds and simplify.

$$V = \pi \left(\frac{4^2}{2} - \frac{0^2}{2} \right) = \pi \left(\frac{16}{2} \right) = \boxed{8\pi}.$$

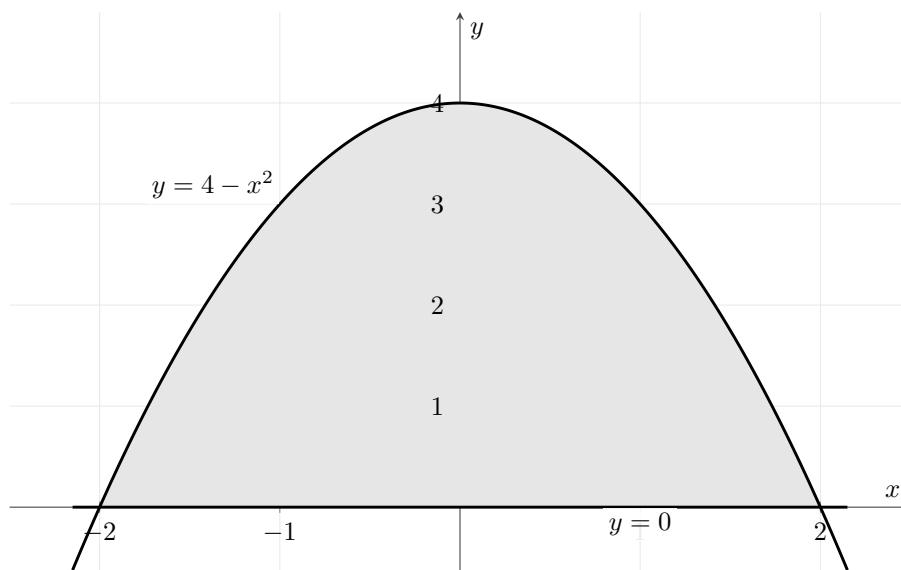
All volumes are in cubic units.

Question 2

Find the volume of the solid obtained by rotating the region bounded by

$$y = 4 - x^2 \quad \text{and} \quad y = 0,$$

from $x = -2$ to $x = 2$, about the x -axis.



Rotate the shaded region about the x -axis.

Question 2 – Solution**1. Choose slices and confirm the bounds.**

Use vertical slices, perpendicular to the x -axis. The curve meets $y = 0$ where

$$4 - x^2 = 0 \implies x = \pm 2.$$

2. Identify the radii.

On $[-2, 2]$, $4 - x^2 \geq 0$. The region touches the axis, so each cross-section is a disk:

$$R(x) = 4 - x^2, \quad r(x) = 0.$$

3. Set up the volume and expand the square.

$$V = \pi \int_{-2}^2 (R^2 - r^2) dx = \pi \int_{-2}^2 (4 - x^2)^2 dx.$$
$$(4 - x^2)^2 = 16 - 8x^2 + x^4.$$

4. Use symmetry and integrate.

The integrand is even, so double the integral over $[0, 2]$:

$$V = 2\pi \int_0^2 (16 - 8x^2 + x^4) dx = 2\pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_0^2.$$

5. Substitute and combine the fractions.

$$V = 2\pi \left(32 - \frac{64}{3} + \frac{32}{5} \right)$$
$$= 2\pi \left(\frac{480 - 320 + 96}{15} \right) = \boxed{\frac{512\pi}{15}}.$$

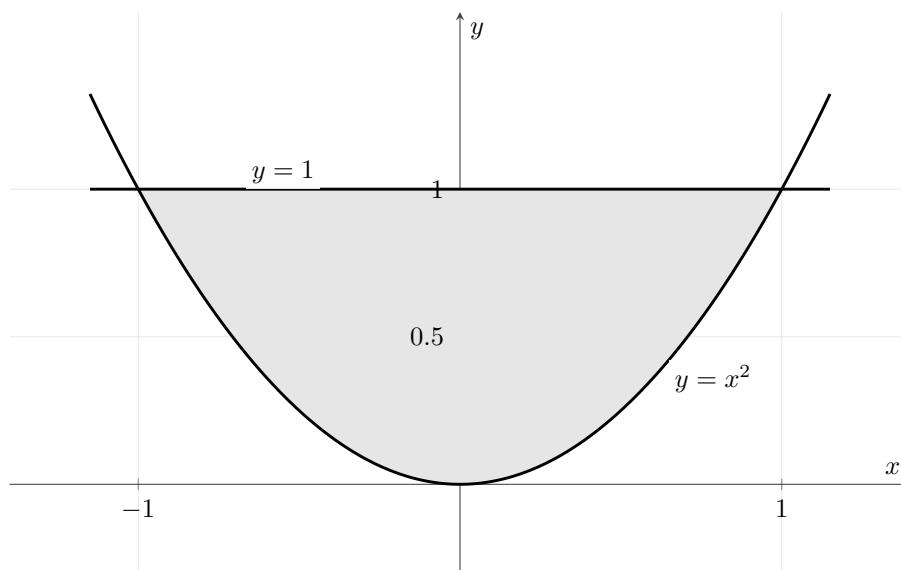
All volumes are in cubic units.

Question 3

Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2 \quad \text{and} \quad y = 1,$$

from $x = -1$ to $x = 1$, about the x -axis.



Rotate the shaded region about the x -axis.

Question 3 – Solution

1. Find the bounds and choose slices.

The intersections satisfy $x^2 = 1$, so $x = -1$ and $x = 1$. Use vertical slices of thickness dx because the rotation axis is horizontal.

2. Identify outer and inner radii.

On $[-1, 1]$, $0 \leq x^2 \leq 1$. Measured from the x -axis,

$$R(x) = 1, \quad r(x) = x^2.$$

The missing inner disk must be subtracted from the outer disk.

3. Set up the washer integral.

$$A(x) = \pi(R(x)^2 - r(x)^2) = \pi(1^2 - (x^2)^2) = \pi(1 - x^4).$$

$$V = \pi \int_{-1}^1 (1 - x^4) dx.$$

4. Integrate term by term.

$$\int 1 dx = x, \quad \int x^4 dx = \frac{x^5}{5}.$$

$$V = \pi \left[x - \frac{x^5}{5} \right]_{-1}^1.$$

5. Evaluate both endpoints.

$$\begin{aligned} V &= \pi \left[\left(1 - \frac{1}{5} \right) - \left(-1 + \frac{1}{5} \right) \right] \\ &= \pi \left(\frac{4}{5} + \frac{4}{5} \right) = \boxed{\frac{8\pi}{5}}. \end{aligned}$$

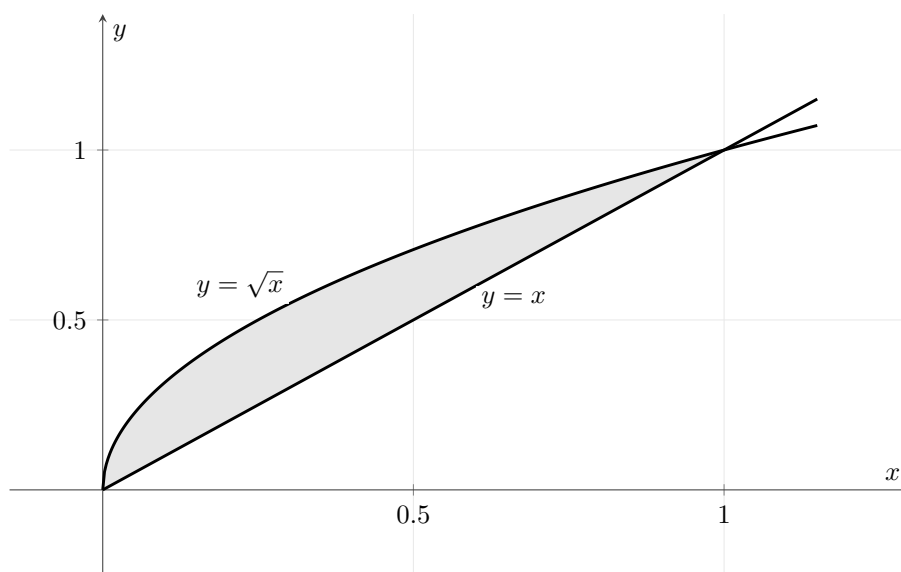
All volumes are in cubic units.

Question 4

Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x} \quad \text{and} \quad y = x,$$

from $x = 0$ to $x = 1$, about the x -axis.



Rotate the shaded region about the x -axis.

Question 4 – Solution

1. Find the bounds and choose slices.

The curves meet where $\sqrt{x} = x$. With $x \geq 0$, squaring gives $x = x^2$, so $x = 0$ or $x = 1$. Use vertical slices for rotation about the x -axis.

2. Measure the radii.

For $0 \leq x \leq 1$, $\sqrt{x} \geq x \geq 0$. Thus the square-root curve is farther from the axis:

$$R(x) = \sqrt{x}, \quad r(x) = x.$$

3. Set up the washer integral.

Subtract the squared radii, then multiply by π :

$$V = \pi \int_0^1 (R^2 - r^2) dx = \pi \int_0^1 ((\sqrt{x})^2 - x^2) dx = \pi \int_0^1 (x - x^2) dx.$$

4. Find the antiderivative.

$$\int x dx = \frac{x^2}{2}, \quad \int x^2 dx = \frac{x^3}{3}.$$

$$V = \pi \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1.$$

5. Apply the bounds and simplify.

$$V = \pi \left[\left(\frac{1}{2} - \frac{1}{3} \right) - 0 \right] = \pi \left(\frac{3-2}{6} \right) = \boxed{\frac{\pi}{6}}.$$

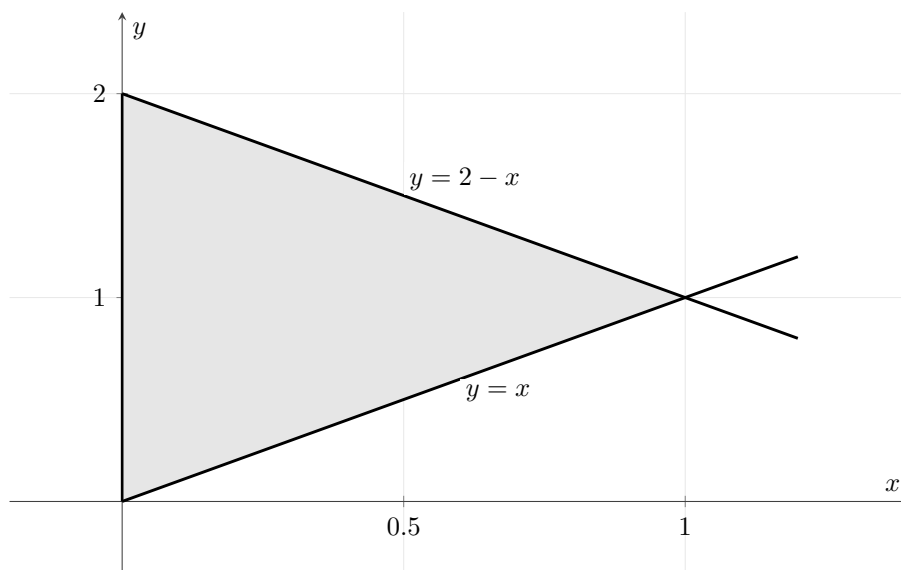
All volumes are in cubic units.

Question 5

Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - x \quad \text{and} \quad y = x,$$

from $x = 0$ to $x = 1$, about the x -axis.



Rotate the shaded region about the x -axis.

Question 5 – Solution

1. Identify the bounds and slice direction.

The given left boundary is $x = 0$. The lines meet where $2 - x = x$, giving $x = 1$. Use vertical slices, perpendicular to the x -axis.

2. Measure the radii from the x -axis.

On $[0, 1]$, $2 - x \geq x \geq 0$, so

$$R(x) = 2 - x, \quad r(x) = x.$$

The rotated slice is a washer with area $\pi(R^2 - r^2)$.

3. Set up and simplify the volume integral.

$$V = \pi \int_0^1 [(2 - x)^2 - x^2] dx.$$

$$(2 - x)^2 - x^2 = (4 - 4x + x^2) - x^2 = 4 - 4x.$$

4. Integrate term by term.

$$\int 4 dx = 4x, \quad \int 4x dx = 4 \frac{x^2}{2} = 2x^2.$$

$$V = \pi [4x - 2x^2]_0^1.$$

5. Apply the bounds.

$$V = \pi [(4(1) - 2(1)^2) - (4(0) - 2(0)^2)] = \pi(4 - 2) = \boxed{2\pi}.$$

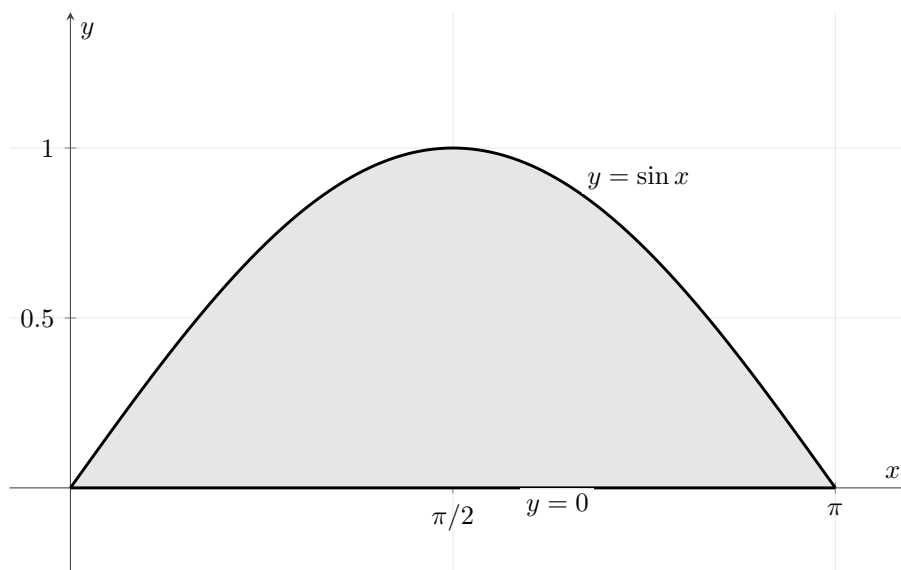
All volumes are in cubic units.

Question 6

Find the volume of the solid obtained by rotating the region bounded by

$$y = \sin x \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = \pi$, about the x -axis.



Rotate the shaded region about the x -axis.

Question 6 – Solution

1. Choose slices and identify the bounds.

Use vertical slices over $0 \leq x \leq \pi$. Since $\sin x \geq 0$ on this interval and the region touches the x -axis, rotation gives disks.

2. Identify radii and set up the volume.

$$R(x) = \sin x, \quad r(x) = 0.$$
$$V = \pi \int_0^\pi ((\sin x)^2 - 0^2) dx = \pi \int_0^\pi \sin^2 x \, dx.$$

3. Use the power-reduction identity.

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \implies V = \frac{\pi}{2} \int_0^\pi (1 - \cos(2x)) \, dx.$$

4. Integrate, including the chain-rule factor.

Because $\frac{d}{dx} \sin(2x) = 2 \cos(2x)$,

$$\int \cos(2x) \, dx = \frac{\sin(2x)}{2}, \quad V = \frac{\pi}{2} \left[x - \frac{\sin(2x)}{2} \right]_0^\pi.$$

5. Evaluate at π and 0.

$$V = \frac{\pi}{2} \left[\left(\pi - \frac{\sin(2\pi)}{2} \right) - \left(0 - \frac{\sin 0}{2} \right) \right]$$
$$= \frac{\pi}{2} (\pi - 0) = \boxed{\frac{\pi^2}{2}}.$$

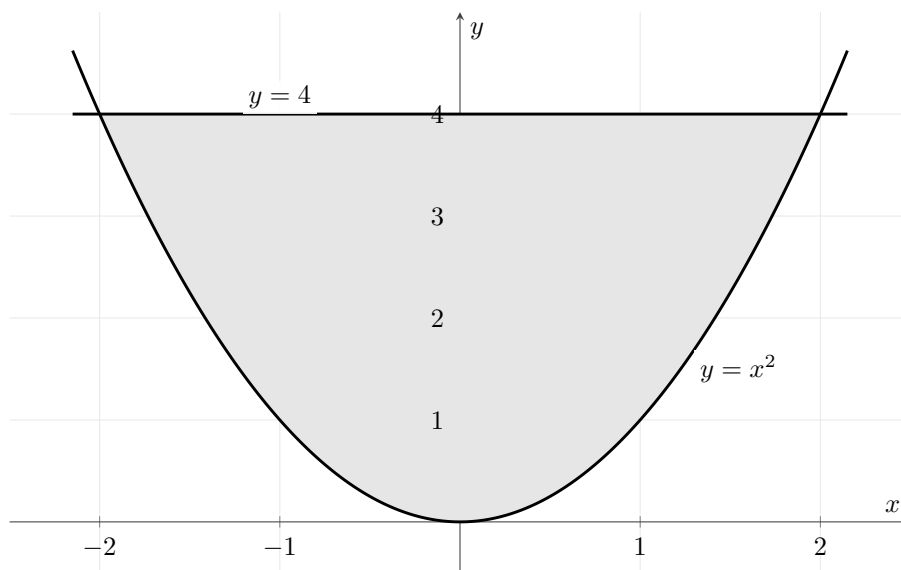
All volumes are in cubic units.

Question 7

Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2 \quad \text{and} \quad y = 4,$$

about the y -axis.



Rotate the shaded region about the y -axis.

Question 7 – Solution**1. Use horizontal slices and find the y -bounds.**

Rotation about the y -axis requires slices of thickness dy . The region lies above $y = x^2$ and below $y = 4$, so $0 \leq y \leq 4$.

2. Express the horizontal boundaries in terms of y .

$$y = x^2 \implies x = \pm\sqrt{y}.$$

At height y , the region extends from $-\sqrt{y}$ to $\sqrt{y}$ and crosses the rotation axis.

3. Identify the disk radius.

Rotating the entire horizontal segment gives one disk, not two separate disks. Its radius is the distance from the axis to either edge:

$$R(y) = \sqrt{y}, \quad r(y) = 0.$$

4. Set up the volume and integrate.

$$\begin{aligned} V &= \pi \int_0^4 (R^2 - r^2) dy = \pi \int_0^4 ((\sqrt{y})^2 - 0^2) dy = \pi \int_0^4 y dy. \\ \int y dy &= \frac{y^2}{2}. \end{aligned}$$

5. Apply the bounds.

$$V = \pi \left[\frac{y^2}{2} \right]_0^4 = \pi \left(\frac{16}{2} - 0 \right) = \boxed{8\pi}.$$

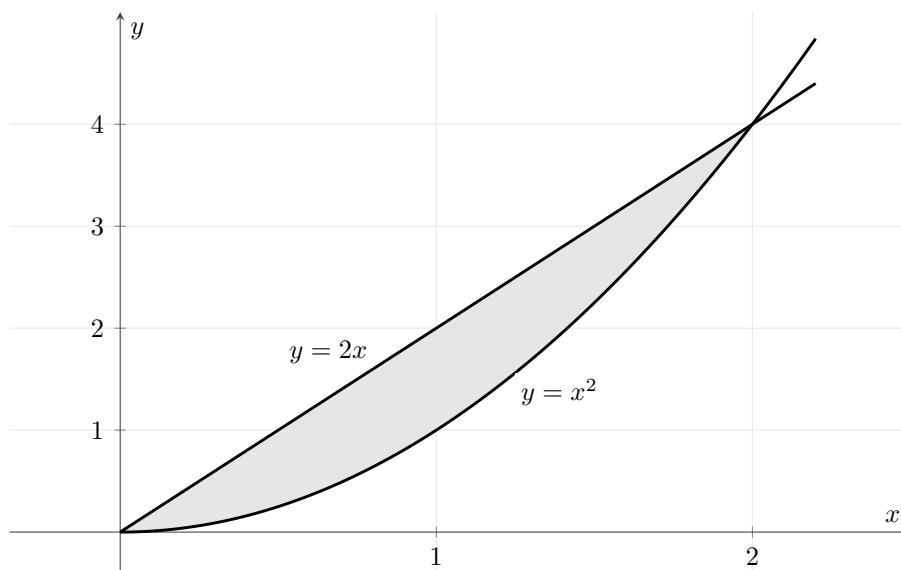
All volumes are in cubic units.

Question 8

Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2 \quad \text{and} \quad y = 2x,$$

about the y -axis.



Rotate the shaded region about the y -axis.

Question 8 – Solution

1. Find the intersections and y -bounds.

$$x^2 = 2x \implies x(x - 2) = 0 \implies x = 0, 2.$$

The intersection points are $(0, 0)$ and $(2, 4)$, so integrate from $y = 0$ to $y = 4$.

2. Use horizontal slices and identify radii.

Rewrite the curves as $x = \sqrt{y}$ and $x = y/2$. For $0 \leq y \leq 4$, $\sqrt{y} \geq y/2$. Distances from the y -axis give

$$R(y) = \sqrt{y}, \quad r(y) = \frac{y}{2}.$$

3. Set up the washer integral.

$$V = \pi \int_0^4 \left[(\sqrt{y})^2 - \left(\frac{y}{2}\right)^2 \right] dy = \pi \int_0^4 \left(y - \frac{y^2}{4} \right) dy.$$

4. Integrate term by term.

$$\int y \, dy = \frac{y^2}{2}, \quad \int \frac{y^2}{4} \, dy = \frac{1}{4} \frac{y^3}{3} = \frac{y^3}{12}.$$

$$V = \pi \left[\frac{y^2}{2} - \frac{y^3}{12} \right]_0^4.$$

5. Apply the bounds and simplify.

$$V = \pi \left(\frac{16}{2} - \frac{64}{12} - 0 \right) = \pi \left(8 - \frac{16}{3} \right) = \boxed{\frac{8\pi}{3}}.$$

All volumes are in cubic units.

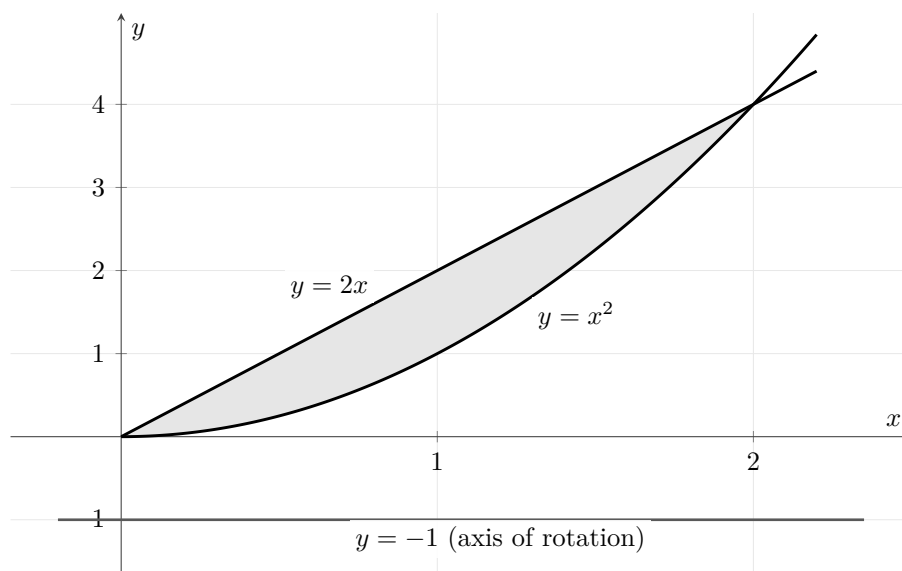
Question 9

Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2 \quad \text{and} \quad y = 2x,$$

about the horizontal line

$$y = -1.$$



Rotate the shaded region about the line $y = -1$.

Question 9 – Solution**1. Find the bounds and choose vertical slices.**

The curves meet where $x^2 = 2x$, or $x(x - 2) = 0$. Thus $0 \leq x \leq 2$. The horizontal rotation axis $y = -1$ requires vertical slices of thickness dx .

2. Measure both radii from $y = -1$.

On $[0, 2]$, $2x \geq x^2$. The outer radius reaches the line, and the inner radius reaches the parabola:

$$R(x) = 2x - (-1) = 2x + 1, \quad r(x) = x^2 - (-1) = x^2 + 1.$$

3. Set up the washer integral and expand.

$$\begin{aligned} V &= \pi \int_0^2 [(2x + 1)^2 - (x^2 + 1)^2] dx. \\ (2x + 1)^2 - (x^2 + 1)^2 &= (4x^2 + 4x + 1) - (x^4 + 2x^2 + 1) \\ &= -x^4 + 2x^2 + 4x. \end{aligned}$$

4. Integrate each power.

$$V = \pi \left[-\frac{x^5}{5} + \frac{2x^3}{3} + 2x^2 \right]_0^2.$$

The coefficients follow from $\int x^n dx = x^{n+1}/(n+1)$.

5. Evaluate and use a common denominator.

$$\begin{aligned} V &= \pi \left(-\frac{32}{5} + \frac{16}{3} + 8 - 0 \right) \\ &= \pi \left(\frac{-96 + 80 + 120}{15} \right) = \boxed{\frac{104\pi}{15}}. \end{aligned}$$

All volumes are in cubic units.

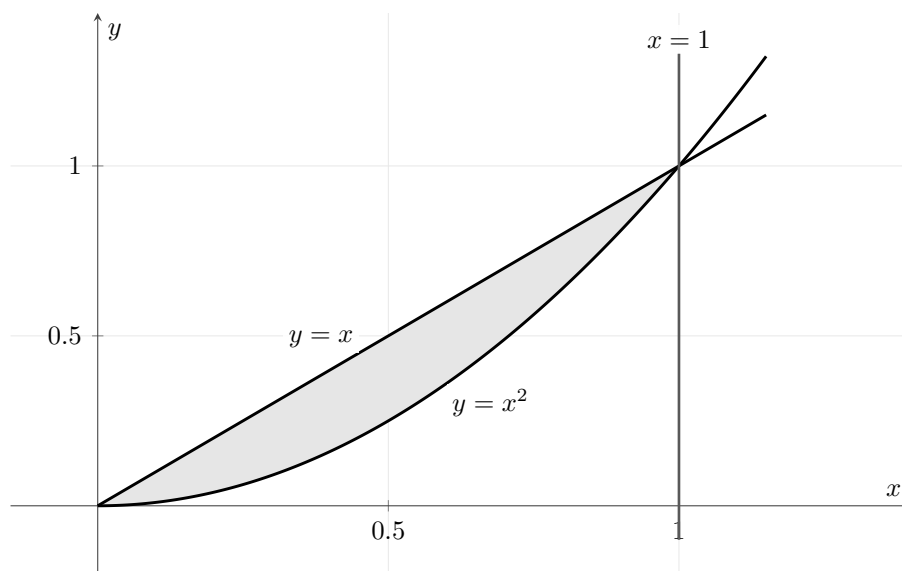
Question 10

Find the volume of the solid obtained by rotating the region bounded by

$$y = x \quad \text{and} \quad y = x^2,$$

about the vertical line

$$x = 1.$$



Rotate the shaded region about the line $x = 1$.

Question 10 – Solution

1. Find the bounds and choose horizontal slices.

The intersections satisfy $x = x^2$, so $x = 0, 1$ and $y = 0, 1$. Rotation about the vertical line $x = 1$ requires horizontal slices of thickness dy .

2. Rewrite the curves and measure radii.

The boundaries are $x = y$ on the left and $x = \sqrt{y}$ on the right. The axis is to their right, so the farther, left boundary gives the outer radius:

$$R(y) = 1 - y, \quad r(y) = 1 - \sqrt{y}.$$

3. Set up the washer integral and expand.

$$\begin{aligned} V &= \pi \int_0^1 [(1 - y)^2 - (1 - \sqrt{y})^2] dy. \\ (1 - y)^2 - (1 - \sqrt{y})^2 &= (1 - 2y + y^2) - (1 - 2\sqrt{y} + y) \\ &= y^2 - 3y + 2y^{1/2}. \end{aligned}$$

4. Integrate using the power rule.

In particular, $\int 2y^{1/2} dy = 2 \cdot \frac{2}{3}y^{3/2} = \frac{4}{3}y^{3/2}$. Therefore,

$$V = \pi \left[\frac{y^3}{3} - \frac{3y^2}{2} + \frac{4}{3}y^{3/2} \right]_0^1.$$

5. Apply the bounds and simplify.

$$V = \pi \left(\frac{1}{3} - \frac{3}{2} + \frac{4}{3} - 0 \right) = \pi \left(\frac{2 - 9 + 8}{6} \right) = \boxed{\frac{\pi}{6}}.$$

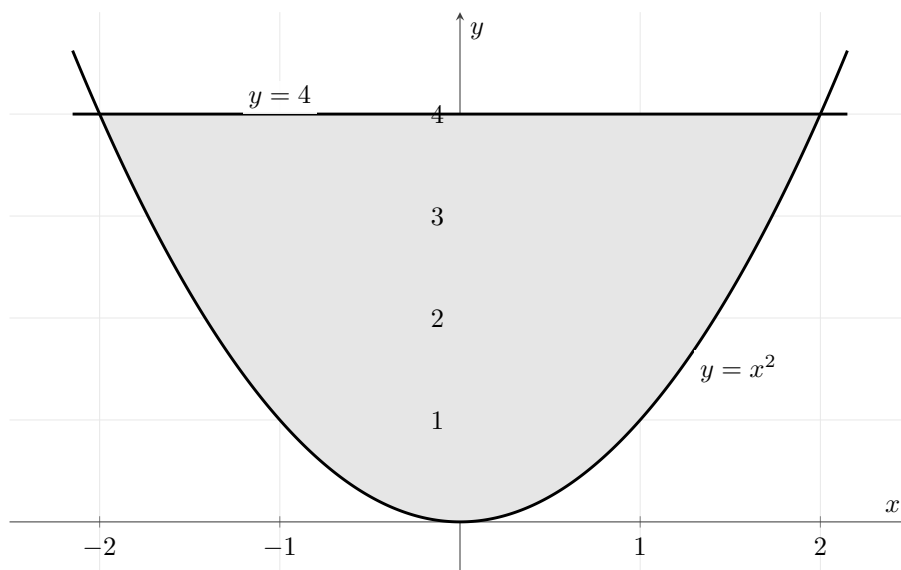
All volumes are in cubic units.

Question 1

Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2 \quad \text{and} \quad y = 4$$

about the y -axis.



Rotate the shaded region about the y -axis.

Question 1 – Solution

1. Find the intersections and choose slices.

For shells about the y -axis, use vertical slices parallel to the axis. The curves meet where $x^2 = 4$, so $x = \pm 2$.

2. Choose an interval that counts each shell once.

The half-region $0 \leq x \leq 2$ generates the entire solid. The negative half generates the same shells, so do not double the volume or integrate over both halves.

3. Identify radius, height, and the shell integral.

The radius is the distance to the y -axis; the height is top minus bottom:

$$r(x) = x, \quad h(x) = 4 - x^2.$$
$$V = 2\pi \int_0^2 r(x)h(x) dx = 2\pi \int_0^2 x(4 - x^2) dx.$$

4. Expand and integrate.

$$x(4 - x^2) = 4x - x^3, \quad \int (4x - x^3) dx = 2x^2 - \frac{x^4}{4}.$$

5. Apply the bounds and simplify.

$$\begin{aligned} V &= 2\pi \left[2x^2 - \frac{x^4}{4} \right]_0^2 \\ &= 2\pi \left[\left(8 - \frac{16}{4} \right) - 0 \right] = 2\pi(4) = \boxed{8\pi}. \end{aligned}$$

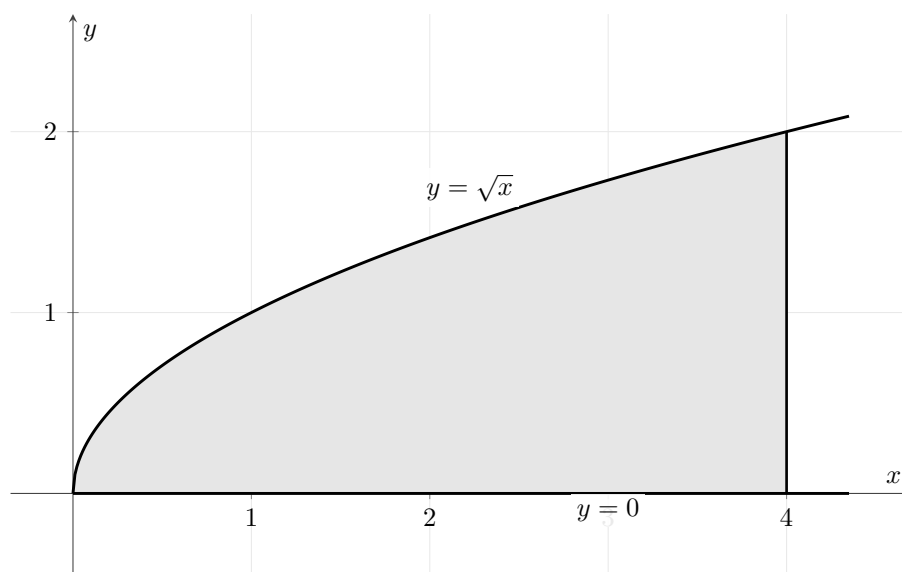
All volumes are in cubic units.

Question 2

Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x} \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = 4$, about the x -axis.



Rotate the shaded region about the x -axis.

Question 2 – Solution

1. Choose horizontal slices and y -bounds.

Shell slices run parallel to the x -axis. Rewrite $y = \sqrt{x}$ as $x = y^2$. The region extends from $y = 0$ to $y = \sqrt{4} = 2$.

2. Identify the shell radius and height.

The radius is the distance to $y = 0$. For a horizontal slice, shell height is its horizontal length, right minus left:

$$r(y) = y, \quad h(y) = 4 - y^2.$$

3. Set up the shell integral.

A thin shell has volume approximately $(2\pi r)(h) dy$. Thus

$$V = 2\pi \int_0^2 y(4 - y^2) dy = 2\pi \int_0^2 (4y - y^3) dy.$$

4. Integrate with respect to y .

$$\int 4y dy = 2y^2, \quad \int y^3 dy = \frac{y^4}{4}.$$
$$V = 2\pi \left[2y^2 - \frac{y^4}{4} \right]_0^2.$$

5. Evaluate both endpoints.

$$V = 2\pi \left[\left(2(2)^2 - \frac{2^4}{4} \right) - 0 \right] = 2\pi(8 - 4) = \boxed{8\pi}.$$

All volumes are in cubic units.

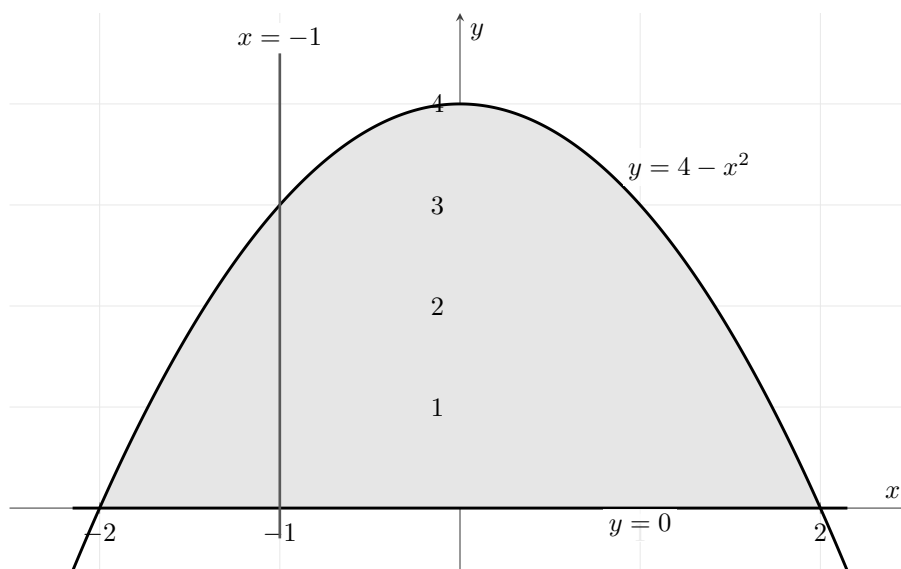
Question 3

Find the volume of the solid obtained by rotating the region bounded by

$$y = 4 - x^2 \quad \text{and} \quad y = 0,$$

about the vertical line

$$x = -1.$$



Rotate the shaded region about the line $x = -1$.

Question 3 – Solution**1. Locate the region and rotation axis.**

The parabola meets $y = 0$ at $x = \pm 2$. The axis $x = -1$ passes through the region. Vertical strips on opposite sides can rotate into the same shell.

2. Count each radius once.

Let r be distance from $x = -1$. The right strip is at $x = -1 + r$, with $0 \leq r \leq 3$. Its height is $4 - (r - 1)^2$. For $0 \leq r \leq 1$, a left strip also exists at $x = -1 - r$, with height $4 - (r + 1)^2$.

$$[4 - (r - 1)^2] - [4 - (r + 1)^2] = 4r \geq 0.$$

Both strips start at $y = 0$, so the taller right strip includes the entire left strip after rotation. Use only the right strip.

3. Set up the shell integral.

$$\begin{aligned} h(r) &= 4 - (r - 1)^2 = 3 + 2r - r^2. \\ V &= 2\pi \int_0^3 r(3 + 2r - r^2) dr = 2\pi \int_0^3 (3r + 2r^2 - r^3) dr. \end{aligned}$$

4. Integrate each power.

$$V = 2\pi \left[\frac{3}{2}r^2 + \frac{2}{3}r^3 - \frac{1}{4}r^4 \right]_0^3.$$

5. Apply the bounds.

$$V = 2\pi \left(\frac{27}{2} + 18 - \frac{81}{4} \right) = 2\pi \left(\frac{54 + 72 - 81}{4} \right) = \boxed{\frac{45\pi}{2}}.$$

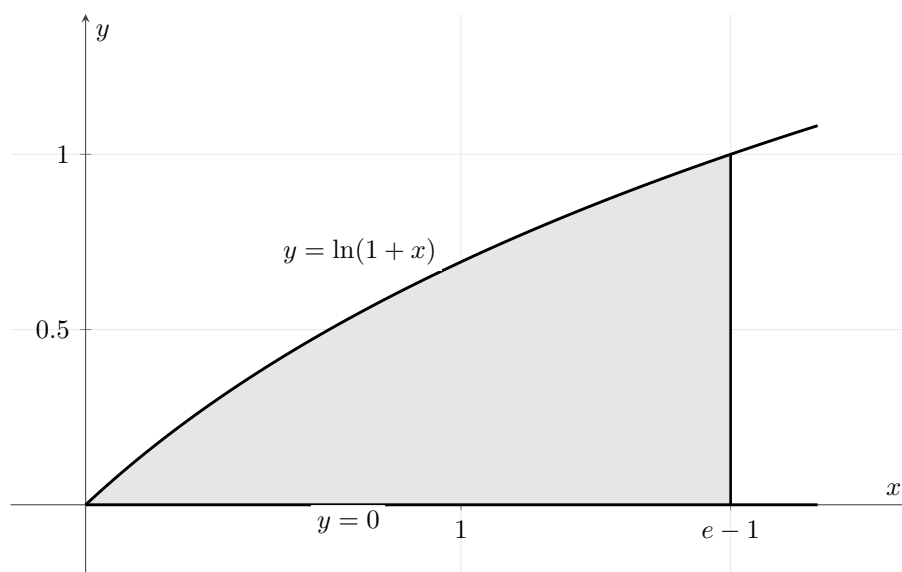
All volumes are in cubic units.

Question 4

Find the volume of the solid obtained by rotating the region bounded by

$$y = \ln(1 + x) \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = e - 1$, about the y -axis.



Rotate the shaded region about the y -axis.

Question 4 – Solution

1. Identify shell radius and height.

Use vertical slices about the y -axis, with $0 \leq x \leq e - 1$:

$$r(x) = x, \quad h(x) = \ln(1 + x) - 0.$$

2. Set up the volume.

$$V = 2\pi \int_0^{e-1} x \ln(1 + x) dx.$$

3. Apply integration by parts.

Choose $u = \ln(1 + x)$ and $dv = x dx$. Then $du = dx/(1 + x)$ and $v = x^2/2$:

$$\int x \ln(1 + x) dx = \frac{x^2}{2} \ln(1 + x) - \frac{1}{2} \int \frac{x^2}{1 + x} dx.$$

4. Divide and integrate the remaining terms.

Since $x^2 = (x + 1)(x - 1) + 1$,

$$\frac{x^2}{1 + x} = x - 1 + \frac{1}{1 + x}.$$

An antiderivative is therefore

$$F(x) = \frac{x^2}{2} \ln(1 + x) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2} \ln(1 + x).$$

5. Apply the bounds and multiply by 2π .

Using $\ln e = 1$ and $F(0) = 0$,

$$\begin{aligned} F(e - 1) - F(0) &= \frac{(e - 1)^2}{4} + \frac{e - 1}{2} - \frac{1}{2} = \frac{e^2 - 3}{4}, \\ V &= 2\pi \cdot \frac{e^2 - 3}{4} = \boxed{\frac{\pi}{2}(e^2 - 3)}. \end{aligned}$$

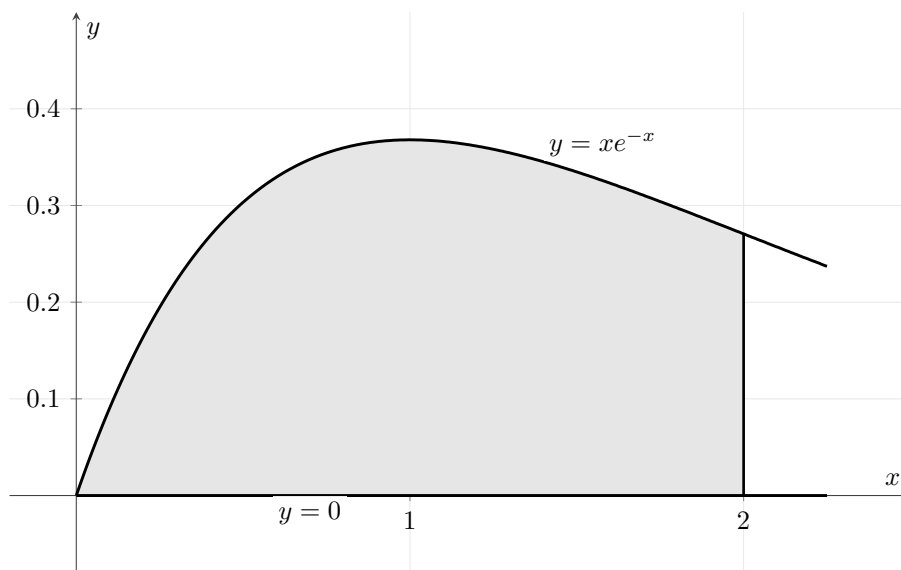
All volumes are in cubic units.

Question 5

Find the volume of the solid obtained by rotating the region bounded by

$$y = xe^{-x} \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = 2$, about the y -axis.



Rotate the shaded region about the y -axis.

Question 5 – Solution

1. Identify the shell dimensions.

Use vertical slices for rotation about the y -axis. On $0 \leq x \leq 2$,

$$r(x) = x, \quad h(x) = xe^{-x}.$$

2. Set up the volume integral.

$$V = 2\pi \int_0^2 r(x)h(x) dx = 2\pi \int_0^2 x^2 e^{-x} dx.$$

3. Integrate by parts once.

Let $u = x^2$, $dv = e^{-x}dx$, so $du = 2x dx$ and $v = -e^{-x}$:

$$\int x^2 e^{-x} dx = -x^2 e^{-x} + 2 \int x e^{-x} dx.$$

4. Integrate by parts a second time.

For the remaining integral, choose $u = x$ and $dv = e^{-x}dx$:

$$\int x e^{-x} dx = -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x}.$$

Substitution gives $F(x) = -e^{-x}(x^2 + 2x + 2)$.

5. Evaluate at 2 and 0.

$$\begin{aligned} V &= 2\pi \left[-e^{-x}(x^2 + 2x + 2) \right]_0^2 \\ &= 2\pi \left[-10e^{-2} - (-2) \right] \\ &= 2\pi(2 - 10e^{-2}) = \boxed{4\pi(1 - 5e^{-2})}. \end{aligned}$$

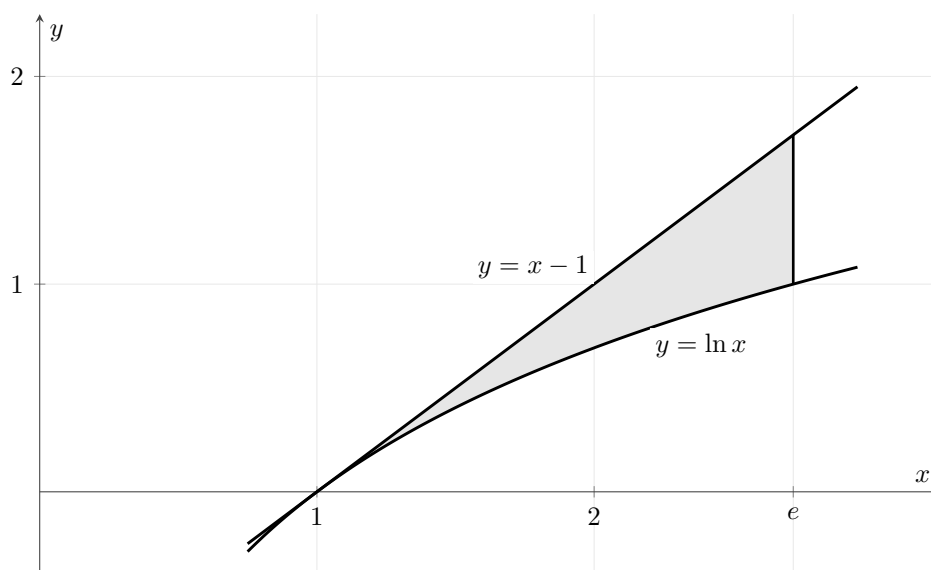
All volumes are in cubic units.

Question 6

Use cylindrical shells to find the volume obtained by revolving the region bounded by

$$y = \ln x, \quad y = x - 1, \quad x = 1, \quad x = e$$

about the y -axis.



Rotate the shaded region about the y -axis.

Question 6 – Solution

1. Determine the upper curve.

Let $g(x) = x - 1 - \ln x$. On $[1, e]$, $g(1) = 0$ and $g'(x) = 1 - 1/x \geq 0$. Hence $x - 1 \geq \ln x$.

2. Identify shell dimensions and set up the volume.

Use vertical slices parallel to the y -axis:

$$r(x) = x, \quad h(x) = x - 1 - \ln x.$$

$$V = 2\pi \int_1^e x(x - 1 - \ln x) dx = 2\pi \int_1^e (x^2 - x - x \ln x) dx.$$

3. Integrate the logarithmic term by parts.

Let $u = \ln x$, $dv = x dx$, so $du = dx/x$ and $v = x^2/2$:

$$\int x \ln x dx = \frac{x^2}{2} \ln x - \frac{1}{2} \int x dx = \frac{x^2}{2} \ln x - \frac{x^2}{4}.$$

4. Combine the antiderivatives.

$$F(x) = \frac{x^3}{3} - \frac{x^2}{2} - \left(\frac{x^2}{2} \ln x - \frac{x^2}{4} \right) = \frac{x^3}{3} - \frac{x^2}{4} - \frac{x^2}{2} \ln x.$$

5. Apply the bounds.

$$F(e) = \frac{e^3}{3} - \frac{3e^2}{4}, \quad F(1) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

$$V = 2\pi(F(e) - F(1)) = \boxed{\pi \left(\frac{2e^3}{3} - \frac{3e^2}{2} - \frac{1}{6} \right)}.$$

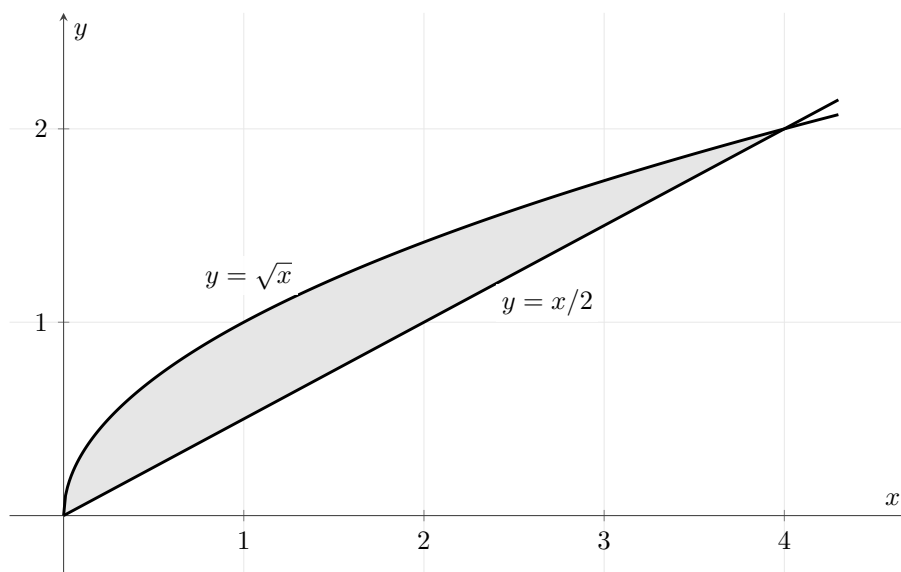
All volumes are in cubic units.

Question 7

Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = \frac{x}{2},$$

between their points of intersection, about the y -axis.



Rotate the shaded region about the y -axis.

Question 7 – Solution**1. Find the intersections.**

Both sides of $\sqrt{x} = x/2$ are nonnegative. Squaring and factoring gives

$$x = \frac{x^2}{4} \implies x(x - 4) = 0 \implies x = 0, 4.$$

Both values satisfy the original equation.

2. Identify radius and height.

Use vertical shells about the y -axis. On $[0, 4]$, $\sqrt{x} \geq x/2$:

$$r(x) = x, \quad h(x) = \sqrt{x} - \frac{x}{2}.$$

3. Set up and expand the shell integral.

$$V = 2\pi \int_0^4 x \left(\sqrt{x} - \frac{x}{2} \right) dx = 2\pi \int_0^4 \left(x^{3/2} - \frac{x^2}{2} \right) dx.$$

4. Integrate using the power rule.

$$\int x^{3/2} dx = \frac{2}{5} x^{5/2}, \quad \int \frac{x^2}{2} dx = \frac{x^3}{6}.$$

$$V = 2\pi \left[\frac{2}{5} x^{5/2} - \frac{x^3}{6} \right]_0^4.$$

5. Apply the bounds and simplify.

Since $4^{5/2} = (\sqrt{4})^5 = 32$,

$$V = 2\pi \left(\frac{64}{5} - \frac{64}{6} \right) = 2\pi \left(\frac{192 - 160}{15} \right) = \boxed{\frac{64\pi}{15}}.$$

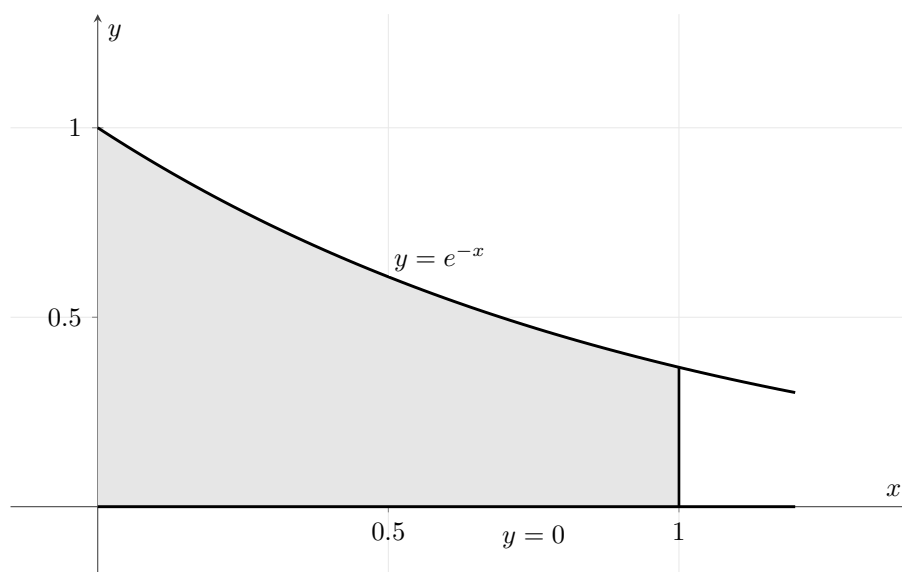
All volumes are in cubic units.

Question 8

Find the volume of the solid obtained by rotating the region bounded by

$$y = e^{-x}, \quad y = 0,$$

from $x = 0$ to $x = 1$, about the y -axis.



Rotate the shaded region about the y -axis.

Question 8 – Solution

1. Identify the shell radius and height.

Use vertical slices over $0 \leq x \leq 1$, parallel to the y -axis:

$$r(x) = x, \quad h(x) = e^{-x} - 0 = e^{-x}.$$

2. Set up the volume integral.

Circumference times height times thickness gives

$$V = 2\pi \int_0^1 r(x)h(x) dx = 2\pi \int_0^1 xe^{-x} dx.$$

3. Apply integration by parts.

Let $u = x$ and $dv = e^{-x}dx$, so $du = dx$ and $v = -e^{-x}$:

$$\int xe^{-x} dx = -xe^{-x} + \int e^{-x} dx = -xe^{-x} - e^{-x}.$$

4. Evaluate the definite integral.

Factor the antiderivative and substitute the bounds:

$$\int_0^1 xe^{-x} dx = [-(x+1)e^{-x}]_0^1 = -\frac{2}{e} - (-1) = 1 - \frac{2}{e}.$$

5. Multiply by the circumference factor.

$$V = \boxed{2\pi \left(1 - \frac{2}{e}\right)}.$$

Since $e > 2$, the expression is positive, as a volume must be.

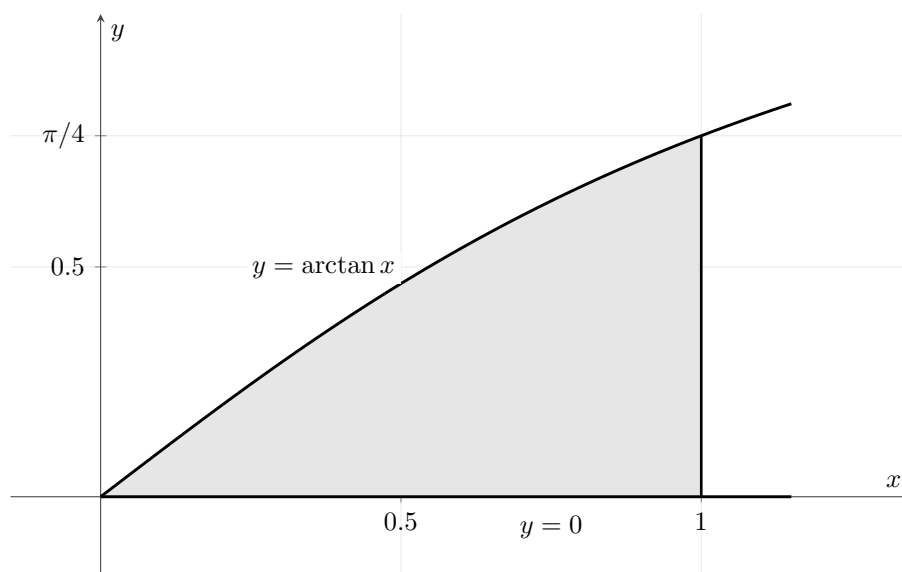
All volumes are in cubic units.

Question 9

Find the volume of the solid obtained by rotating the region bounded by

$$y = \arctan x \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = 1$, about the y -axis.



Rotate the shaded region about the y -axis.

Question 9 – Solution**1. Identify shell radius and height.**

Use vertical slices about the y -axis on $[0, 1]$:

$$r(x) = x, \quad h(x) = \arctan x.$$

Angles are measured in radians.

2. Set up the volume integral.

$$V = 2\pi \int_0^1 x \arctan x \, dx.$$

3. Apply integration by parts.

Let $u = \arctan x$, $dv = x \, dx$, so $du = dx/(1 + x^2)$ and $v = x^2/2$:

$$\int x \arctan x \, dx = \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2}{1 + x^2} \, dx.$$

4. Simplify the remaining rational function.

$$\frac{x^2}{1 + x^2} = 1 - \frac{1}{1 + x^2}.$$

Since $\int (1 + x^2)^{-1} dx = \arctan x$, an antiderivative is

$$F(x) = \frac{x^2}{2} \arctan x - \frac{x}{2} + \frac{1}{2} \arctan x.$$

5. Apply the bounds and simplify.

Using $\arctan 1 = \pi/4$ and $\arctan 0 = 0$,

$$F(1) - F(0) = \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}.$$

$$V = 2\pi \left(\frac{\pi}{4} - \frac{1}{2} \right) = \boxed{\pi \left(\frac{\pi}{2} - 1 \right)}.$$

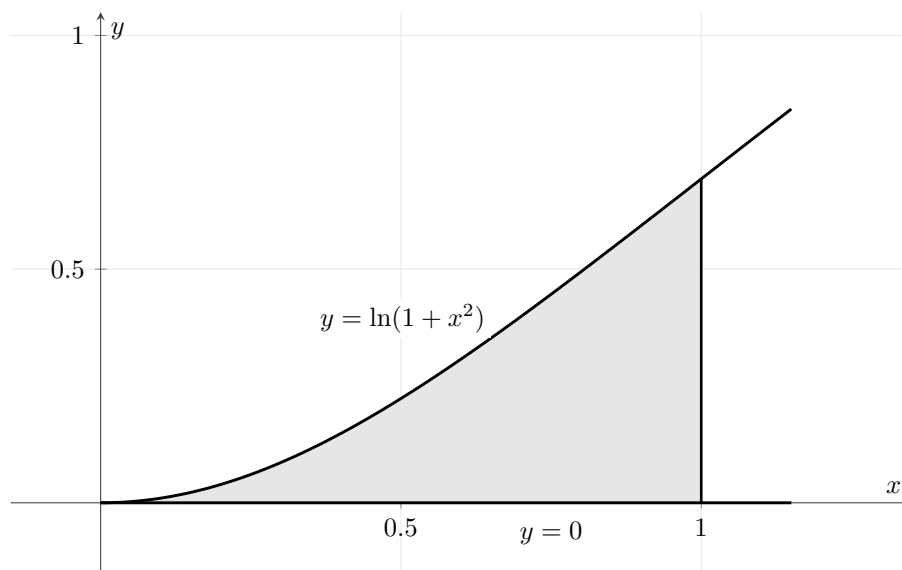
All volumes are in cubic units.

Question 10

Find the volume of the solid obtained by rotating the region bounded by

$$y = \ln(1 + x^2) \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = 1$, about the y -axis.



Rotate the shaded region about the y -axis.

Question 10 – Solution**1. Identify shell dimensions and set up the integral.**

Use vertical shells about the y -axis, with $0 \leq x \leq 1$:

$$r(x) = x, \quad h(x) = \ln(1 + x^2).$$

$$V = 2\pi \int_0^1 x \ln(1 + x^2) dx.$$

2. Substitute and change the bounds.

Let $u = 1 + x^2$. Then $du = 2x dx$, so $x dx = du/2$. The bounds change as follows:

$$x = 0 \Rightarrow u = 1, \quad x = 1 \Rightarrow u = 2.$$

3. Rewrite the volume in terms of u .

$$V = 2\pi \cdot \frac{1}{2} \int_1^2 \ln u du = \pi \int_1^2 \ln u du.$$

4. Integrate the logarithm by parts.

Choose $a = \ln u$ and $db = du$, giving $da = du/u$ and $b = u$:

$$\int \ln u du = u \ln u - \int u \frac{1}{u} du = u \ln u - u.$$

5. Evaluate the endpoints.

$$\begin{aligned} V &= \pi [u \ln u - u]_1^2 \\ &= \pi [(2 \ln 2 - 2) - (\ln 1 - 1)] \\ &= \pi(2 \ln 2 - 1) = \boxed{2\pi \left(\ln 2 - \frac{1}{2} \right)}. \end{aligned}$$

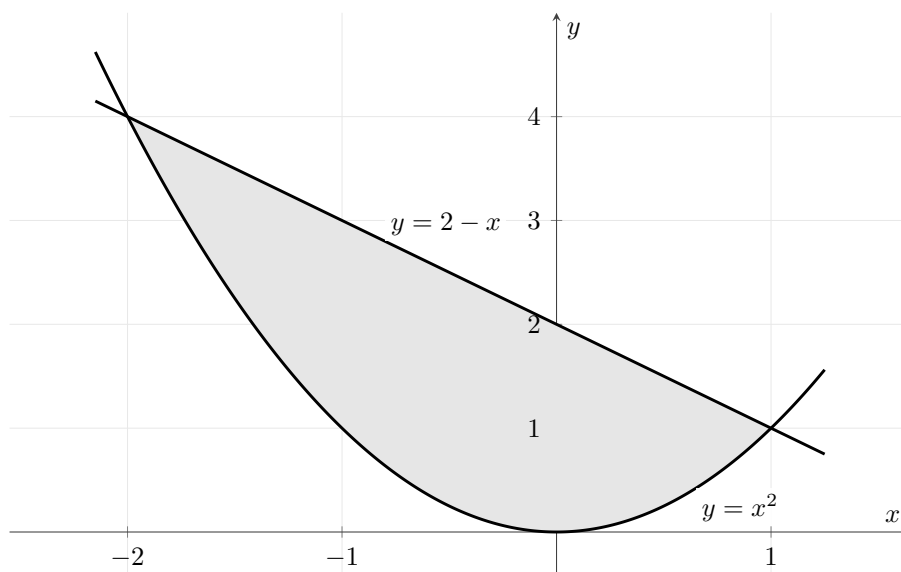
All volumes are in cubic units.

Question 1

Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = 2 - x,$$

about the x -axis.



Rotate the shaded region about the x -axis.

Question 1 – Solution**1. Find the intersections.**

$$x^2 = 2 - x \implies x^2 + x - 2 = (x + 2)(x - 1) = 0.$$

Thus $x = -2$ and $x = 1$, giving intersection points $(-2, 4)$ and $(1, 1)$.

2. Choose washers and identify the radii.

Use vertical slices perpendicular to the x -axis. On $[-2, 1]$, $0 \leq x^2 \leq 2 - x$, so

$$R(x) = 2 - x, \quad r(x) = x^2.$$

3. Set up the volume and expand.

$$V = \pi \int_{-2}^1 (R^2 - r^2) dx = \pi \int_{-2}^1 ((2 - x)^2 - x^4) dx.$$
$$(2 - x)^2 - x^4 = 4 - 4x + x^2 - x^4.$$

4. Find an antiderivative.

Using the power rule for each term,

$$F(x) = 4x - 2x^2 + \frac{x^3}{3} - \frac{x^5}{5}, \quad V = \pi[F(x)]_{-2}^1.$$

5. Evaluate the endpoints and subtract.

$$F(1) = 4 - 2 + \frac{1}{3} - \frac{1}{5} = \frac{32}{15}, \quad F(-2) = -8 - 8 - \frac{8}{3} + \frac{32}{5} = -\frac{184}{15}.$$

$$V = \pi \left(\frac{32}{15} + \frac{184}{15} \right) = \boxed{\frac{72\pi}{5}}.$$

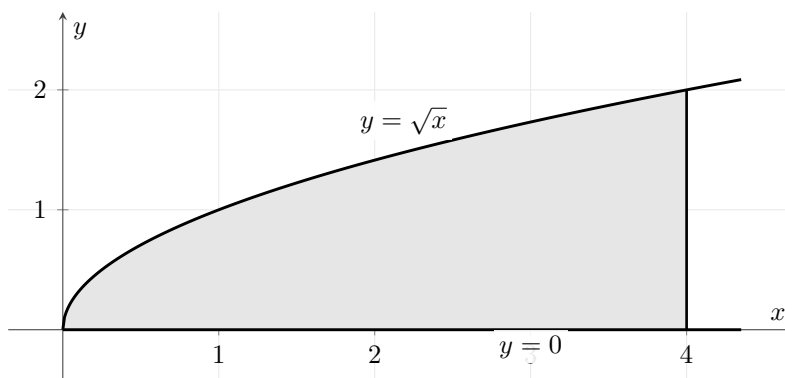
All volumes are in cubic units.

Question 2

The base of a solid is the region bounded by

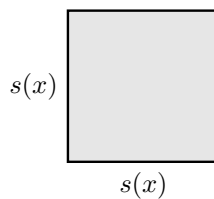
$$y = \sqrt{x} \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = 4$. Cross-sections perpendicular to the x -axis are squares. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 2 – Solution**1. Choose slices and bounds.**

The cross-sections are perpendicular to the x -axis, so use slices of thickness dx on the given interval $0 \leq x \leq 4$.

2. Find the square side length.

The vertical segment in the base is the side of the square:

$$s(x) = y_{\text{top}} - y_{\text{bottom}} = \sqrt{x} - 0 = \sqrt{x}.$$

3. Write the cross-sectional area.

For a square, area is side squared:

$$A(x) = s(x)^2 = (\sqrt{x})^2 = x.$$

4. Set up and integrate the volume.

Add the areas of the thin cross-sections:

$$V = \int_0^4 A(x) dx = \int_0^4 x dx = \left[\frac{x^2}{2} \right]_0^4.$$

5. Evaluate both bounds.

$$V = \frac{4^2}{2} - \frac{0^2}{2} = \frac{16}{2} - 0 = \boxed{8}.$$

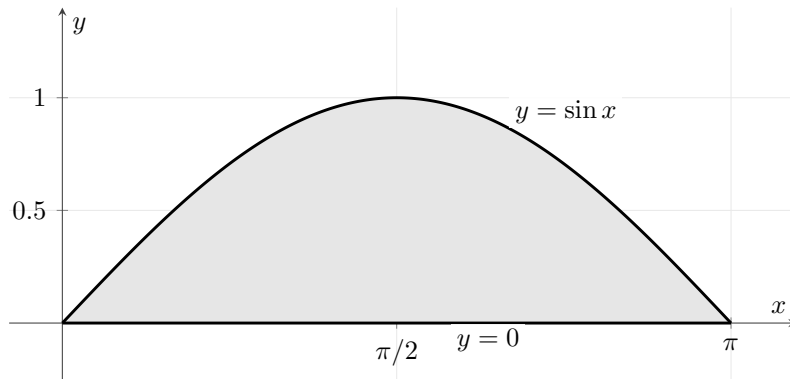
All volumes are in cubic units.

Question 3

The base of a solid is the region enclosed by

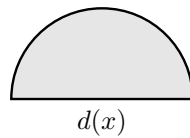
$$y = \sin x \quad \text{and} \quad y = 0$$

from $x = 0$ to $x = \pi$. Cross-sections perpendicular to the x -axis are semicircles whose diameters lie in the base region. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 3 – Solution**1. Identify the diameter and radius.**

Use dx slices for $0 \leq x \leq \pi$. The base segment is a diameter, so

$$d(x) = \sin x - 0 = \sin x, \quad r(x) = \frac{d(x)}{2} = \frac{\sin x}{2}.$$

2. Find the area of a semicircle.

$$A(x) = \frac{1}{2}\pi r(x)^2 = \frac{1}{2}\pi \left(\frac{\sin x}{2}\right)^2 = \frac{\pi}{8} \sin^2 x.$$

3. Set up the volume and use a trigonometric identity.

$$V = \frac{\pi}{8} \int_0^\pi \sin^2 x \, dx, \quad \sin^2 x = \frac{1 - \cos(2x)}{2}.$$
$$V = \frac{\pi}{16} \int_0^\pi (1 - \cos(2x)) \, dx.$$

4. Integrate with the chain-rule factor.

Since $\int \cos(2x) \, dx = \sin(2x)/2$,

$$V = \frac{\pi}{16} \left[x - \frac{\sin(2x)}{2} \right]_0^\pi.$$

5. Apply the bounds.

$$V = \frac{\pi}{16} \left[\left(\pi - \frac{\sin(2\pi)}{2} \right) - \left(0 - \frac{\sin 0}{2} \right) \right] = \boxed{\frac{\pi^2}{16}}.$$

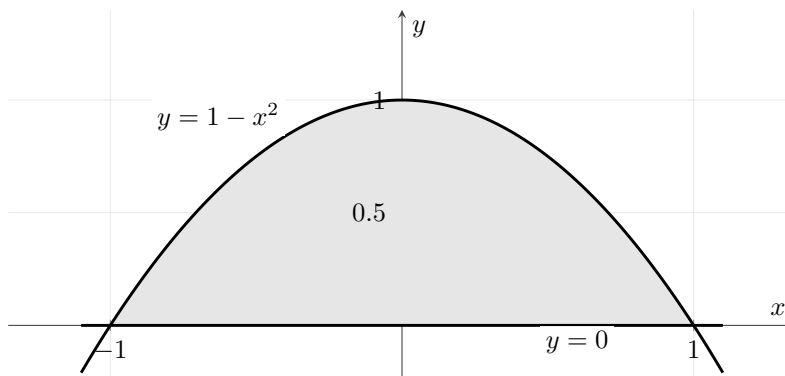
All volumes are in cubic units.

Question 4

The base of a solid is the region enclosed by

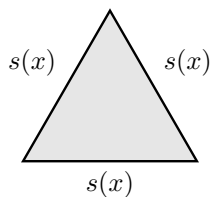
$$y = 1 - x^2 \quad \text{and} \quad y = 0$$

between their intersection points. Cross-sections perpendicular to the x -axis are equilateral triangles. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 4 – Solution**1. Find the bounds and side length.**

The intersections satisfy $1 - x^2 = 0$, so $x = -1, 1$. A vertical base segment gives

$$s(x) = 1 - x^2 \quad (-1 \leq x \leq 1).$$

2. Find the area of an equilateral triangle.

The altitude is $\sqrt{s^2 - (s/2)^2} = \sqrt{3} s/2$. Hence

$$A = \frac{1}{2}s \left(\frac{\sqrt{3}}{2}s \right) = \frac{\sqrt{3}}{4}s^2, \quad A(x) = \frac{\sqrt{3}}{4}(1 - x^2)^2.$$

3. Set up the volume and expand.

$$V = \frac{\sqrt{3}}{4} \int_{-1}^1 (1 - x^2)^2 dx = \frac{\sqrt{3}}{4} \int_{-1}^1 (1 - 2x^2 + x^4) dx.$$

4. Use symmetry and integrate.

The integrand is even, so

$$V = \frac{\sqrt{3}}{2} \int_0^1 (1 - 2x^2 + x^4) dx = \frac{\sqrt{3}}{2} \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_0^1.$$

5. Evaluate and simplify.

$$V = \frac{\sqrt{3}}{2} \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{\sqrt{3}}{2} \left(\frac{15 - 10 + 3}{15} \right) = \boxed{\frac{4\sqrt{3}}{15}}.$$

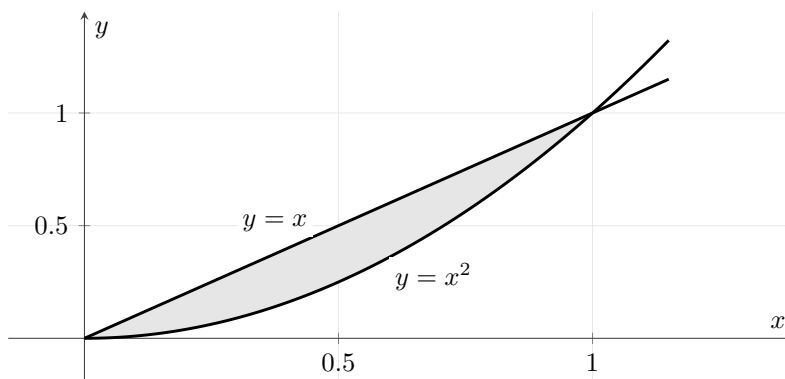
All volumes are in cubic units.

Question 5

The base of a solid is the region bounded by

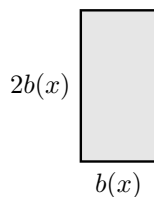
$$y = x \quad \text{and} \quad y = x^2$$

between their intersection points. Cross-sections perpendicular to the x -axis are rectangles whose height is *twice* the base in the region. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 5 – Solution

1. Find the interval and upper curve.

$$x = x^2 \implies x(x - 1) = 0 \implies x = 0, 1.$$

On $[0, 1]$, $x \geq x^2$, so the line lies above the parabola.

2. Identify the rectangle dimensions.

The base is the vertical distance between curves; its height is twice that distance:

$$b(x) = x - x^2, \quad h(x) = 2b(x) = 2(x - x^2).$$

3. Find the area and expand.

$$A(x) = b(x)h(x) = 2(x - x^2)^2 = 2x^2 - 4x^3 + 2x^4.$$

4. Set up and integrate the volume.

$$V = \int_0^1 A(x) dx = \left[\frac{2}{3}x^3 - x^4 + \frac{2}{5}x^5 \right]_0^1.$$

Each term follows from $\int x^n dx = x^{n+1}/(n+1)$.

5. Evaluate and combine the fractions.

$$V = \left(\frac{2}{3} - 1 + \frac{2}{5} \right) - 0 = \frac{10 - 15 + 6}{15} = \boxed{\frac{1}{15}}.$$

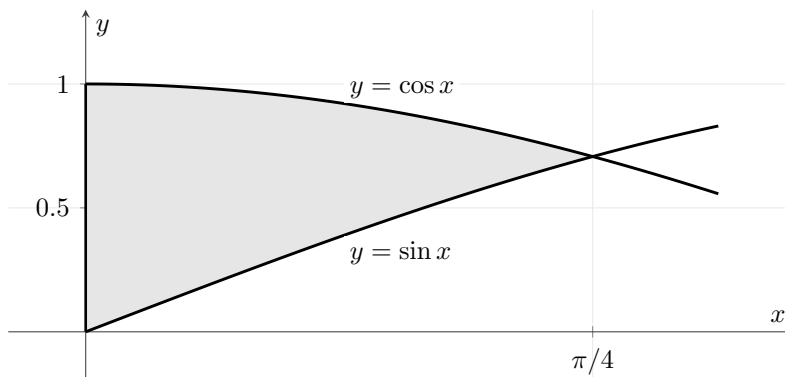
All volumes are in cubic units.

Question 6

The base of a solid is the region bounded by

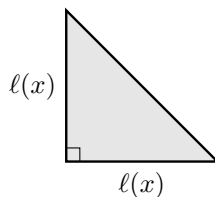
$$y = \cos x \quad \text{and} \quad y = \sin x$$

on the interval where $\cos x \geq \sin x$ in the first quadrant. Cross-sections perpendicular to the x -axis are isosceles right triangles with one leg lying in the base region. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 6 – Solution**1. Determine the bounds.**

In the first quadrant, $\cos x = \sin x$ at $x = \pi/4$. The specified order $\cos x \geq \sin x$ holds for $0 \leq x \leq \pi/4$.

2. Identify the triangle leg and area.

The segment in the base is a leg, not the hypotenuse. Both perpendicular legs have length

$$\ell(x) = \cos x - \sin x, \quad A(x) = \frac{1}{2}\ell(x)^2.$$

3. Set up the volume and simplify.

$$V = \frac{1}{2} \int_0^{\pi/4} (\cos x - \sin x)^2 dx.$$
$$(\cos x - \sin x)^2 = \cos^2 x + \sin^2 x - 2 \sin x \cos x = 1 - \sin(2x).$$

4. Integrate each term.

$$V = \frac{1}{2} \int_0^{\pi/4} (1 - \sin(2x)) dx = \frac{1}{2} \left[x + \frac{1}{2} \cos(2x) \right]_0^{\pi/4}.$$

5. Apply the bounds.

$$V = \frac{1}{2} \left[\left(\frac{\pi}{4} + \frac{1}{2} \cos \frac{\pi}{2} \right) - \left(0 + \frac{1}{2} \cos 0 \right) \right] = \boxed{\frac{\pi}{8} - \frac{1}{4}}.$$

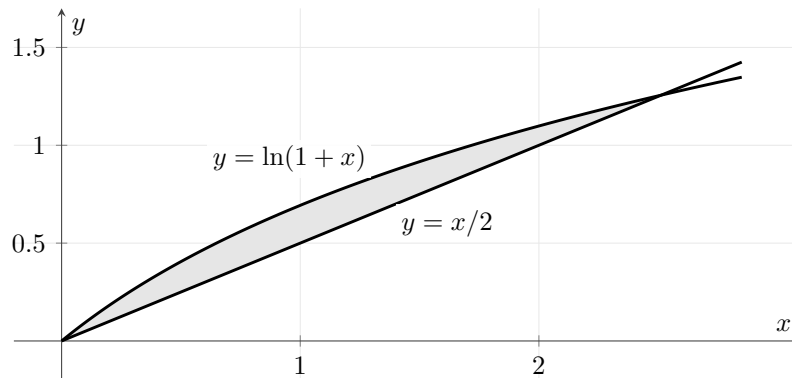
All volumes are in cubic units.

Question 7

The base of a solid is the region bounded by

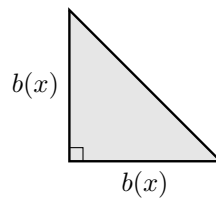
$$y = \ln(x + 1) \quad \text{and} \quad y = \frac{x}{2},$$

between their points of intersection. Cross-sections perpendicular to the x -axis are right triangles whose height is equal to the base. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 7 – Solution

1. Define the intersection bounds.

Let $b(x) = \ln(1+x) - x/2$. Since $b'(x) = (1-x)/(2(1+x))$, b increases to $x = 1$ and then decreases. Its zeros are 0 and a unique $a > 1$ with

$$\ln(1+a) = a/2, \quad a \approx 2.512862417.$$

This equation defines a exactly; $b(x) > 0$ on $(0, a)$.

2. Find the triangular area and set up the volume.

Both perpendicular legs have length $b(x)$, so

$$V = \frac{1}{2} \int_0^a \left(\ln(1+x) - \frac{x}{2} \right)^2 dx.$$

3. Expand and find the needed antiderivatives.

Write $L = \ln(1+x)$ and $u = 1+x$. The square is $L^2 - xL + x^2/4$. Integration by parts gives

$$\int L^2 dx = u(L^2 - 2L + 2), \quad \int xL dx = \frac{x^2}{2}L - \frac{x^2}{4} + \frac{x}{2} - \frac{L}{2}.$$

For the first identity, use $\int \ln^2 u du = u \ln^2 u - 2 \int \ln u du$.

4. Combine the terms and apply the endpoints.

An antiderivative of the squared expression is

$$H(x) = uL^2 - \left(u + \frac{u^2}{2}\right)L + u + \frac{u^2}{4} + \frac{x^3}{12}.$$

At $x = 0$, $H(0) = 5/4$. Substituting $u = 1+a$ and $L = a/2$ gives

$$H(a) - H(0) = \frac{a^3}{12} - \frac{a^2}{2} + \frac{3a}{4} = \frac{a(a-3)^2}{12}.$$

5. Include the triangular-area factor.

$$V = \frac{1}{2}[H(a) - H(0)] = \boxed{\frac{a(a-3)^2}{24}} \approx 0.024846244.$$

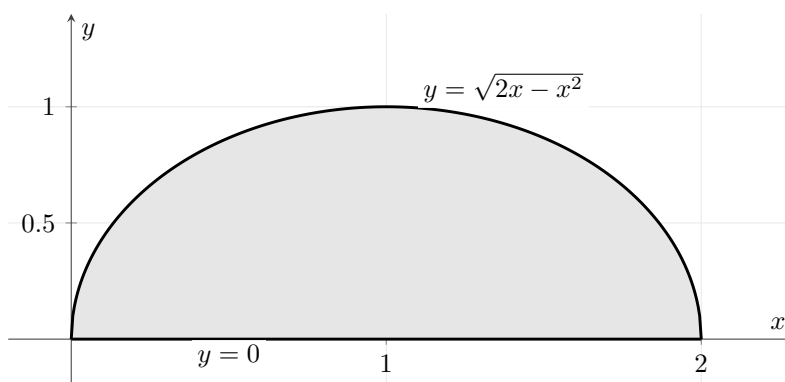
All volumes are in cubic units.

Question 8

The base of a solid is the region bounded by

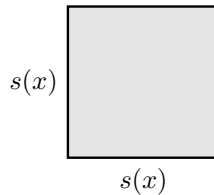
$$y = \sqrt{2x - x^2} \quad \text{and} \quad y = 0.$$

Cross-sections perpendicular to the x -axis are squares. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 8 – Solution

1. Determine the bounds.

The square root is real when $2x - x^2 = x(2 - x) \geq 0$, giving $0 \leq x \leq 2$. These endpoints are also the intersections with $y = 0$.

2. Find the square side length.

The cross-sections are perpendicular to the x -axis. Their side length is the vertical segment in the base:

$$s(x) = \sqrt{2x - x^2} - 0.$$

3. Find the cross-sectional area.

$$A(x) = s(x)^2 = \left(\sqrt{2x - x^2}\right)^2 = 2x - x^2.$$

4. Set up and integrate the volume.

$$V = \int_0^2 A(x) dx = \int_0^2 (2x - x^2) dx = \left[x^2 - \frac{x^3}{3} \right]_0^2.$$

5. Evaluate and simplify.

$$V = \left(4 - \frac{8}{3} \right) - 0 = \frac{12 - 8}{3} = \boxed{\frac{4}{3}}.$$

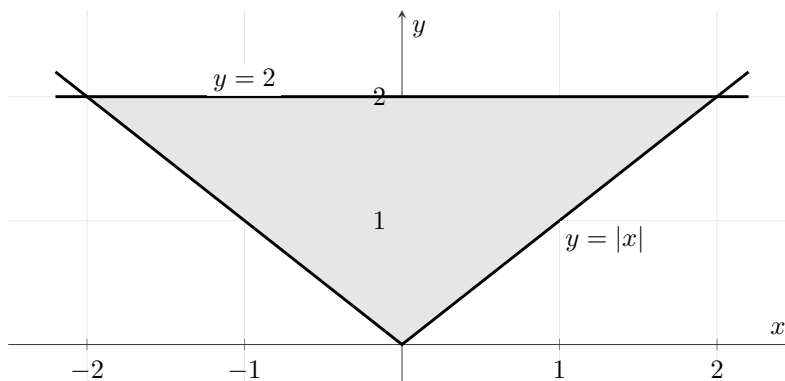
All volumes are in cubic units.

Question 9

The base of a solid is the region bounded by

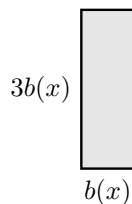
$$y = |x| \quad \text{and} \quad y = 2,$$

between their intersection points. Cross-sections perpendicular to the x -axis are rectangles whose height is three times their base. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 9 – Solution**1. Find the bounds.**

The intersections satisfy $|x| = 2$, giving $x = -2$ and $x = 2$. Use dx slices because the cross-sections are perpendicular to the x -axis.

2. Identify the rectangle dimensions.

The base length is top minus bottom, and the height is three times the base:

$$b(x) = 2 - |x|, \quad h(x) = 3(2 - |x|).$$

3. Set up the area and volume.

$$A(x) = b(x)h(x) = 3(2 - |x|)^2, \quad V = 3 \int_{-2}^2 (2 - |x|)^2 dx.$$

4. Use symmetry and integrate.

The integrand is even; on $[0, 2]$, $|x| = x$. Thus

$$V = 6 \int_0^2 (2 - x)^2 dx = 6 \int_0^2 (4 - 4x + x^2) dx.$$

$$V = 6 \left[4x - 2x^2 + \frac{x^3}{3} \right]_0^2.$$

5. Apply the bounds.

$$V = 6 \left[\left(8 - 8 + \frac{8}{3} \right) - 0 \right] = 6 \cdot \frac{8}{3} = \boxed{16}.$$

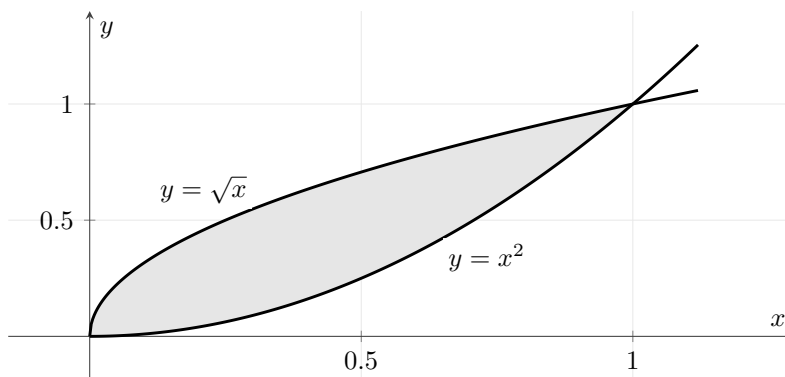
All volumes are in cubic units.

Question 10

The base of a solid is the region bounded by

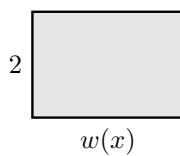
$$y = \sqrt{x} \quad \text{and} \quad y = x^2,$$

between their points of intersection. Cross-sections parallel to the y -axis are rectangles whose width is equal to the distance between the curves and whose height is constant and equal to 2. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 10 – Solution

1. Interpret the slice direction and find the bounds.

The base segments are parallel to the y -axis, so they are vertical; integrate with respect to x . Solving $\sqrt{x} = x^2$ gives $x = x^4$, or $x(x^3 - 1) = 0$. The real intersections occur at $x = 0, 1$.

2. Identify width and height.

For $0 \leq x \leq 1$, $\sqrt{x} \geq x^2$. Thus each rectangle has

$$w(x) = \sqrt{x} - x^2, \quad h = 2.$$

The height of 2 extends out of the base plane.

3. Find the area and set up the integral.

$$A(x) = 2(\sqrt{x} - x^2), \quad V = \int_0^1 A(x) dx = 2 \int_0^1 (x^{1/2} - x^2) dx.$$

4. Integrate using the power rule.

$$\int x^{1/2} dx = \frac{2}{3}x^{3/2}, \quad \int x^2 dx = \frac{x^3}{3}.$$
$$V = 2 \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3} \right]_0^1.$$

5. Apply the bounds and simplify.

$$V = 2 \left[\left(\frac{2}{3} - \frac{1}{3} \right) - 0 \right] = \boxed{\frac{2}{3}}.$$

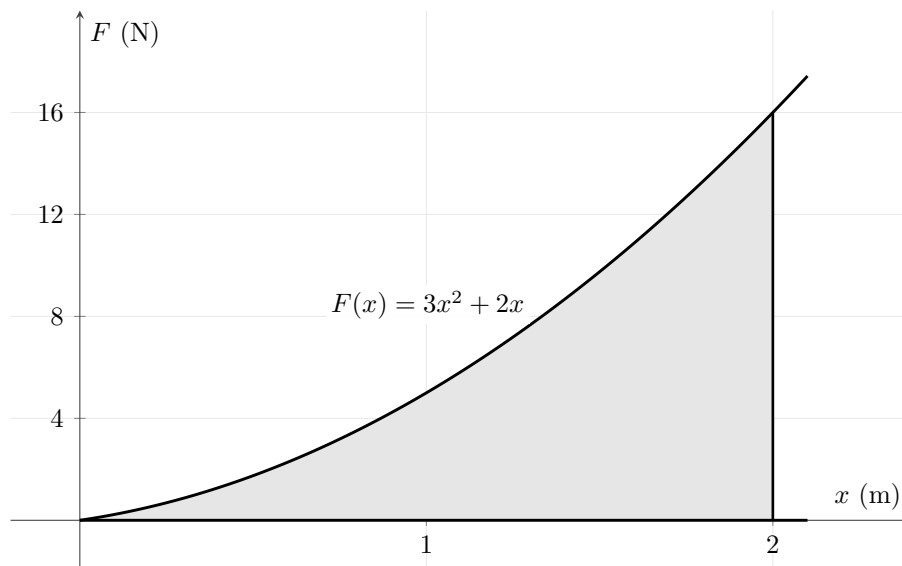
All volumes are in cubic units.

Question 1

A force acting along a straight line is given by

$$F(x) = 3x^2 + 2x \quad (\text{newtons}),$$

where x is the displacement in meters from the origin. Find the work done by the force in moving an object from $x = 0$ to $x = 2$.



The shaded area represents work over the specified interval.

Question 1 – Solution

1. Identify the force and interval.

The force acts along the displacement and is $F(x) = 3x^2 + 2x$ newtons. The object moves from $x = 0$ to $x = 2$ meters.

2. Write the work integral.

Over a small displacement dx , $dW = F(x) dx$. Therefore,

$$W = \int_0^2 F(x) dx = \int_0^2 (3x^2 + 2x) dx.$$

3. Integrate term by term.

By the power rule,

$$\int 3x^2 dx = 3\frac{x^3}{3} = x^3, \quad \int 2x dx = 2\frac{x^2}{2} = x^2.$$

4. Apply the bounds.

$$W = [x^3 + x^2]_0^2 = (2^3 + 2^2) - (0^3 + 0^2) = 8 + 4 = 12.$$

5. State the result with units.

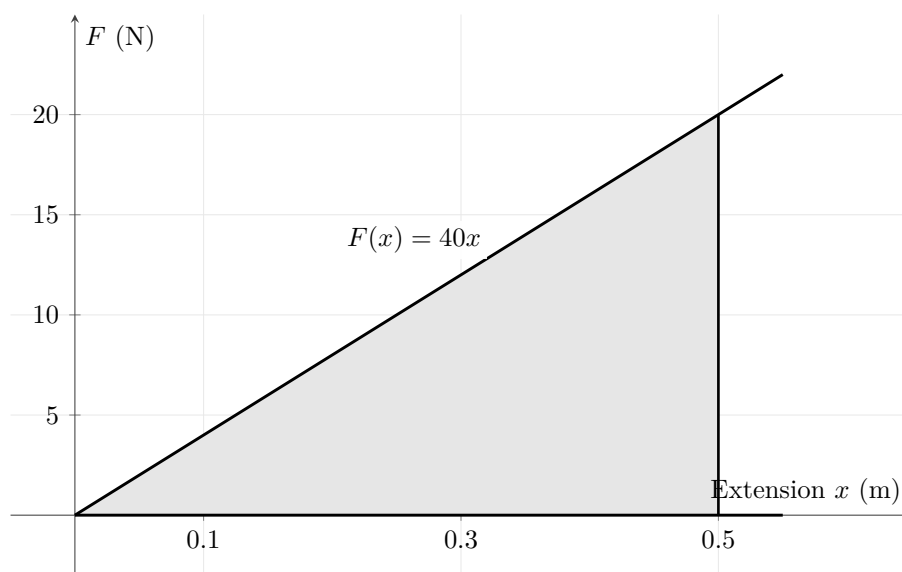
$$\boxed{W = 12 \text{ J}}.$$

A newton times a meter is a joule. Since $F(x) \geq 0$ on $[0, 2]$, the work is the positive shaded area under the force curve.

Question 2

A spring has natural length 0.5 m. A force of 12 N is required to stretch the spring to a length of 0.8 m.

Find the work required to stretch the spring from its natural length to a length of 1.0 m.



The shaded area represents work over the specified interval.

Question 2 – Solution

1. Use extension beyond natural length.

Let x be extension, not total spring length. The calibration force produces

$$x = 0.8 - 0.5 = 0.3 \text{ m.}$$

2. Find the spring constant.

Hooke's law for the required stretching force is $F = kx$:

$$12 = k(0.3) \implies k = \frac{12}{0.3} = 40 \text{ N/m.}$$

3. Translate the requested lengths into bounds.

The initial and final extensions are

$$x_0 = 0.5 - 0.5 = 0, \quad x_1 = 1.0 - 0.5 = 0.5 \text{ m.}$$

Thus $F(x) = 40x$ and $W = \int_0^{0.5} 40x \, dx$.

4. Integrate and evaluate.

$$W = [20x^2]_0^{0.5} = 20(0.5)^2 - 0 = 20(0.25) = 5.$$

5. State and check the work.

$$\boxed{W = 5 \text{ J}}.$$

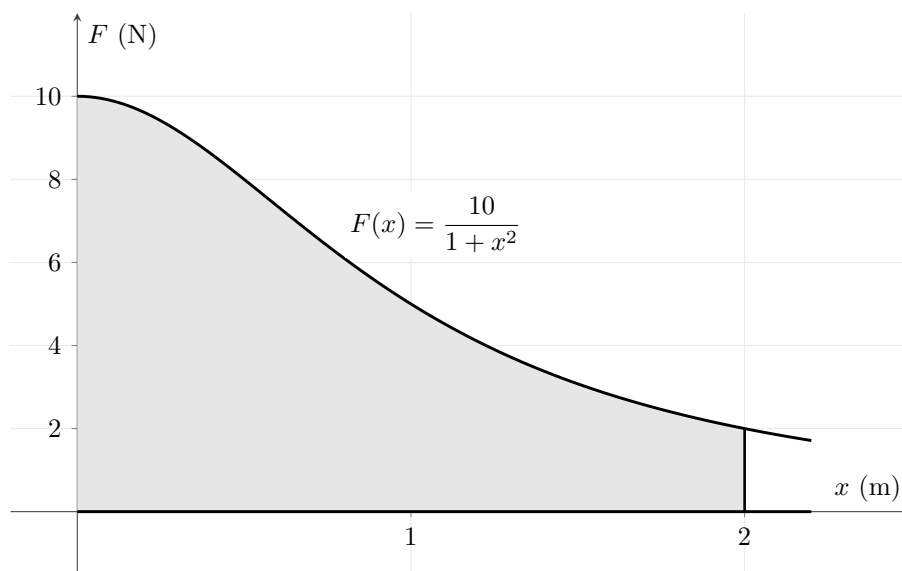
The force rises linearly from 0 to 20 N, so the triangular area is also $\frac{1}{2}(0.5)(20) = 5 \text{ J}$.

Question 3

A force acting on an object is given by

$$F(x) = \frac{10}{1+x^2} \quad (\text{newtons}),$$

where x is the distance (in meters) the object has moved along a straight line. Find the work done in moving the object from $x = 0$ to $x = 2$.



The shaded area represents work over the specified interval.

Question 3 – Solution

1. Identify the work model.

The force is positive over $0 \leq x \leq 2$, so the work equals the area under the force curve:

$$W = \int_0^2 F(x) dx.$$

2. Substitute the force.

$$W = \int_0^2 \frac{10}{1+x^2} dx = 10 \int_0^2 \frac{1}{1+x^2} dx.$$

3. Use the inverse-tangent antiderivative.

Since $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$,

$$W = 10 [\arctan x]_0^2.$$

Inverse-trigonometric values are in radians.

4. Apply the endpoints.

$$W = 10(\arctan 2 - \arctan 0) = 10 \arctan 2.$$

5. State the exact and approximate work.

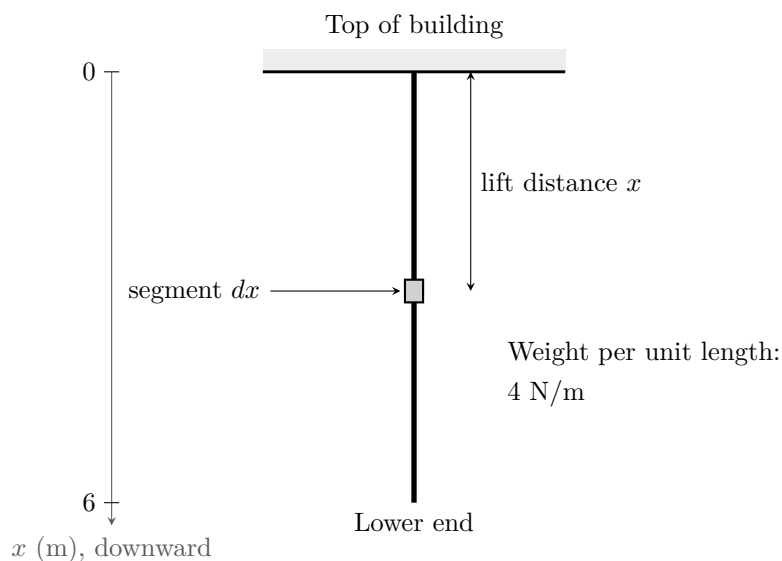
$$\boxed{W = 10 \arctan 2 \text{ J}} \approx 11.0715 \text{ J}.$$

The force decreases from 10 N to 2 N; the resulting work lies between $2(2) = 4 \text{ J}$ and $10(2) = 20 \text{ J}$.

Question 4

A 6-meter cable hangs vertically from the top of a building. The cable has constant linear weight density 4 N/m .

Find the work required to lift the entire cable to the top of the building.



Initial configuration; x is measured down from the top.

Question 4 – Solution

1. Choose a coordinate for each cable segment.

Let x measure the initial distance downward from the top of the building. The cable occupies $0 \leq x \leq 6$ meters.

2. Find the weight of a small segment.

The given quantity is weight per unit length. A segment of length dx weighs

$$dF = (4 \text{ N/m}) dx.$$

No additional factor of gravity is needed.

3. Find its lifting distance and work.

The segment initially at depth x must rise x meters. Thus

$$dW = (\text{weight})(\text{distance}) = 4x dx.$$

4. Add the work for all segments.

$$W = \int_0^6 4x dx = [2x^2]_0^6 = 2(36) - 0 = 72.$$

5. State and check the result.

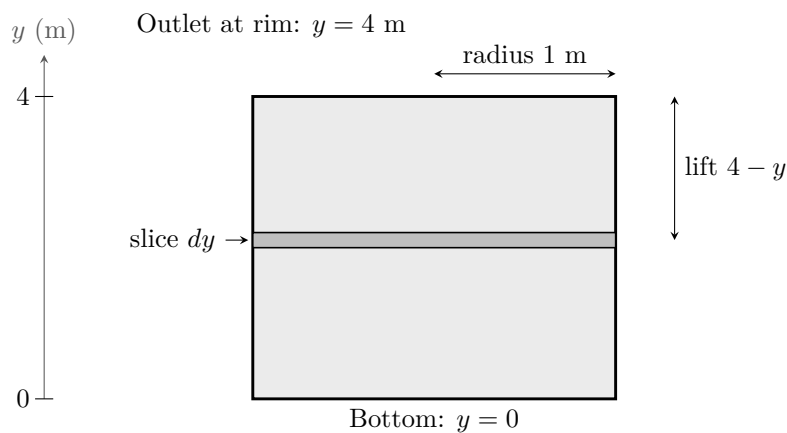
$$\boxed{W = 72 \text{ J}}.$$

The cable weighs $4(6) = 24$ N, and its center of mass rises 3 m. This gives the same work: $24(3) = 72$ J.

Question 5

A cylindrical tank of radius 1 m and height 4 m is full of water. Water weighs 9800 N/m^3 .

Find the work required to pump all the water to the top of the tank.



Vertical section through the tank; the shaded band represents a horizontal slice.

Question 5 – Solution

1. Choose horizontal slices.

Let y be height above the tank bottom, with $0 \leq y \leq 4$. A slice at height y has thickness dy .

2. Find the slice volume and weight.

The horizontal circular area is $A = \pi(1)^2 = \pi \text{ m}^2$. Hence

$$dV = \pi dy, \quad dF = 9800 dV = 9800\pi dy.$$

The density is already a weight density.

3. Find the distance to the outlet.

The outlet is at the top, $y = 4$, so the slice must rise $4 - y$ meters:

$$dW = 9800\pi(4 - y) dy.$$

4. Set up and evaluate the work integral.

$$W = 9800\pi \int_0^4 (4 - y) dy = 9800\pi \left[4y - \frac{y^2}{2} \right]_0^4.$$
$$W = 9800\pi(16 - 8) = 78400\pi.$$

5. State and check the work.

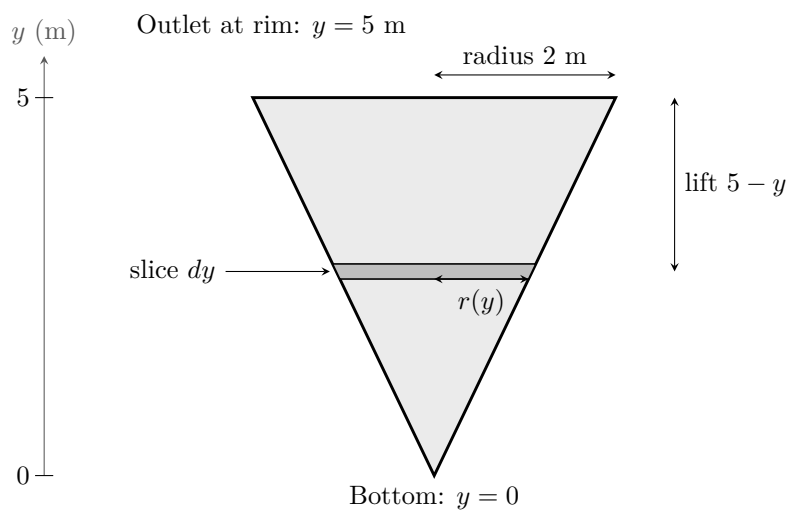
$$\boxed{W = 78400\pi \text{ J}}.$$

The total water weight is $9800(4\pi) = 39200\pi \text{ N}$. Its average lifting distance is 2 m, giving $78400\pi \text{ J}$.

Question 6

A conical tank with its vertex at the bottom, height 5 m, and top radius 2 m is full of water. Water weighs 9800 N/m^3 .

Find the work required to pump all the water to the top of the tank.



Vertical section through the tank; the shaded band represents a horizontal slice.

Question 6 – Solution

1. Choose a height coordinate and determine the radius.

Measure y upward from the bottom vertex, so $0 \leq y \leq 5$. Similar triangles give

$$\frac{r(y)}{y} = \frac{2}{5} \implies r(y) = \frac{2y}{5}.$$

2. Find the volume and weight of a slice.

$$dV = \pi r(y)^2 dy = \frac{4\pi}{25} y^2 dy, \quad dF = 9800 dV = \frac{39200\pi}{25} y^2 dy.$$

3. Include the lifting distance.

The outlet is at $y = 5$, so a slice at y rises $5 - y$ meters:

$$W = \frac{39200\pi}{25} \int_0^5 y^2 (5 - y) dy.$$

4. Expand and integrate.

$$y^2(5 - y) = 5y^2 - y^3, \quad W = \frac{39200\pi}{25} \left[\frac{5}{3} y^3 - \frac{y^4}{4} \right]_0^5.$$

5. Apply the bounds and simplify.

$$\begin{aligned} W &= \frac{39200\pi}{25} \left(\frac{625}{3} - \frac{625}{4} \right) \\ &= \frac{39200\pi}{25} \cdot \frac{625}{12} = \boxed{\frac{245000\pi}{3} \text{ J}}. \end{aligned}$$

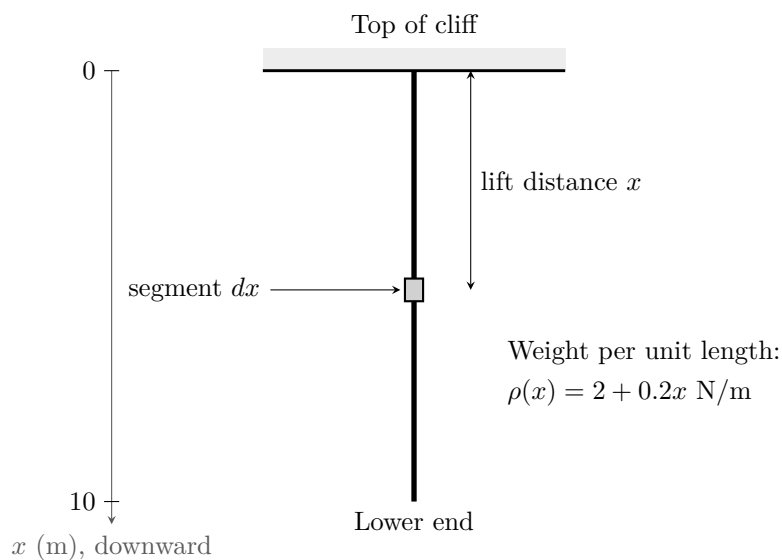
Question 7

A 10-meter rope hangs vertically from the edge of a cliff. The rope has variable linear weight density given by

$$\rho(x) = 2 + 0.2x \quad \text{N/m},$$

where x is the distance (in meters) measured downward from the top of the cliff.

Find the work required to pull the entire rope to the top of the cliff.



Initial configuration; x is measured down from the top.

Question 7 – Solution

1. Label each segment by its initial depth.

Let x be distance downward from the cliff top, as in the question. The rope occupies $0 \leq x \leq 10$ meters; $\rho(x)$ describes the initial weight distribution.

2. Find the segment weight.

For a short piece of length dx ,

$$dF = \rho(x) dx = (2 + 0.2x) dx.$$

The units are newtons because ρ is given in N/m.

3. Multiply by the distance lifted.

The segment at depth x is lifted x meters:

$$dW = x(2 + 0.2x) dx, \quad W = \int_0^{10} x(2 + 0.2x) dx.$$

4. Expand and integrate.

$$W = \int_0^{10} \left(2x + \frac{x^2}{5} \right) dx = \left[x^2 + \frac{x^3}{15} \right]_0^{10}.$$

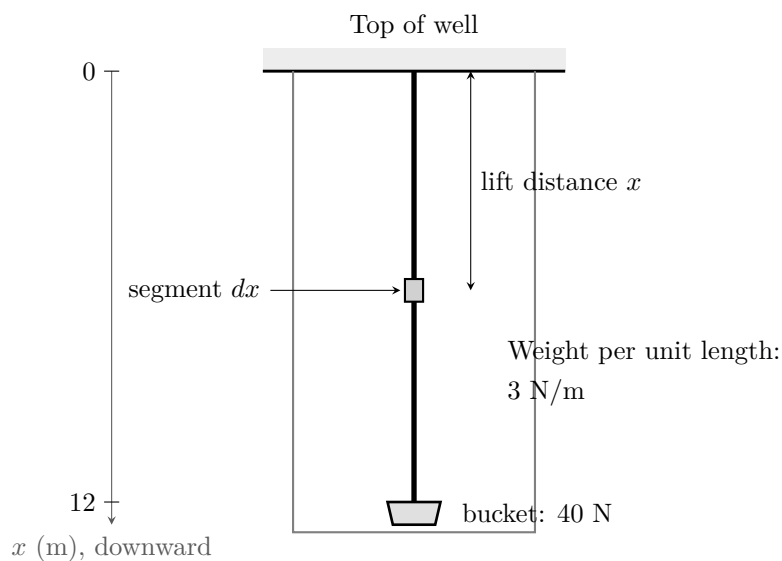
5. Evaluate and state the work.

$$W = 100 + \frac{1000}{15} = 100 + \frac{200}{3} = \boxed{\frac{500}{3} \text{ J}} \approx 166.667 \text{ J}.$$

Question 8

A bucket is lifted from the bottom of a well that is 12 m deep. The bucket itself weighs 40 N. Attached to the bucket is a rope that weighs 3 N/m.

Find the work required to lift the bucket to the top of the well.



Initial configuration; x is measured down from the top.

Question 8 – Solution

1. Separate the bucket and rope contributions.

The bucket has constant weight, while different pieces of rope rise different distances:

$$W_{\text{total}} = W_{\text{bucket}} + W_{\text{rope}}.$$

2. Calculate the bucket work.

The 40 N bucket rises the full 12 m:

$$W_{\text{bucket}} = 40(12) = 480 \text{ J}.$$

3. Define a consistent coordinate for the rope.

Let x be the initial depth below the top of the well, with $0 \leq x \leq 12$. A segment of length dx weighs $3 dx$ newtons and rises x meters.

4. Integrate the rope work.

$$W_{\text{rope}} = \int_0^{12} 3x \, dx = \left[\frac{3}{2}x^2 \right]_0^{12} = \frac{3}{2}(144) = 216 \text{ J}.$$

5. Add and check the total.

$$W_{\text{total}} = 480 + 216 = \boxed{696 \text{ J}}.$$

As a check, the rope weighs $3(12) = 36$ N and its center of mass rises 6 m, giving $36(6) = 216$ J.

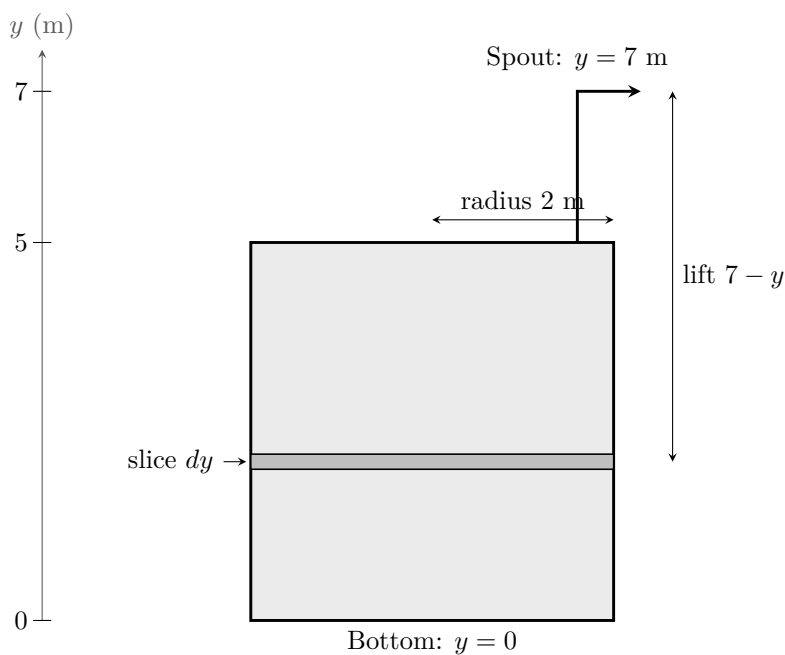
Question 9

A cylindrical tank of radius 2 m and height 5 m is full of oil. The oil has weight density

$$w(y) = 9000(1 + 0.1y) \text{ N/m}^3,$$

where y is the height (in meters) above the bottom of the tank.

Find the work required to pump all the oil to a spout located 2 m above the top of the tank.



Vertical section through the tank; the shaded band represents a horizontal slice.

Question 9 – Solution

1. Locate the slices and outlet.

Let y be height above the bottom. Oil occupies $0 \leq y \leq 5$, and the spout is at height $5 + 2 = 7$ m. The lift distance is $7 - y$.

2. Find the slice volume and weight.

The tank radius is 2 m, so

$$dV = \pi(2)^2 dy = 4\pi dy.$$

$$dF = w(y) dV = 9000(1 + 0.1y)(4\pi) dy = 36000\pi(1 + 0.1y) dy.$$

3. Set up the work integral.

Multiply each slice weight by its lifting distance:

$$W = 36000\pi \int_0^5 (1 + 0.1y)(7 - y) dy.$$

4. Expand and integrate exactly.

$$(1 + 0.1y)(7 - y) = 7 - \frac{3}{10}y - \frac{1}{10}y^2.$$

$$W = 36000\pi \left[7y - \frac{3}{20}y^2 - \frac{1}{30}y^3 \right]_0^5.$$

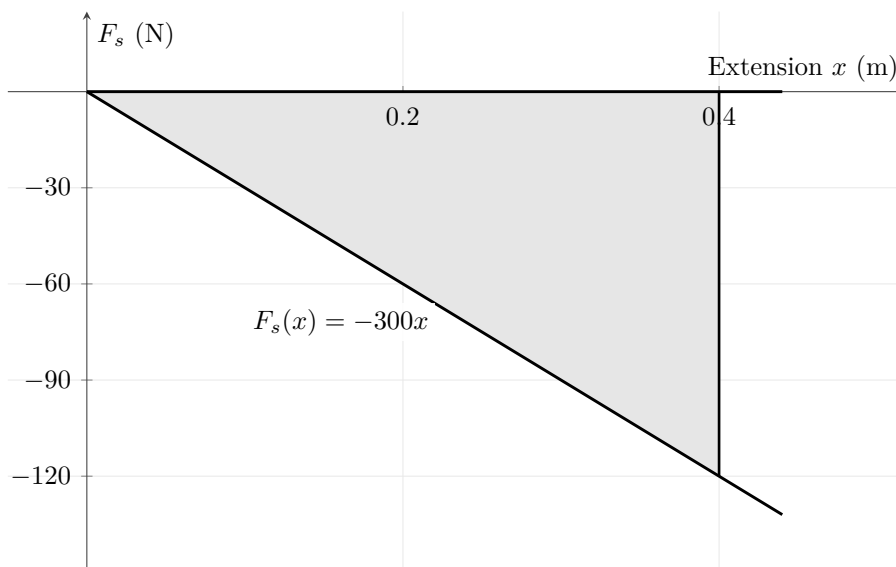
5. Apply the bounds and simplify.

$$\begin{aligned} W &= 36000\pi \left(35 - \frac{15}{4} - \frac{25}{6} \right) \\ &= 36000\pi \left(\frac{420 - 45 - 50}{12} \right) = \boxed{975000\pi \text{ J}}. \end{aligned}$$

Question 10

A vertical spring attached to a ceiling has natural length 1 m and spring constant $k = 300$ N/m. An external agent stretches it downward from length 1 m to 1.4 m.

Find the work done by the spring during this motion.



The shaded area represents negative work done by the spring.

Question 10 – Solution

1. Define displacement from natural length.

Let x be extension measured positively downward. The spring lengths correspond to

$$x_0 = 1 - 1 = 0, \quad x_1 = 1.4 - 1 = 0.4 \text{ m.}$$

2. Use the signed force exerted by the spring.

The spring pulls upward, opposite the positive direction. Therefore,

$$F_s(x) = -kx = -300x \text{ N.}$$

The question asks for work done by the spring itself.

3. Set up the signed work integral.

$$W_s = \int_0^{0.4} F_s(x) dx = \int_0^{0.4} -300x dx.$$

4. Integrate and apply the bounds.

$$W_s = [-150x^2]_0^{0.4} = -150(0.4)^2 = -150(0.16) = -24.$$

5. State the result and interpret its sign.

$$\boxed{W_s = -24 \text{ J.}}$$

The negative sign indicates that the spring force opposes the motion. Its elastic potential energy increases by $\frac{1}{2}k(0.4)^2 = 24 \text{ J}$, consistent with $W_s = -\Delta U$.