

Question 1

Evaluate the indefinite integral

$$\int \left(x^2 e^{x^3} + \frac{1}{x} \sqrt{1 + \ln x} - \frac{3x^2}{1 + x^3} \right) dx.$$

Question 1 - Solution

We integrate term by term.

First term:

$$\int x^2 e^{x^3} dx.$$

Let $u = x^3$, so $du = 3x^2 dx$:

$$\int x^2 e^{x^3} dx = \frac{1}{3} \int e^u du = \frac{1}{3} e^{x^3}.$$

Second term:

$$\int \frac{1}{x} \sqrt{1 + \ln x} dx.$$

Let $u = 1 + \ln x$, so $du = \frac{1}{x} dx$:

$$\int \sqrt{u} du = \frac{2}{3} u^{3/2} = \frac{2}{3} (1 + \ln x)^{3/2}.$$

Third term:

$$\int \frac{3x^2}{1 + x^3} dx.$$

Let $u = 1 + x^3$, so $du = 3x^2 dx$:

$$\int \frac{3x^2}{1 + x^3} dx = \int \frac{du}{u} = \ln(1 + x^3).$$

Final Answer:

$\frac{1}{3} e^{x^3} + \frac{2}{3} (1 + \ln x)^{3/2} - \ln(1 + x^3) + C$
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Question 2

Find the most general antiderivative of

$$\int \left(e^{2x} \sin x + \frac{1}{(1-x)^2} \right) dx.$$

Question 2 - Solution

We evaluate each term separately.

First term:

$$\int e^{2x} \sin x \, dx.$$

Use integration by parts. Let

$$u = \sin x, \quad dv = e^{2x} \, dx.$$

Then

$$du = \cos x \, dx, \quad v = \frac{1}{2}e^{2x}.$$

So,

$$\int e^{2x} \sin x \, dx = \frac{1}{2}e^{2x} \sin x - \frac{1}{2} \int e^{2x} \cos x \, dx.$$

Apply integration by parts again to the remaining integral. Let

$$u = \cos x, \quad dv = e^{2x} \, dx.$$

Then

$$du = -\sin x \, dx, \quad v = \frac{1}{2}e^{2x}.$$

Thus,

$$\int e^{2x} \cos x \, dx = \frac{1}{2}e^{2x} \cos x + \frac{1}{2} \int e^{2x} \sin x \, dx.$$

Substitute this back:

$$\int e^{2x} \sin x \, dx = \frac{1}{2}e^{2x} \sin x - \frac{1}{4}e^{2x} \cos x - \frac{1}{4} \int e^{2x} \sin x \, dx.$$

Add $\frac{1}{4} \int e^{2x} \sin x \, dx$ to both sides:

$$\frac{5}{4} \int e^{2x} \sin x \, dx = \frac{1}{2}e^{2x} \sin x - \frac{1}{4}e^{2x} \cos x.$$

Solve:

$$\int e^{2x} \sin x \, dx = \frac{e^{2x}}{5} (2 \sin x - \cos x).$$

Second term:

$$\int \frac{1}{(1-x)^2} \, dx.$$

Let

$$u = 1 - x, \quad du = -dx.$$

Then

$$\int \frac{1}{(1-x)^2} \, dx = - \int u^{-2} \, du = u^{-1} = \frac{1}{1-x}.$$

Final Answer:

$$\boxed{\frac{e^{2x}}{5} (2 \sin x - \cos x) + \frac{1}{1-x} + C}$$

Question 3

Determine an antiderivative of

$$\int \frac{x^3}{\sqrt{1-x^2}} dx.$$

Question 3 - Solution

For $|x| < 1$, set $u = 1 - x^2$, so $du = -2x \, dx$ and $x^2 = 1 - u$.

$$\int \frac{x^3}{\sqrt{1-x^2}} \, dx = -\frac{1}{2} \int (1-u)u^{-1/2} \, du = -\sqrt{u} + \frac{1}{3}u^{3/2} + C.$$

$$-\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2} + C.$$

Differentiation gives $x/\sqrt{1-x^2} - x\sqrt{1-x^2} = x^3/\sqrt{1-x^2}$.

Question 4

Evaluate the integral

$$\int \left(\sqrt{x} \ln x + \frac{1}{\sqrt{x}} \right) dx.$$

Question 4 - Solution

We first rewrite the integrand using exponents:

$$\sqrt{x} \ln x + \frac{1}{\sqrt{x}} = x^{1/2} \ln x + x^{-1/2}.$$

For the first term, apply integration by parts with

$$u = \ln x, \quad dv = x^{1/2} dx.$$

Then

$$du = \frac{1}{x} dx, \quad v = \frac{2}{3} x^{3/2}.$$

Thus,

$$\int x^{1/2} \ln x dx = \frac{2}{3} x^{3/2} \ln x - \frac{2}{3} \int x^{1/2} dx = \frac{2}{3} x^{3/2} \ln x - \frac{4}{9} x^{3/2}.$$

For the second term,

$$\int x^{-1/2} dx = 2x^{1/2}.$$

Combining results:

$$\boxed{\frac{2}{3} x^{3/2} \ln x - \frac{4}{9} x^{3/2} + 2x^{1/2} + C}$$

Question 5

Find an antiderivative of

$$\int \left(\frac{1}{x} \arctan x + \frac{x^2}{1+x^2} \right) dx.$$

Question 5 - Solution

Split the integral:

$$\int \left(\frac{1}{x} \arctan x + \frac{x^2}{1+x^2} \right) dx = \int \frac{\arctan x}{x} dx + \int \frac{x^2}{1+x^2} dx.$$

1) The rational part

Rewrite:

$$\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}.$$

So

$$\int \frac{x^2}{1+x^2} dx = \int \left(1 - \frac{1}{1+x^2} \right) dx = x - \arctan x.$$

2) The non-elementary part $\int \frac{\arctan x}{x} dx$

A common attempt is integration by parts, which produces

$$\int \frac{\arctan x}{x} dx = \arctan x \ln|x| - \int \frac{\ln|x|}{1+x^2} dx,$$

and then the substitution $x = \tan \theta$ turns the remaining integral into

$$\int \frac{\ln|x|}{1+x^2} dx = \int \ln(\tan \theta) d\theta.$$

The key clarification is:

$\int \ln(\tan \theta) d\theta$ **is not an elementary antiderivative.** So any step claiming

$$\int \ln(\tan \theta) d\theta = \theta \ln(\tan \theta) - \tan \theta + C$$

is incorrect (differentiate it to check).

A correct closed form uses the dilogarithm function Li_2 . Start from

$$\arctan x = \frac{1}{2i} (\ln(1+ix) - \ln(1-ix)).$$

Then

$$\int \frac{\arctan x}{x} dx = \frac{1}{2i} \int \frac{\ln(1+ix) - \ln(1-ix)}{x} dx.$$

Use the identity

$$\frac{d}{dx} \text{Li}_2(u) = -\ln(1-u) \frac{u'}{u},$$

so, for real x ,

$$\frac{d}{dx} \text{Li}_2(-ix) = -\frac{\ln(1+ix)}{x}, \quad \frac{d}{dx} \text{Li}_2(ix) = -\frac{\ln(1-ix)}{x}.$$

Therefore,

$$\int \frac{\ln(1+ix)}{x} dx = -\text{Li}_2(-ix) + C, \quad \int \frac{\ln(1-ix)}{x} dx = -\text{Li}_2(ix) + C,$$

and hence

$$\int \frac{\arctan x}{x} dx = \frac{1}{2i} (\text{Li}_2(ix) - \text{Li}_2(-ix)) + C.$$

Final Answer

Combine both parts:

$$\int \left(\frac{1}{x} \arctan x + \frac{x^2}{1+x^2} \right) dx = (x - \arctan x) + \frac{1}{2i} \left(\operatorname{Li}_2(ix) - \operatorname{Li}_2(-ix) \right) + C.$$

Question 6

Find the general antiderivative of

$$\int \left(x e^{-x^2} + \ln(1+x) \right) dx.$$

Question 6 - Solution

We evaluate each term separately.

First term:

$$\int x e^{-x^2} dx.$$

Let $u = -x^2$, so $du = -2x dx$:

$$\int x e^{-x^2} dx = -\frac{1}{2} \int e^u du = -\frac{1}{2} e^{-x^2}.$$

Second term:

$$\int \ln(1+x) dx.$$

Use integration by parts with

$$u = \ln(1+x), \quad dv = dx.$$

Then

$$du = \frac{1}{1+x} dx, \quad v = x.$$

Thus,

$$\int \ln(1+x) dx = x \ln(1+x) - \int \frac{x}{1+x} dx.$$

Rewrite the integrand:

$$\frac{x}{1+x} = 1 - \frac{1}{1+x}.$$

Hence,

$$\int \frac{x}{1+x} dx = x - \ln(1+x).$$

Combining results:

$$-\frac{1}{2} e^{-x^2} + x \ln(1+x) - x + \ln(1+x) + C$$

Question 7

Find an antiderivative of

$$\int \left(\frac{\sqrt{1+2x}}{x} - \frac{1}{x\sqrt{1+2x}} \right) dx.$$

Question 7 - Solution

Combine the terms into a single fraction:

$$\frac{\sqrt{1+2x}}{x} - \frac{1}{x\sqrt{1+2x}} = \frac{(1+2x) - 1}{x\sqrt{1+2x}} = \frac{2x}{x\sqrt{1+2x}} = \frac{2}{\sqrt{1+2x}}.$$

Thus the integral simplifies to

$$\int \frac{2}{\sqrt{1+2x}} dx.$$

Let

$$u = 1 + 2x, \quad du = 2 dx.$$

Then

$$\int \frac{2}{\sqrt{1+2x}} dx = \int u^{-1/2} du = 2u^{1/2}.$$

Substitute back:

$$\boxed{2\sqrt{1+2x} + C}$$

Question 8

Find the general antiderivative of

$$\int \left(\ln(x + \sqrt{1 + x^2}) + \frac{x^2}{\sqrt{1 + x^2}} \right) dx.$$

Question 8 - Solution

Split the integral:

$$\int \ln(x + \sqrt{1+x^2}) \, dx + \int \frac{x^2}{\sqrt{1+x^2}} \, dx.$$

Part A: $\int \ln(x + \sqrt{1+x^2}) \, dx$

Use integration by parts with

$$u = \ln(x + \sqrt{1+x^2}), \quad dv = dx, \quad v = x.$$

Let

$$g(x) = x + \sqrt{1+x^2}.$$

Then

$$du = \frac{g'(x)}{g(x)} \, dx, \quad g'(x) = 1 + \frac{x}{\sqrt{1+x^2}}.$$

Thus

$$du = \frac{1 + \frac{x}{\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} \, dx = \frac{1}{\sqrt{1+x^2}} \, dx.$$

Therefore,

$$\int \ln(x + \sqrt{1+x^2}) \, dx = x \ln(x + \sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}} \, dx.$$

Let $w = 1 + x^2$, $dw = 2x \, dx$:

$$\int \frac{x}{\sqrt{1+x^2}} \, dx = \frac{1}{2} \int w^{-1/2} \, dw = \sqrt{1+x^2}.$$

Hence,

$$\int \ln(x + \sqrt{1+x^2}) \, dx = x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2}.$$

Part B: $\int \frac{x^2}{\sqrt{1+x^2}} \, dx$ (use an easier trigonometric substitution)

Let

$$x = \tan \theta, \quad dx = \sec^2 \theta \, d\theta, \quad \sqrt{1+x^2} = \sec \theta.$$

Then

$$\int \frac{x^2}{\sqrt{1+x^2}} \, dx = \int \frac{\tan^2 \theta}{\sec \theta} \sec^2 \theta \, d\theta = \int \tan^2 \theta \sec \theta \, d\theta.$$

Use $\tan^2 \theta = \sec^2 \theta - 1$:

$$\int \tan^2 \theta \sec \theta \, d\theta = \int (\sec^3 \theta - \sec \theta) \, d\theta.$$

Evaluate each term directly:

$$\int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta|,$$

and for $\int \sec^3 \theta \, d\theta$, integrate by parts:

$$\int \sec^3 \theta \, d\theta = \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|).$$

Thus,

$$\int (\sec^3 \theta - \sec \theta) \, d\theta = \frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta|.$$

Return to x using $\tan \theta = x$, $\sec \theta = \sqrt{1+x^2}$:

$$\int \frac{x^2}{\sqrt{1+x^2}} dx = \frac{1}{2}x\sqrt{1+x^2} - \frac{1}{2}\ln\left(x + \sqrt{1+x^2}\right).$$

Combine Parts A and B:

$$x \ln\left(x + \sqrt{1+x^2}\right) - \sqrt{1+x^2} + \frac{1}{2}x\sqrt{1+x^2} - \frac{1}{2}\ln\left(x + \sqrt{1+x^2}\right) + C$$

Question 9

Find the general antiderivative of

$$\int \frac{\arctan x}{x^2} dx.$$

Question 9 - Solution

We use integration by parts.

Let

$$u = \arctan x, \quad dv = \frac{1}{x^2} dx.$$

Then

$$du = \frac{1}{1+x^2} dx, \quad v = -\frac{1}{x}.$$

Applying integration by parts,

$$\int \frac{\arctan x}{x^2} dx = -\frac{\arctan x}{x} + \int \frac{1}{x(1+x^2)} dx.$$

Now decompose the integrand:

$$\frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}.$$

Thus,

$$\int \frac{1}{x(1+x^2)} dx = \ln|x| - \frac{1}{2} \ln(1+x^2).$$

Final Answer:

$-\frac{\arctan x}{x} + \ln x - \frac{1}{2} \ln(1+x^2) + C$
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Question 10

Find an antiderivative of

$$\int \frac{x \ln(1+x^2)}{1+x^2} dx.$$

Question 10 - Solution

Use the substitution

$$u = \ln(1 + x^2).$$

Then

$$du = \frac{2x}{1+x^2} dx, \quad \frac{1}{2} du = \frac{x}{1+x^2} dx.$$

Rewrite the integral:

$$\int \frac{x \ln(1+x^2)}{1+x^2} dx = \frac{1}{2} \int u du.$$

Integrate:

$$\frac{1}{2} \int u du = \frac{1}{4} u^2.$$

Substitute back:

$$\boxed{\frac{1}{4} (\ln(1+x^2))^2 + C}$$

Question 1

Compute the indefinite integral

$$\int x^2 e^x dx.$$

Question 1 - Solution

Use integration by parts with

$$u = x^2, \quad dv = e^x dx.$$

Then

$$du = 2x dx, \quad v = e^x.$$

So

$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx.$$

Apply integration by parts again to $\int 2x e^x dx$:

$$u = 2x, \quad dv = e^x dx \Rightarrow du = 2 dx, \quad v = e^x.$$

Thus,

$$\int 2x e^x dx = 2x e^x - 2 \int e^x dx = 2x e^x - 2e^x.$$

Hence,

$$\int x^2 e^x dx = e^x(x^2 - 2x + 2) + C.$$

Final Answer:

$$\int x^2 e^x dx = e^x(x^2 - 2x + 2) + C$$

Question 2

Evaluate the integral

$$\int x^2 \ln(x^2 + 1) dx.$$

Question 2 - Solution

Use integration by parts. Let

$$u = \ln(x^2 + 1), \quad dv = x^2 dx.$$

Then

$$du = \frac{2x}{x^2 + 1} dx, \quad v = \frac{x^3}{3}.$$

Apply integration by parts:

$$\int x^2 \ln(x^2 + 1) dx = \frac{x^3}{3} \ln(x^2 + 1) - \frac{2}{3} \int \frac{x^4}{x^2 + 1} dx.$$

Simplify the integrand by division:

$$\frac{x^4}{x^2 + 1} = x^2 - 1 + \frac{1}{x^2 + 1}.$$

Thus,

$$\int \frac{x^4}{x^2 + 1} dx = \int (x^2 - 1) dx + \int \frac{1}{x^2 + 1} dx = \frac{x^3}{3} - x + \arctan x.$$

Substitute back:

$$\int x^2 \ln(x^2 + 1) dx = \frac{x^3}{3} \ln(x^2 + 1) - \frac{2}{3} \left(\frac{x^3}{3} - x + \arctan x \right).$$

Final Answer:

$$\frac{x^3}{3} \ln(x^2 + 1) - \frac{2x^3}{9} + \frac{2x}{3} - \frac{2}{3} \arctan x + C$$

Question 3

Evaluate the integral

$$\int \frac{x^3}{(x^2 + 1)^2} dx.$$

Question 3 - Solution

Rewrite the integrand by separating powers of x :

$$\frac{x^3}{(x^2 + 1)^2} = \frac{x(x^2 + 1 - 1)}{(x^2 + 1)^2} = \frac{x}{x^2 + 1} - \frac{x}{(x^2 + 1)^2}.$$

Split the integral:

$$\int \frac{x}{x^2 + 1} dx - \int \frac{x}{(x^2 + 1)^2} dx.$$

For the first integral, let $u = x^2 + 1$, $du = 2x dx$:

$$\int \frac{x}{x^2 + 1} dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln(x^2 + 1).$$

For the second integral, use the same substitution:

$$\int \frac{x}{(x^2 + 1)^2} dx = \frac{1}{2} \int u^{-2} du = -\frac{1}{2u} = -\frac{1}{2(x^2 + 1)}.$$

Combine results:

$$\boxed{\frac{1}{2} \ln(x^2 + 1) + \frac{1}{2(x^2 + 1)} + C}$$

Question 4

Evaluate the integral

$$\int x \sqrt{x^2 + 4} \ln(x^2 + 4) dx.$$

Question 4 - Solution

Use substitution. Let

$$u = x^2 + 4, \quad du = 2x \, dx.$$

Then

$$\int x\sqrt{x^2 + 4} \ln(x^2 + 4) \, dx = \frac{1}{2} \int u^{1/2} \ln u \, du.$$

Apply integration by parts with

$$w = \ln u, \quad dv = u^{1/2} \, du.$$

Then

$$dw = \frac{1}{u} \, du, \quad v = \frac{2}{3} u^{3/2}.$$

Thus,

$$\int u^{1/2} \ln u \, du = \frac{2}{3} u^{3/2} \ln u - \frac{2}{3} \int u^{1/2} \, du = \frac{2}{3} u^{3/2} \ln u - \frac{4}{9} u^{3/2}.$$

Multiply by $\frac{1}{2}$ and substitute back:

$$\frac{1}{3} (x^2 + 4)^{3/2} \ln(x^2 + 4) - \frac{2}{9} (x^2 + 4)^{3/2} + C$$

Question 5

Evaluate

$$\int \frac{x^2}{\sqrt{x^2 + 1}} dx.$$

Question 5 - Solution**Step 1: Rewrite the integrand**

Write $x^2 = (x^2 + 1) - 1$:

$$\frac{x^2}{\sqrt{x^2 + 1}} = \frac{(x^2 + 1) - 1}{\sqrt{x^2 + 1}} = \sqrt{x^2 + 1} - \frac{1}{\sqrt{x^2 + 1}}.$$

Thus,

$$\int \frac{x^2}{\sqrt{x^2 + 1}} dx = \int \sqrt{x^2 + 1} dx - \int \frac{1}{\sqrt{x^2 + 1}} dx.$$

Step 2: Trigonometric substitution

Let

$$x = \tan \theta, \quad dx = \sec^2 \theta d\theta.$$

Then

$$\sqrt{x^2 + 1} = \sqrt{\tan^2 \theta + 1} = \sec \theta.$$

Step 3: Evaluate $\int \sqrt{x^2 + 1} dx$

Substitute:

$$\int \sqrt{x^2 + 1} dx = \int \sec \theta (\sec^2 \theta d\theta) = \int \sec^3 \theta d\theta.$$

To evaluate $\int \sec^3 \theta d\theta$, write

$$\sec^3 \theta = \sec \theta \sec^2 \theta.$$

Let

$$u = \sec \theta, \quad dv = \sec^2 \theta d\theta.$$

Then

$$du = \sec \theta \tan \theta d\theta, \quad v = \tan \theta.$$

By integration by parts,

$$\int \sec^3 \theta d\theta = \sec \theta \tan \theta - \int \tan \theta (\sec \theta \tan \theta) d\theta.$$

Simplify the integrand:

$$\tan^2 \theta = \sec^2 \theta - 1,$$

so

$$\int \sec^3 \theta d\theta = \sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1) d\theta.$$

Distribute:

$$= \sec \theta \tan \theta - \int \sec^3 \theta d\theta + \int \sec \theta d\theta.$$

Move the repeated integral to the left:

$$2 \int \sec^3 \theta d\theta = \sec \theta \tan \theta + \int \sec \theta d\theta.$$

Now compute $\int \sec \theta d\theta$:

$$\int \sec \theta d\theta = \int \frac{\sec \theta (\sec \theta + \tan \theta)}{\sec \theta + \tan \theta} d\theta.$$

Let

$$u = \sec \theta + \tan \theta, \quad du = (\sec \theta \tan \theta + \sec^2 \theta) d\theta = \sec \theta (\sec \theta + \tan \theta) d\theta.$$

Hence,

$$\int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C.$$

Substitute back:

$$2 \int \sec^3 \theta \, d\theta = \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|.$$

Therefore,

$$\int \sec^3 \theta \, d\theta = \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) + C.$$

Returning to x :

$$\sec \theta = \sqrt{x^2 + 1}, \quad \tan \theta = x.$$

Thus,

$$\int \sqrt{x^2 + 1} \, dx = \frac{1}{2} \left(x\sqrt{x^2 + 1} + \ln(x + \sqrt{x^2 + 1}) \right) + C.$$

Step 4: Evaluate $\int \frac{1}{\sqrt{x^2 + 1}} \, dx$

Substitute again:

$$\int \frac{1}{\sqrt{x^2 + 1}} \, dx = \int \frac{1}{\sec \theta} (\sec^2 \theta \, d\theta) = \int \sec \theta \, d\theta.$$

From above,

$$\int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C = \ln(x + \sqrt{x^2 + 1}) + C.$$

Step 5: Combine results

$$\int \frac{x^2}{\sqrt{x^2 + 1}} \, dx = \frac{1}{2} \left(x\sqrt{x^2 + 1} + \ln(x + \sqrt{x^2 + 1}) \right) - \ln(x + \sqrt{x^2 + 1}) + C.$$

Combine logarithms:

$$= \frac{x}{2} \sqrt{x^2 + 1} - \frac{1}{2} \ln(x + \sqrt{x^2 + 1}) + C.$$

Final Answer:

$$\boxed{\int \frac{x^2}{\sqrt{x^2 + 1}} \, dx = \frac{x}{2} \sqrt{x^2 + 1} - \frac{1}{2} \ln(x + \sqrt{x^2 + 1}) + C}$$

Question 6

Evaluate the integral

$$\int \frac{x^3 \ln x}{1+x^2} dx.$$

Question 6 - Solution

Rewrite the integrand by splitting the rational expression:

$$\frac{x^3}{1+x^2} = x - \frac{x}{1+x^2}.$$

Thus,

$$\int \frac{x^3 \ln x}{1+x^2} dx = \int x \ln x dx - \int \frac{x \ln x}{1+x^2} dx.$$

First integral: Use integration by parts with

$$u = \ln x, \quad dv = x dx.$$

Then

$$du = \frac{1}{x} dx, \quad v = \frac{x^2}{2}.$$

So

$$\int x \ln x dx = \frac{x^2}{2} \ln x - \frac{x^2}{4}.$$

Second integral: Use integration by parts with

$$u = \ln x, \quad dv = \frac{x}{1+x^2} dx.$$

Then

$$du = \frac{1}{x} dx, \quad v = \frac{1}{2} \ln(1+x^2).$$

Hence,

$$\int \frac{x \ln x}{1+x^2} dx = \frac{1}{2} \ln x \ln(1+x^2) - \frac{1}{2} \int \frac{\ln(1+x^2)}{x} dx.$$

The remaining integral is non-elementary and is expressed using the dilogarithm function:

$$\int \frac{\ln(1+x^2)}{x} dx = -\frac{1}{2} \text{Li}_2(-x^2).$$

Therefore,

$$\int \frac{x \ln x}{1+x^2} dx = \frac{1}{2} \ln x \ln(1+x^2) + \frac{1}{4} \text{Li}_2(-x^2).$$

Final Answer:

$$\boxed{\frac{x^2}{2} \ln x - \frac{x^2}{4} - \frac{1}{2} \ln x \ln(1+x^2) - \frac{1}{4} \text{Li}_2(-x^2) + C}$$

Question 7

Evaluate the integral

$$\int \frac{\sqrt{x^2 + 1}}{x} dx.$$

Question 7 - Solution

For $x \neq 0$, write $s = \sqrt{1+x^2}$, so $ds = (x/s) dx$ and $x^2 = s^2 - 1$.

$$\int \frac{\sqrt{1+x^2}}{x} dx = \int \frac{s^2}{s^2-1} ds = s + \frac{1}{2} \ln \left| \frac{s-1}{s+1} \right| + C.$$

Since $(s-1)(s+1) = x^2$, this becomes

$$\boxed{\sqrt{1+x^2} + \ln \left| \frac{x}{1+\sqrt{1+x^2}} \right| + C.}$$

The formula is valid on either interval $(-\infty, 0)$ or $(0, \infty)$; constants may differ between intervals.

Question 8

Evaluate the integral

$$\int (x^2 + 1) \ln(x^2 + 1) \, dx.$$

Question 8 - Solution

Using substitution does not work!

$$u = x^2 + 1, \quad du = 2x \, dx.$$

Rewrite the integrand:

$$(x^2 + 1) \ln(x^2 + 1) \, dx = u \ln u \cdot \frac{du}{2x}.$$

Instead, apply integration by parts directly.

Let

$$u = \ln(x^2 + 1), \quad dv = (x^2 + 1) \, dx.$$

Then

$$du = \frac{2x}{x^2 + 1} \, dx, \quad v = \frac{x^3}{3} + x.$$

Apply integration by parts:

$$\int (x^2 + 1) \ln(x^2 + 1) \, dx = \left(\frac{x^3}{3} + x \right) \ln(x^2 + 1) - \int \left(\frac{x^3}{3} + x \right) \frac{2x}{x^2 + 1} \, dx.$$

Simplify the remaining integral:

$$\left(\frac{x^3}{3} + x \right) \frac{2x}{x^2 + 1} = \frac{2x^4}{3(x^2 + 1)} + \frac{2x^2}{x^2 + 1}.$$

Use algebraic division:

$$\frac{x^4}{x^2 + 1} = x^2 - 1 + \frac{1}{x^2 + 1}.$$

Thus,

$$\int (x^2 + 1) \ln(x^2 + 1) \, dx = \left(\frac{x^3}{3} + x \right) \ln(x^2 + 1) - \int \left(\frac{2}{3}(x^2 - 1) + \frac{2}{3(x^2 + 1)} + \frac{2x^2}{x^2 + 1} \right) \, dx.$$

Compute term by term:

$$\begin{aligned} \int (x^2 - 1) \, dx &= \frac{x^3}{3} - x, \\ \int \frac{1}{x^2 + 1} \, dx &= \arctan x, \quad \int \frac{x^2}{x^2 + 1} \, dx = x - \arctan x. \end{aligned}$$

Putting everything together,

$$\boxed{\int (x^2 + 1) \ln(x^2 + 1) \, dx = \left(\frac{x^3}{3} + x \right) \ln(x^2 + 1) - \frac{2x^3}{9} - \frac{4x}{3} + \frac{4}{3} \arctan x + C}$$

Question 9

Evaluate the integral

$$\int \frac{x^4}{(1+x^2)^{3/2}} dx.$$

Question 9 - Solution

Use the trigonometric substitution

$$x = \tan \theta, \quad dx = \sec^2 \theta \, d\theta.$$

Then

$$1 + x^2 = \sec^2 \theta, \quad (1 + x^2)^{3/2} = \sec^3 \theta.$$

Substitute into the integral:

$$\int \frac{\tan^4 \theta}{\sec^3 \theta} \sec^2 \theta \, d\theta = \int \tan^4 \theta \cos \theta \, d\theta.$$

Rewrite $\tan^4 \theta$:

$$\tan^4 \theta = (\sec^2 \theta - 1)^2 = \sec^4 \theta - 2\sec^2 \theta + 1.$$

Thus,

$$\int (\sec^3 \theta - 2\sec \theta + \cos \theta) \, d\theta.$$

Integrate term by term:

$$\int \sec^3 \theta \, d\theta = \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln(\sec \theta + \tan \theta),$$

$$\int \sec \theta \, d\theta = \ln(\sec \theta + \tan \theta), \quad \int \cos \theta \, d\theta = \sin \theta.$$

Combining,

$$\frac{1}{2} \sec \theta \tan \theta - \frac{3}{2} \ln(\sec \theta + \tan \theta) + \sin \theta + C.$$

Substitute back:

$$\sec \theta = \sqrt{1 + x^2}, \quad \tan \theta = x, \quad \sin \theta = \frac{x}{\sqrt{1 + x^2}}.$$

Therefore,

$$\frac{x}{2} \sqrt{1 + x^2} - \frac{3}{2} \ln(x + \sqrt{1 + x^2}) + \frac{x}{\sqrt{1 + x^2}} + C$$

Question 10

Evaluate the indefinite integral:

$$\int \frac{x^2 + 2x + 2}{\sqrt{x^2 + 2x + 5}} dx.$$

Question 10 - Solution

We are given:

$$\int \frac{x^2 + 2x + 2}{\sqrt{x^2 + 2x + 5}} dx$$

Rewrite the numerator:

$$x^2 + 2x + 2 = (x^2 + 2x + 5) - 3.$$

Then

$$\int \frac{x^2 + 2x + 2}{\sqrt{x^2 + 2x + 5}} dx = \int \sqrt{x^2 + 2x + 5} dx - 3 \int \frac{dx}{\sqrt{x^2 + 2x + 5}}.$$

Complete the square:

$$x^2 + 2x + 5 = (x + 1)^2 + 4.$$

Let

$$t = x + 1.$$

Then

$$x^2 + 2x + 5 = t^2 + 4$$

and

$$dt = dx.$$

So the integral becomes:

$$\int \sqrt{t^2 + 4} dt - 3 \int \frac{dt}{\sqrt{t^2 + 4}}.$$

Using the formulas

$$\int \sqrt{t^2 + a^2} dt = \frac{t}{2} \sqrt{t^2 + a^2} + \frac{a^2}{2} \ln |t + \sqrt{t^2 + a^2}|$$

and

$$\int \frac{dt}{\sqrt{t^2 + a^2}} = \ln |t + \sqrt{t^2 + a^2}|,$$

with $a = 2$, we get:

$$\int \sqrt{t^2 + 4} dt = \frac{t}{2} \sqrt{t^2 + 4} + 2 \ln |t + \sqrt{t^2 + 4}|$$

and

$$3 \int \frac{dt}{\sqrt{t^2 + 4}} = 3 \ln |t + \sqrt{t^2 + 4}|.$$

Therefore,

$$\int \frac{x^2 + 2x + 2}{\sqrt{x^2 + 2x + 5}} dx = \frac{t}{2} \sqrt{t^2 + 4} + 2 \ln |t + \sqrt{t^2 + 4}| - 3 \ln |t + \sqrt{t^2 + 4}|.$$

Simplify:

$$= \frac{t}{2} \sqrt{t^2 + 4} - \ln |t + \sqrt{t^2 + 4}| + C.$$

Substitute back $t = x + 1$:

$$\boxed{\frac{x+1}{2} \sqrt{x^2 + 2x + 5} - \ln |x + 1 + \sqrt{x^2 + 2x + 5}| + C}$$

Question 1

Evaluate the integral

$$\int \frac{x e^{\sqrt{1+x^2}}}{\sqrt{1+x^2}} dx.$$

Question 1 - Solution

Use substitution. Let

$$u = \sqrt{1+x^2}.$$

Then

$$u^2 = 1 + x^2 \quad \Rightarrow \quad 2u \, du = 2x \, dx \quad \Rightarrow \quad x \, dx = u \, du.$$

Substitute into the integral:

$$\int \frac{x e^{\sqrt{1+x^2}}}{\sqrt{1+x^2}} \, dx = \int \frac{u e^u}{u} \, du = \int e^u \, du.$$

Integrate:

$$\int e^u \, du = e^u.$$

Substitute back:

$$\boxed{e^{\sqrt{1+x^2}} + C}$$

Question 2

Evaluate the integral

$$\int \frac{\ln(x^2 + 1)}{x^2 + 1} x \, dx.$$

Question 2 - Solution

Use substitution. Let

$$u = \ln(x^2 + 1).$$

Then

$$du = \frac{2x}{x^2 + 1} dx \quad \Rightarrow \quad \frac{x}{x^2 + 1} dx = \frac{1}{2} du.$$

Substitute:

$$\int \frac{\ln(x^2 + 1)}{x^2 + 1} x dx = \frac{1}{2} \int u du.$$

Integrate:

$$\frac{1}{2} \int u du = \frac{1}{4} u^2.$$

Substitute back:

$$\boxed{\frac{1}{4} (\ln(x^2 + 1))^2 + C}$$

Question 3

Evaluate the integral

$$\int \frac{x^3}{\sqrt{1+x^4}} dx.$$

Question 3 - Solution

Use substitution. Let

$$u = 1 + x^4.$$

Then

$$du = 4x^3 dx \quad \Rightarrow \quad x^3 dx = \frac{1}{4} du.$$

Substitute into the integral:

$$\int \frac{x^3}{\sqrt{1+x^4}} dx = \frac{1}{4} \int u^{-1/2} du.$$

Integrate:

$$\frac{1}{4} \int u^{-1/2} du = \frac{1}{4} \cdot 2u^{1/2} = \frac{1}{2} \sqrt{u}.$$

Substitute back:

$$\boxed{\frac{1}{2} \sqrt{1+x^4} + C}$$

Question 4

Evaluate the integral

$$\int \frac{x \sin(x^2)}{\sqrt{1 + \cos(x^2)}} dx.$$

Question 4 - Solution

Use substitution. Let

$$u = \cos(x^2).$$

Then

$$du = -\sin(x^2) \cdot 2x \, dx \quad \Rightarrow \quad -\frac{1}{2} du = x \sin(x^2) \, dx.$$

Substitute into the integral:

$$\int \frac{x \sin(x^2)}{\sqrt{1 + \cos(x^2)}} \, dx = -\frac{1}{2} \int \frac{du}{\sqrt{1 + u}}.$$

Integrate:

$$-\frac{1}{2} \int (1 + u)^{-1/2} \, du = -\frac{1}{2} \cdot 2(1 + u)^{1/2} = -\sqrt{1 + u}.$$

Substitute back:

$$\boxed{-\sqrt{1 + \cos(x^2)} + C}$$

Question 5

Evaluate the integral

$$\int \frac{2x}{1 + \sqrt{1 + x^2}} dx.$$

Question 5 - Solution

Use substitution. Let

$$u = \sqrt{1 + x^2}.$$

Then

$$u^2 = 1 + x^2 \quad \Rightarrow \quad 2u \, du = 2x \, dx \quad \Rightarrow \quad 2x \, dx = 2u \, du.$$

Substitute into the integral:

$$\int \frac{2x}{1 + \sqrt{1 + x^2}} \, dx = \int \frac{2u}{1 + u} \, du.$$

Rewrite the integrand:

$$\frac{2u}{1 + u} = 2 \left(1 - \frac{1}{1 + u} \right).$$

Integrate:

$$\int 2 \left(1 - \frac{1}{1 + u} \right) \, du = 2u - 2 \ln(1 + u).$$

Substitute back:

$$2\sqrt{1 + x^2} - 2 \ln(1 + \sqrt{1 + x^2}) + C$$

Question 6

Evaluate the integral

$$\int \frac{x e^{1/(1+x^2)}}{(1+x^2)^2} dx.$$

Question 6 - Solution

Use substitution. Let

$$u = \frac{1}{1+x^2}.$$

Then

$$du = -\frac{2x}{(1+x^2)^2} dx \quad \Rightarrow \quad -\frac{1}{2} du = \frac{x}{(1+x^2)^2} dx.$$

Substitute into the integral:

$$\int \frac{x e^{1/(1+x^2)}}{(1+x^2)^2} dx = -\frac{1}{2} \int e^u du.$$

Integrate:

$$-\frac{1}{2} \int e^u du = -\frac{1}{2} e^u.$$

Substitute back:

$$\boxed{-\frac{1}{2} e^{1/(1+x^2)} + C}$$

Question 7

Evaluate

$$\int \frac{x}{\sqrt{1-x^2}} \ln(1 + \sqrt{1-x^2}) \, dx.$$

Question 7 - Solution

Let

$$u = \sqrt{1 - x^2}.$$

Then

$$du = \frac{-x}{\sqrt{1 - x^2}} dx \quad \Rightarrow \quad \frac{x}{\sqrt{1 - x^2}} dx = -du.$$

So the integral becomes

$$-\int \ln(1 + u) du.$$

Now integrate by parts. Take

$$w = \ln(1 + u), \quad dv = du,$$

so

$$dw = \frac{1}{1 + u} du, \quad v = u.$$

Then

$$-\int \ln(1 + u) du = -\left(u \ln(1 + u) - \int \frac{u}{1 + u} du\right).$$

Simplify

$$\frac{u}{1 + u} = 1 - \frac{1}{1 + u},$$

hence

$$\int \frac{u}{1 + u} du = \int \left(1 - \frac{1}{1 + u}\right) du = u - \ln(1 + u).$$

Therefore,

$$-(u \ln(1 + u) - u + \ln(1 + u)) = -u \ln(1 + u) + u - \ln(1 + u) + C.$$

Substitute back $u = \sqrt{1 - x^2}$:

$$\boxed{-\sqrt{1 - x^2} \ln(1 + \sqrt{1 - x^2}) + \sqrt{1 - x^2} - \ln(1 + \sqrt{1 - x^2}) + C}$$

Question 8

Evaluate the integral

$$\int \frac{3x^2}{1+x^3} \sqrt{\ln(1+x^3)} dx.$$

Question 8 - Solution

Use substitution. Let

$$u = \ln(1 + x^3).$$

Then

$$du = \frac{3x^2}{1 + x^3} dx.$$

Substitute into the integral:

$$\int \frac{3x^2}{1 + x^3} \sqrt{\ln(1 + x^3)} dx = \int \sqrt{u} du.$$

Integrate:

$$\int u^{1/2} du = \frac{2}{3} u^{3/2}.$$

Substitute back:

$$\boxed{\frac{2}{3} (\ln(1 + x^3))^{3/2} + C}$$

Question 9

Evaluate the integral

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx.$$

Question 9 - Solution

Rewrite the integrand to reveal a derivative pattern:

$$2 \cos(\sqrt{x}) \cdot \frac{d}{dx}(\sqrt{x}).$$

Let

$$u = \sqrt{x}.$$

Then

$$du = \frac{1}{2\sqrt{x}} dx \quad \Rightarrow \quad dx = 2\sqrt{x} du.$$

Substitute into the integral:

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx = \int \frac{\cos(u)}{\sqrt{x}} \cdot 2\sqrt{x} du = 2 \int \cos u du.$$

Integrate:

$$2 \int \cos u du = 2 \sin u.$$

Substitute back:

$$\boxed{2 \sin(\sqrt{x}) + C}$$

Question 10

Evaluate the integral

$$\int x \ln(1 - x^2) dx.$$

Question 10 - Solution

Let

$$u = 1 - x^2.$$

Then

$$du = -2x \, dx.$$

Rewrite the integral:

$$\int x \ln(1 - x^2) \, dx = -\frac{1}{2} \int \ln u \, du.$$

Now integrate:

$$\int \ln u \, du = u \ln u - u.$$

Therefore,

$$-\frac{1}{2}(u \ln u - u) + C.$$

Substitute back:

$$-\frac{1}{2}[(1 - x^2) \ln(1 - x^2) - (1 - x^2)] + C$$

Question 1

Evaluate the integral

$$\int \frac{x}{1+x^2} \sqrt{1+\ln(1+x^2)} \, dx.$$

Question 1 - Solution**Step 1: Choose a substitution**

Set $u = 1 + \ln(1 + x^2)$, so $du = 2x/(1 + x^2) dx$.

$$\int \frac{x}{1 + x^2} \sqrt{1 + \ln(1 + x^2)} dx = \frac{1}{2} \int u^{1/2} du = \frac{1}{3} u^{3/2} + C.$$

$$\frac{1}{3} (1 + \ln(1 + x^2))^{3/2} + C.$$

This is valid for every real x .

Question 2

Evaluate the integral

$$\int \frac{x^2}{\sqrt{4-x^2}} dx.$$

Question 2 - Solution

Because the integrand contains $\sqrt{4-x^2}$, a trigonometric substitution is natural.

Let

$$x = 2 \sin \theta.$$

Then

$$dx = 2 \cos \theta \, d\theta, \quad \sqrt{4-x^2} = \sqrt{4-4\sin^2 \theta} = 2 \cos \theta.$$

Substitute into the integral:

$$\int \frac{x^2}{\sqrt{4-x^2}} dx = \int \frac{4 \sin^2 \theta}{2 \cos \theta} \cdot 2 \cos \theta \, d\theta = \int 4 \sin^2 \theta \, d\theta.$$

Use the identity

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}.$$

Then

$$\int 4 \sin^2 \theta \, d\theta = 4 \int \frac{1 - \cos(2\theta)}{2} d\theta = 2 \int (1 - \cos(2\theta)) \, d\theta.$$

Integrate:

$$2 \left(\theta - \frac{1}{2} \sin(2\theta) \right) = 2\theta - \sin(2\theta).$$

Return to x using

$$\theta = \arcsin\left(\frac{x}{2}\right), \quad \sin(2\theta) = 2 \sin \theta \cos \theta = \frac{x}{2} \sqrt{4-x^2}.$$

Thus,

$$2 \arcsin\left(\frac{x}{2}\right) - \frac{x}{2} \sqrt{4-x^2} + C$$

Question 3

Evaluate the integral

$$\int \frac{x^2}{\sqrt{x^2 - 9}} dx, \quad x > 3.$$

Question 3 - Solution

Because the integrand involves $\sqrt{x^2 - a^2}$, a *hyperbolic substitution* is natural.

Let

$$x = 3 \cosh t.$$

Then

$$dx = 3 \sinh t \, dt, \quad \sqrt{x^2 - 9} = \sqrt{9 \cosh^2 t - 9} = 3 \sinh t.$$

Substitute into the integral:

$$\int \frac{x^2}{\sqrt{x^2 - 9}} dx = \int \frac{9 \cosh^2 t}{3 \sinh t} \cdot 3 \sinh t \, dt = \int 9 \cosh^2 t \, dt.$$

Use the identity

$$\cosh^2 t = \frac{1 + \cosh(2t)}{2}.$$

Then

$$\int 9 \cosh^2 t \, dt = \frac{9}{2} \int (1 + \cosh(2t)) \, dt = \frac{9}{2} \left(t + \frac{1}{2} \sinh(2t) \right).$$

Simplify:

$$\frac{9}{2}t + \frac{9}{4} \sinh(2t).$$

Return to x . Since

$$x = 3 \cosh t \quad \Rightarrow \quad t = \operatorname{arcosh}\left(\frac{x}{3}\right),$$

and

$$\sinh(2t) = 2 \sinh t \cosh t = 2 \left(\frac{\sqrt{x^2 - 9}}{3} \right) \left(\frac{x}{3} \right) = \frac{2x\sqrt{x^2 - 9}}{9}.$$

Substitute back:

$$\boxed{\frac{9}{2} \operatorname{arcosh}\left(\frac{x}{3}\right) + \frac{x}{2} \sqrt{x^2 - 9} + C}$$

Question 4

Evaluate the integral

$$\int \frac{x}{\sqrt{x^2 + 2x + 5}} dx.$$

Question 4 - Solution

Complete the square:

$$x^2 + 2x + 5 = (x + 1)^2 + 4.$$

Let

$$u = x + 1, \quad du = dx, \quad x = u - 1.$$

Then the integral becomes

$$\int \frac{u-1}{\sqrt{u^2+4}} du = \int \frac{u}{\sqrt{u^2+4}} du - \int \frac{1}{\sqrt{u^2+4}} du.$$

First integral: Let $w = u^2 + 4$, so $dw = 2u du$:

$$\int \frac{u}{\sqrt{u^2+4}} du = \frac{1}{2} \int w^{-1/2} dw = \sqrt{u^2+4}.$$

Second integral (expanded):

$$\int \frac{1}{\sqrt{u^2+4}} du.$$

Use a trigonometric substitution. Let

$$u = 2 \tan \theta.$$

Then

$$du = 2 \sec^2 \theta d\theta, \quad \sqrt{u^2+4} = \sqrt{4 \tan^2 \theta + 4} = 2 \sec \theta.$$

Substitute:

$$\int \frac{1}{\sqrt{u^2+4}} du = \int \frac{2 \sec^2 \theta}{2 \sec \theta} d\theta = \int \sec \theta d\theta.$$

Recall:

$$\int \sec \theta d\theta = \ln |\sec \theta + \tan \theta|.$$

Return to u :

$$\tan \theta = \frac{u}{2}, \quad \sec \theta = \frac{\sqrt{u^2+4}}{2}.$$

Thus,

$$\int \frac{1}{\sqrt{u^2+4}} du = \ln(u + \sqrt{u^2+4}).$$

Final Answer:

$$\sqrt{x^2+2x+5} - \ln(x+1+\sqrt{x^2+2x+5}) + C$$

Question 5

Evaluate the integral

$$\int \frac{dx}{1 + \sin x}.$$

Question 5 - Solution

The integrand is defined wherever $1 + \sin x \neq 0$.

A primitive valid on each such interval is

$$\boxed{-\frac{\cos x}{1 + \sin x} + C.}$$

Indeed, the quotient rule gives

$$\frac{d}{dx} \left(-\frac{\cos x}{1 + \sin x} \right) = \frac{\sin x(1 + \sin x) + \cos^2 x}{(1 + \sin x)^2} = \frac{1}{1 + \sin x}.$$

This expression introduces no extra excluded points.

Question 6

Evaluate the integral

$$\int \sqrt{x^2 + x} \, dx.$$

Question 6 - Solution

Because the integrand involves a square root of a quadratic expression, an *Euler substitution* is effective.

Complete the square:

$$x^2 + x = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}.$$

Use the substitution

$$x + \frac{1}{2} = \frac{1}{2} \left(t - \frac{1}{t}\right), \quad t > 0.$$

Then

$$dx = \frac{1}{2} \left(1 + \frac{1}{t^2}\right) dt,$$

and

$$\sqrt{x^2 + x} = \sqrt{\frac{1}{4} \left(t - \frac{1}{t}\right)^2 - \frac{1}{4}} = \frac{1}{2} \left(t - \frac{1}{t}\right).$$

Substitute into the integral:

$$\int \sqrt{x^2 + x} dx = \int \frac{1}{2} \left(t - \frac{1}{t}\right) \cdot \frac{1}{2} \left(1 + \frac{1}{t^2}\right) dt.$$

Simplify:

$$\frac{1}{4} \int \left(t - \frac{1}{t^3}\right) dt.$$

Integrate:

$$\frac{1}{4} \left(\frac{t^2}{2} + \frac{1}{2t^2}\right) = \frac{1}{8} \left(t^2 + \frac{1}{t^2}\right).$$

Substitute back using

$$t - \frac{1}{t} = 2 \left(x + \frac{1}{2}\right).$$

Thus,

$$\boxed{\frac{1}{2} \left(x + \frac{1}{2}\right) \sqrt{x^2 + x} - \frac{1}{8} \ln \left|2x + 1 + 2\sqrt{x^2 + x}\right| + C}$$

Question 7

Evaluate the integral

$$\int \sqrt{x^2 + x} \, dx.$$

Question 7 - Solution

Begin by completing the square:

$$x^2 + x = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}.$$

Let

$$u = x + \frac{1}{2}.$$

Then

$$du = dx, \quad \sqrt{x^2 + x} = \sqrt{u^2 - \frac{1}{4}}.$$

The integral becomes

$$\int \sqrt{u^2 - \frac{1}{4}} du.$$

Now use a standard substitution for expressions of the form $\sqrt{u^2 - a^2}$. Let

$$u = \frac{1}{2} \sec \theta.$$

Then

$$du = \frac{1}{2} \sec \theta \tan \theta d\theta, \quad \sqrt{u^2 - \frac{1}{4}} = \frac{1}{2} \tan \theta.$$

Substitute:

$$\int \sqrt{u^2 - \frac{1}{4}} du = \int \frac{1}{2} \tan \theta \cdot \frac{1}{2} \sec \theta \tan \theta d\theta = \frac{1}{4} \int \tan^2 \theta \sec \theta d\theta.$$

Use the identity $\tan^2 \theta = \sec^2 \theta - 1$:

$$\frac{1}{4} \int (\sec^3 \theta - \sec \theta) d\theta.$$

Integrate:

$$\int \sec^3 \theta d\theta = \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|),$$

$$\int \sec \theta d\theta = \ln |\sec \theta + \tan \theta|.$$

Thus,

$$\frac{1}{4} \left[\frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| \right].$$

Return to x using

$$\sec \theta = 2u = 2x + 1, \quad \tan \theta = 2\sqrt{x^2 + x}.$$

Final Answer:

$$\boxed{\frac{1}{2} \left(x + \frac{1}{2}\right) \sqrt{x^2 + x} - \frac{1}{8} \ln |2x + 1 + 2\sqrt{x^2 + x}| + C}$$

Question 8

Evaluate the integral

$$\int \frac{x+2}{\sqrt{x^2+4x+8}} dx.$$

Question 8 - Solution

Complete the square in the denominator:

$$x^2 + 4x + 8 = (x + 2)^2 + 4.$$

Let

$$u = x + 2.$$

Then

$$du = dx, \quad x + 2 = u.$$

Substitute into the integral:

$$\int \frac{x + 2}{\sqrt{x^2 + 4x + 8}} dx = \int \frac{u}{\sqrt{u^2 + 4}} du.$$

Now use a direct substitution. Let

$$w = u^2 + 4, \quad dw = 2u du.$$

Then

$$\int \frac{u}{\sqrt{u^2 + 4}} du = \frac{1}{2} \int w^{-1/2} dw = \sqrt{w}.$$

Substitute back:

$$\boxed{\sqrt{x^2 + 4x + 8} + C}$$

Question 9

Evaluate the integral

$$\int \frac{x+1}{(x^2+2x+3)^{3/2}} dx.$$

Question 9 - Solution

Begin by completing the square in the denominator:

$$x^2 + 2x + 3 = (x + 1)^2 + 2.$$

Let

$$u = x + 1.$$

Then

$$du = dx, \quad (x^2 + 2x + 3)^{3/2} = (u^2 + 2)^{3/2}.$$

Substitute into the integral:

$$\int \frac{u}{(u^2 + 2)^{3/2}} du.$$

Now use a direct substitution. Let

$$w = u^2 + 2, \quad dw = 2u du.$$

Then

$$\int \frac{u}{(u^2 + 2)^{3/2}} du = \frac{1}{2} \int w^{-3/2} dw.$$

Integrate:

$$\frac{1}{2} \int w^{-3/2} dw = \frac{1}{2} \left(-2w^{-1/2} \right) = -\frac{1}{\sqrt{w}}.$$

Substitute back:

$$\boxed{-\frac{1}{\sqrt{x^2 + 2x + 3}} + C}$$

Question 10

Evaluate the integral

$$\int \frac{e^x}{\sqrt{1 + e^{2x}}} dx.$$

Question 10 - Solution

Begin with an exponential substitution. Let

$$u = e^x.$$

Then

$$du = e^x dx.$$

Substitute into the integral:

$$\int \frac{e^x}{\sqrt{1+e^{2x}}} dx = \int \frac{1}{\sqrt{1+u^2}} du.$$

Inverse trigonometric step (expanded):

We recognize the integrand as the derivative of $\operatorname{arsinh} u$ or $\arctan u$ -type expressions. Use a trigonometric substitution to justify the result.

Let

$$u = \tan \theta.$$

Then

$$du = \sec^2 \theta d\theta, \quad \sqrt{1+u^2} = \sqrt{1+\tan^2 \theta} = \sec \theta.$$

Substitute:

$$\int \frac{1}{\sqrt{1+u^2}} du = \int \frac{\sec^2 \theta}{\sec \theta} d\theta = \int \sec \theta d\theta.$$

Recall the standard integral:

$$\int \sec \theta d\theta = \ln |\sec \theta + \tan \theta|.$$

Substitute back using

$$\tan \theta = u, \quad \sec \theta = \sqrt{1+u^2}.$$

Thus,

$$\int \frac{1}{\sqrt{1+u^2}} du = \ln(u + \sqrt{1+u^2}).$$

Substitute back $u = e^x$:

$$\boxed{\ln(e^x + \sqrt{1+e^{2x}}) + C}$$

Question 1

Find the exact area of the region enclosed by the curves

$$y = x\sqrt{4 - x^2} \quad \text{and} \quad y = x^2.$$

Question 1 - Solution

First find the points of intersection:

$$x\sqrt{4-x^2} = x^2.$$

Factor:

$$x(\sqrt{4-x^2} - x) = 0.$$

Thus,

$$x = 0 \quad \text{or} \quad \sqrt{4-x^2} = x.$$

Square the second equation:

$$4 - x^2 = x^2 \quad \Rightarrow \quad x^2 = 2 \quad \Rightarrow \quad x = \sqrt{2}.$$

On the interval $[0, \sqrt{2}]$, we have

$$x\sqrt{4-x^2} \geq x^2.$$

Therefore, the area is

$$A = \int_0^{\sqrt{2}} (x\sqrt{4-x^2} - x^2) dx.$$

Split the integral:

$$A = \int_0^{\sqrt{2}} x\sqrt{4-x^2} dx - \int_0^{\sqrt{2}} x^2 dx.$$

For the first integral, use substitution. Let

$$u = 4 - x^2, \quad du = -2x dx.$$

Then

$$\int x\sqrt{4-x^2} dx = -\frac{1}{2} \int u^{1/2} du = -\frac{1}{3} u^{3/2} = -\frac{1}{3} (4-x^2)^{3/2}.$$

Evaluate from 0 to $\sqrt{2}$:

$$\left[-\frac{1}{3} (4-x^2)^{3/2} \right]_0^{\sqrt{2}} = -\frac{1}{3} (2\sqrt{2}) + \frac{1}{3} (8) = \frac{8-2\sqrt{2}}{3}.$$

For the second integral:

$$\int_0^{\sqrt{2}} x^2 dx = \left[\frac{x^3}{3} \right]_0^{\sqrt{2}} = \frac{2\sqrt{2}}{3}.$$

Subtract:

$$A = \frac{8-2\sqrt{2}}{3} - \frac{2\sqrt{2}}{3} = \frac{8-4\sqrt{2}}{3}.$$

$$\boxed{\frac{8-4\sqrt{2}}{3}}$$

Question 2

Find the exact area of the region enclosed by the curves

$$y = \ln x \quad \text{and} \quad y = x - 1,$$

restricted to the interval $0 < x \leq 1$.

Question 2 - Solution

On the interval $0 < x \leq 1$, the curves intersect where

$$\ln x = x - 1,$$

which occurs at

$$x = 1.$$

For $0 < x < 1$, we have

$$x - 1 \geq \ln x,$$

so the area between the curves is

$$A = \int_0^1 ((x - 1) - \ln x) dx.$$

Compute each term:

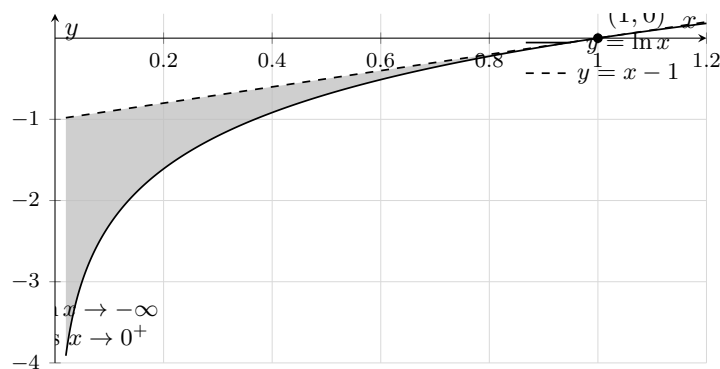
$$\int_0^1 (x - 1) dx = \left[\frac{x^2}{2} - x \right]_0^1 = -\frac{1}{2},$$

$$\int_0^1 \ln x dx = [x \ln x - x]_0^1 = -1.$$

Therefore,

$$A = -\frac{1}{2} - (-1) = \frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$



Question 3

Find the exact area of the region enclosed by the curves

$$y = \sqrt{x} \quad \text{and} \quad y = x^2,$$

restricted to the interval $0 \leq x \leq 1$.

Question 3 - Solution

First determine which curve lies above the other on $[0, 1]$. For $0 < x < 1$,

$$\sqrt{x} \geq x^2.$$

Thus, the enclosed area is

$$A = \int_0^1 (\sqrt{x} - x^2) dx.$$

Evaluate each term:

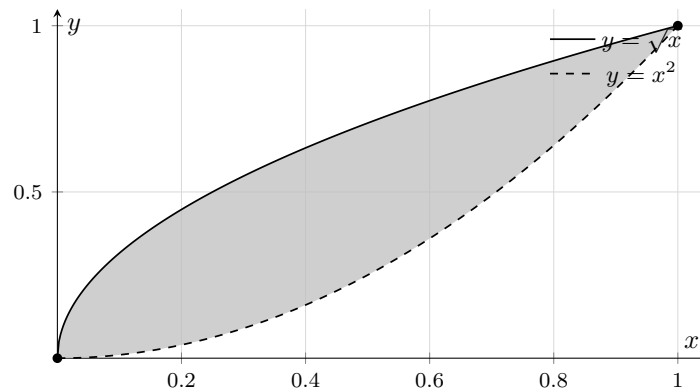
$$\int_0^1 \sqrt{x} dx = \left[\frac{2}{3} x^{3/2} \right]_0^1 = \frac{2}{3},$$

$$\int_0^1 x^2 dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}.$$

Subtract:

$$A = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

$$\boxed{\frac{1}{3}}$$



Question 4

Find the exact area of the region enclosed by the curves

$$y = x\sqrt{2-x} \quad \text{and} \quad y = x.$$

Question 4 - Solution

The intersections satisfy $x\sqrt{2-x} = x$, giving $x = 0, 1$.

For $0 < x < 1$, the square-root curve is above the line, hence

$$A = \int_0^1 (x\sqrt{2-x} - x) dx.$$

With $u = 2 - x$,

$$\int_0^1 x\sqrt{2-x} dx = \int_1^2 (2-u)\sqrt{u} du = \left[\frac{4}{3}u^{3/2} - \frac{2}{5}u^{5/2} \right]_1^2 = \frac{16\sqrt{2} - 14}{15}.$$

$$A = \frac{32\sqrt{2} - 43}{30} > 0.$$

Question 5

Find the exact area of the region enclosed by the curves

$$y = |x| \quad \text{and} \quad y = x^2.$$

Question 5 - Solution

First find the points of intersection by solving

$$|x| = x^2.$$

For $x \geq 0$, this becomes

$$x = x^2 \Rightarrow x = 0 \text{ or } x = 1.$$

For $x < 0$, we have

$$-x = x^2 \Rightarrow x = 0 \text{ or } x = -1.$$

Thus, the curves intersect at

$$x = -1, 0, 1.$$

On $[0, 1]$, we have $|x| = x \geq x^2$, and by symmetry the same holds on $[-1, 0]$. Therefore, the total area is twice the area on $[0, 1]$:

$$A = 2 \int_0^1 (x - x^2) dx.$$

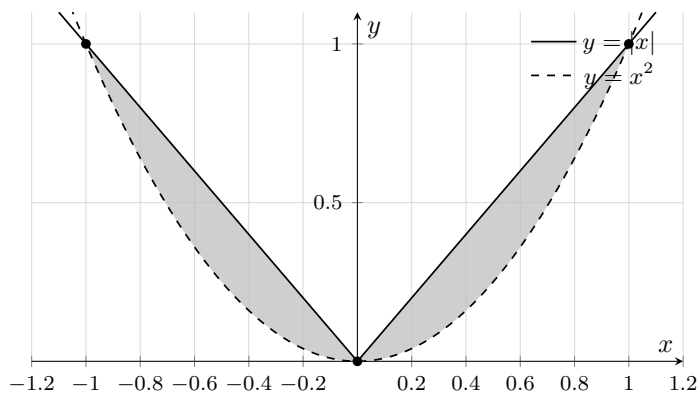
Compute the integral:

$$\int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Thus,

$$A = 2 \cdot \frac{1}{6} = \frac{1}{3}.$$

$$\boxed{\frac{1}{3}}$$



Question 6

Find the exact area of the region enclosed by the curves

$$y = x \quad \text{and} \quad y = x^3.$$

Question 6 - Solution

First find the points of intersection:

$$x = x^3 \Rightarrow x(x^2 - 1) = 0.$$

Thus,

$$x = -1, 0, 1.$$

On the interval $(0, 1)$, we have

$$x \geq x^3,$$

and by symmetry, the same configuration occurs on $(-1, 0)$.

Therefore, the total enclosed area is twice the area on $[0, 1]$:

$$A = 2 \int_0^1 (x - x^3) dx.$$

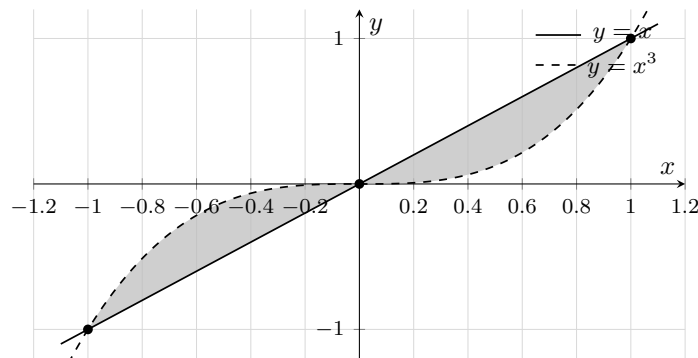
Evaluate the integral:

$$\int_0^1 (x - x^3) dx = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Thus,

$$A = 2 \cdot \frac{1}{4} = \frac{1}{2}.$$

$\frac{1}{2}$



Question 7

Find the exact area of the region enclosed by the curves

$$y = \sin x \quad \text{and} \quad y = \cos x$$

on the interval

$$\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}.$$

Question 7 - Solution

First find the points of intersection by solving

$$\sin x = \cos x.$$

This gives

$$\tan x = 1 \quad \Rightarrow \quad x = \frac{\pi}{4} + k\pi.$$

The smallest two consecutive intersection points occur at

$$x = \frac{\pi}{4} \quad \text{and} \quad x = \frac{5\pi}{4}.$$

On the interval between these points, we have

$$\sin x \geq \cos x.$$

Thus the enclosed area is

$$A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx.$$

Compute:

$$\int \sin x \, dx = -\cos x, \quad \int \cos x \, dx = \sin x.$$

So

$$A = [-\cos x - \sin x]_{\pi/4}^{5\pi/4}.$$

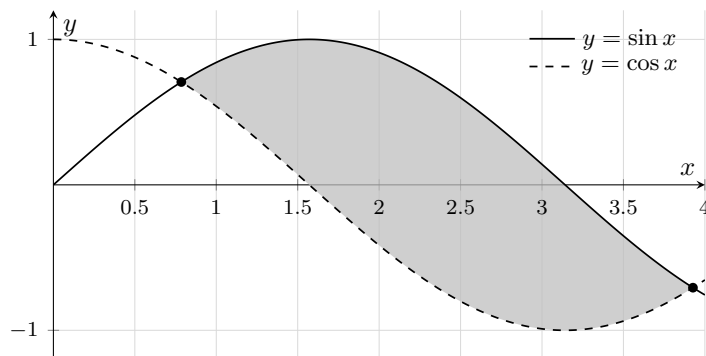
Evaluate:

$$\begin{aligned} -\cos\left(\frac{5\pi}{4}\right) - \sin\left(\frac{5\pi}{4}\right) &= \sqrt{2}, \\ -\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right) &= -\sqrt{2}. \end{aligned}$$

Thus,

$$A = \sqrt{2} - (-\sqrt{2}) = 2\sqrt{2}.$$

$$\boxed{2\sqrt{2}}$$



Question 8

Find the exact area of the region enclosed by the curves

$$y = e^{-x} \quad \text{and} \quad y = 1 + \left(\frac{1}{e} - 1\right)x.$$

Question 8 - Solution

First find the intersection points:

$$e^{-x} = 1 + \left(\frac{1}{e} - 1\right)x.$$

At $x = 0$,

$$e^0 = 1 = 1 + \left(\frac{1}{e} - 1\right) \cdot 0.$$

At $x = 1$,

$$e^{-1} = \frac{1}{e} = 1 + \left(\frac{1}{e} - 1\right) \cdot 1.$$

So the curves intersect at $x = 0$ and $x = 1$.

Since e^{-x} is convex, the chord (the line) lies above e^{-x} on $(0, 1)$, so the area is

$$A = \int_0^1 \left(1 + \left(\frac{1}{e} - 1\right)x - e^{-x}\right) dx.$$

Compute each piece:

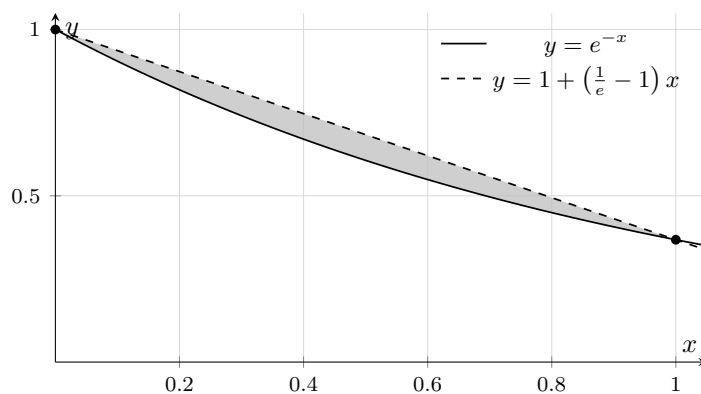
$$\int_0^1 \left(1 + \left(\frac{1}{e} - 1\right)x\right) dx = \left[x + \left(\frac{1}{e} - 1\right)\frac{x^2}{2}\right]_0^1 = 1 + \frac{1}{2}\left(\frac{1}{e} - 1\right) = \frac{1}{2} + \frac{1}{2e},$$

$$\int_0^1 e^{-x} dx = [-e^{-x}]_0^1 = 1 - \frac{1}{e}.$$

Thus

$$A = \left(\frac{1}{2} + \frac{1}{2e}\right) - \left(1 - \frac{1}{e}\right) = -\frac{1}{2} + \frac{3}{2e} = \frac{1}{2}\left(\frac{3}{e} - 1\right).$$

$$\boxed{\frac{1}{2}\left(\frac{3}{e} - 1\right)}$$



Question 9

Find the exact area of the region enclosed by the curves

$$y = \ln(1 + x) \quad \text{and} \quad y = \frac{x}{1 + x},$$

between $x = 0$ and $x = 1$.

Question 9 - Solution

On the interval $0 \leq x \leq 1$, we compare the curves. At $x = 0$,

$$\ln(1) = 0, \quad \frac{0}{1} = 0.$$

At $x = 1$,

$$\ln 2 > \frac{1}{2},$$

so

$$\ln(1+x) \geq \frac{x}{1+x} \quad \text{on } [0, 1].$$

Thus the enclosed area is

$$A = \int_0^1 \left(\ln(1+x) - \frac{x}{1+x} \right) dx.$$

Split the integral:

$$A = \int_0^1 \ln(1+x) dx - \int_0^1 \frac{x}{1+x} dx.$$

Evaluate the first integral:

$$\int \ln(1+x) dx = (1+x) \ln(1+x) - x,$$

so

$$\int_0^1 \ln(1+x) dx = [(1+x) \ln(1+x) - x]_0^1 = 2 \ln 2 - 1.$$

For the second integral, rewrite:

$$\frac{x}{1+x} = 1 - \frac{1}{1+x}.$$

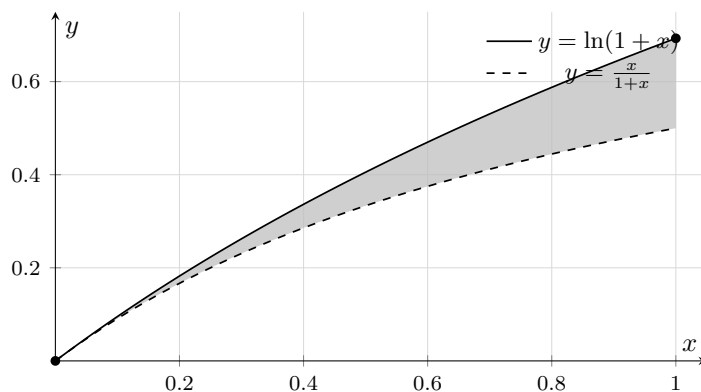
Then

$$\int_0^1 \frac{x}{1+x} dx = \int_0^1 \left(1 - \frac{1}{1+x} \right) dx = 1 - \ln 2.$$

Subtract:

$$A = (2 \ln 2 - 1) - (1 - \ln 2) = 3 \ln 2 - 2.$$

$$\boxed{3 \ln 2 - 2}$$



Question 10

Find the exact area of the region bounded by the curves

$$y = \sqrt{x} \quad \text{and} \quad y = \ln x,$$

between $x = 1$ and $x = e^2$.

Question 10 - Solution

On $1 \leq x \leq e^2$, we have $\sqrt{x} \geq \ln x$, so

$$A = \int_1^{e^2} (\sqrt{x} - \ln x) dx.$$

Compute:

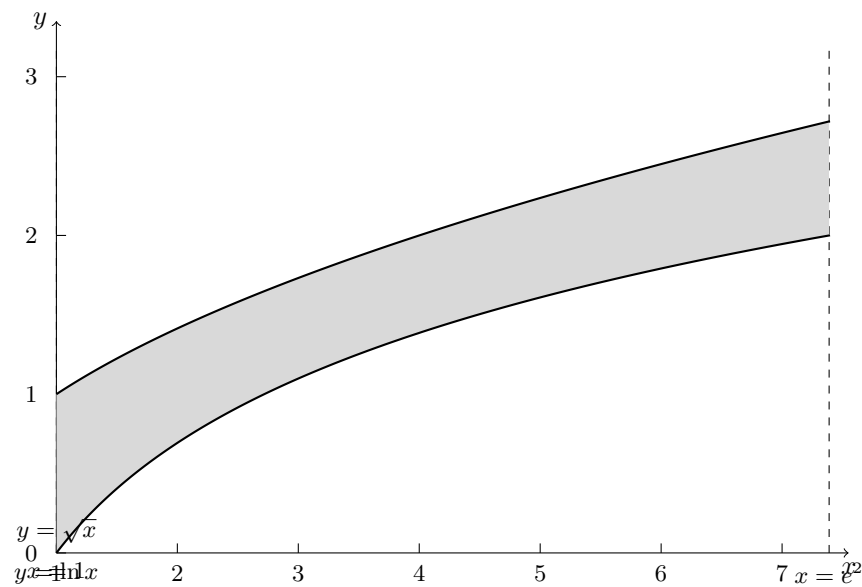
$$\int_1^{e^2} \sqrt{x} dx = \left[\frac{2}{3} x^{3/2} \right]_1^{e^2} = \frac{2}{3} (e^3 - 1),$$

$$\int_1^{e^2} \ln x dx = [x \ln x - x]_1^{e^2} = e^2 + 1.$$

Thus

$$A = \frac{2}{3} (e^3 - 1) - (e^2 + 1) = \frac{2}{3} e^3 - e^2 - \frac{5}{3}.$$

$$\boxed{\frac{2}{3} e^3 - e^2 - \frac{5}{3}}$$



Question 1

Consider the function

$$f(x) = 2x + 1$$

on the interval $[0, 2]$.

Using the definition of the definite integral, express

$$\int_0^2 (2x + 1) \, dx$$

as the limit of a Riemann sum with n equal subintervals.

Question 1 - Solution

Divide the interval $[0, 2]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{2}{n}.$$

Using right endpoints, the i th sample point is

$$x_i = 0 + i\Delta x = \frac{2i}{n}.$$

Evaluate the function at x_i :

$$f(x_i) = 2\left(\frac{2i}{n}\right) + 1 = \frac{4i}{n} + 1.$$

The Riemann sum is

$$\sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n \left(\frac{4i}{n} + 1\right) \frac{2}{n}.$$

Simplify:

$$\frac{2}{n} \sum_{i=1}^n \left(\frac{4i}{n} + 1\right) = \frac{8}{n^2} \sum_{i=1}^n i + \frac{2}{n} \sum_{i=1}^n 1.$$

Using

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n 1 = n,$$

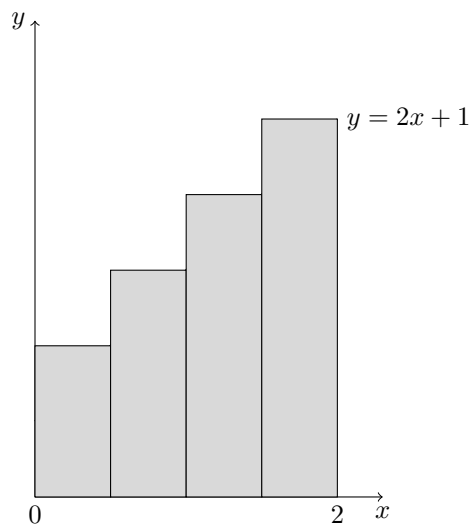
we obtain

$$\frac{8}{n^2} \cdot \frac{n(n+1)}{2} + \frac{2}{n} \cdot n = 4\left(1 + \frac{1}{n}\right) + 2.$$

Thus,

$$\int_0^2 (2x + 1) dx = \lim_{n \rightarrow \infty} \left(6 + \frac{4}{n}\right) = 6.$$

6



Question 2

Let

$$f(x) = x^2$$

on the interval $[1, 3]$.

Using the definition of the definite integral, write

$$\int_1^3 x^2 dx$$

as the limit of a Riemann sum using midpoints.

Question 2 - Solution

Divide $[1, 3]$ into n equal subintervals of width $\Delta x = 2/n$.

The midpoints are $x_i^* = 1 + (2i - 1)/n$.

$$M_n = \frac{2}{n} \sum_{i=1}^n \left(1 + \frac{2i-1}{n} \right)^2.$$

Use $\sum (2i - 1) = n^2$ and $\sum (2i - 1)^2 = n(4n^2 - 1)/3$:

$$M_n = 2 + 4 + \frac{2(4n^2 - 1)}{3n^2} = \frac{26}{3} - \frac{2}{3n^2}.$$

$$\boxed{\int_1^3 x^2 dx = \lim_{n \rightarrow \infty} M_n = \frac{26}{3}.$$

Question 3

Let

$$f(x) = \sqrt{x}$$

on the interval $[0, 4]$.

Using the definition of the definite integral, write

$$\int_0^4 \sqrt{x} \, dx$$

as the limit of a Riemann sum using left endpoints, and evaluate the limit.

Question 3 - Solution

Divide the interval $[0, 4]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{4}{n}.$$

Using left endpoints, the i th sample point is

$$x_{i-1} = (i-1)\Delta x = \frac{4(i-1)}{n}.$$

Evaluate the function:

$$f(x_{i-1}) = \sqrt{\frac{4(i-1)}{n}} = 2\sqrt{\frac{i-1}{n}}.$$

The Riemann sum is

$$\sum_{i=1}^n f(x_{i-1}) \Delta x = \sum_{i=1}^n 2\sqrt{\frac{i-1}{n}} \cdot \frac{4}{n} = \frac{8}{n^{3/2}} \sum_{i=1}^n \sqrt{i-1}.$$

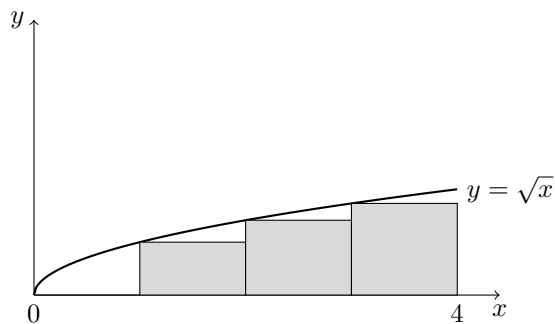
Thus,

$$\int_0^4 \sqrt{x} \, dx = \lim_{n \rightarrow \infty} \frac{8}{n^{3/2}} \sum_{i=1}^n \sqrt{i-1}.$$

As $n \rightarrow \infty$, this Riemann sum converges to the definite integral:

$$\int_0^4 \sqrt{x} \, dx = \left[\frac{2}{3} x^{3/2} \right]_0^4 = \frac{16}{3}.$$

$$\boxed{\frac{16}{3}}$$



Question 4

Let

$$f(x) = \frac{1}{1+x^2}$$

on the interval $[-1, 1]$.

Using the definition of the definite integral, express

$$\int_{-1}^1 \frac{1}{1+x^2} dx$$

as the limit of a Riemann sum using right endpoints, and evaluate the limit.

Question 4 - Solution

Divide the interval $[-1, 1]$ into n equal subintervals. The width of each subinterval is

$$\Delta x = \frac{1 - (-1)}{n} = \frac{2}{n}.$$

Using right endpoints, the i th sample point is

$$x_i = -1 + i\Delta x = -1 + \frac{2i}{n}.$$

Evaluate the function:

$$f(x_i) = \frac{1}{1 + \left(-1 + \frac{2i}{n}\right)^2}.$$

The Riemann sum is

$$\sum_{i=1}^n f(x_i) \Delta x = \frac{2}{n} \sum_{i=1}^n \frac{1}{1 + \left(-1 + \frac{2i}{n}\right)^2}.$$

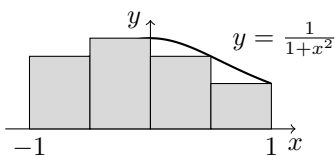
Thus,

$$\int_{-1}^1 \frac{1}{1+x^2} dx = \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n \frac{1}{1 + \left(-1 + \frac{2i}{n}\right)^2}.$$

Evaluating the definite integral,

$$\int_{-1}^1 \frac{1}{1+x^2} dx = [\arctan x]_{-1}^1 = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$$

$$\boxed{\frac{\pi}{2}}$$



Question 5

Let

$$f(x) = e^x$$

on the interval $[0, 1]$.

Using the definition of the definite integral, express

$$\int_0^1 e^x dx$$

as the limit of a Riemann sum using right endpoints, and evaluate the limit.

Question 5- Solution

Partition the interval $[0, 1]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{1}{n}.$$

Using right endpoints, the i th sample point is

$$x_i = i\Delta x = \frac{i}{n}.$$

Evaluate the function at each sample point:

$$f(x_i) = e^{i/n}.$$

The Riemann sum is

$$\sum_{i=1}^n f(x_i) \Delta x = \frac{1}{n} \sum_{i=1}^n e^{i/n}.$$

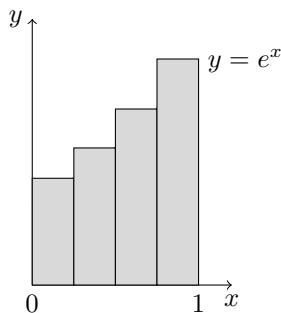
Thus,

$$\int_0^1 e^x dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n e^{i/n}.$$

Evaluating the definite integral,

$$\int_0^1 e^x dx = [e^x]_0^1 = e - 1.$$

$e - 1$



Question 6

Let

$$f(x) = |x|$$

on the interval $[-2, 2]$.

Using the definition of the definite integral, express

$$\int_{-2}^2 |x| dx$$

as the limit of a Riemann sum using equal subintervals, and evaluate the limit.

Question 6 - Solution

Divide the interval $[-2, 2]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{2 - (-2)}{n} = \frac{4}{n}.$$

Using right endpoints, the i th sample point is

$$x_i = -2 + i\Delta x = -2 + \frac{4i}{n}.$$

Evaluate the function:

$$f(x_i) = \left| -2 + \frac{4i}{n} \right|.$$

The Riemann sum is

$$\sum_{i=1}^n f(x_i) \Delta x = \frac{4}{n} \sum_{i=1}^n \left| -2 + \frac{4i}{n} \right|.$$

Thus,

$$\int_{-2}^2 |x| dx = \lim_{n \rightarrow \infty} \frac{4}{n} \sum_{i=1}^n \left| -2 + \frac{4i}{n} \right|.$$

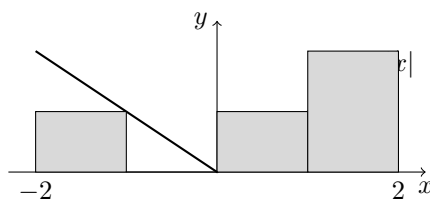
To evaluate the integral, use symmetry:

$$\int_{-2}^2 |x| dx = 2 \int_0^2 x dx.$$

Compute:

$$2 \left[\frac{x^2}{2} \right]_0^2 = 4.$$

$$\boxed{4}$$



Question 7

Let

$$f(x) = 3 - 2x$$

on the interval $[0, 1]$.

Using the definition of the definite integral, express

$$\int_0^1 (3 - 2x) \, dx$$

as the limit of a Riemann sum using left endpoints, and evaluate the limit.

Question 7 - Solution

Partition the interval $[0, 1]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{1}{n}.$$

Using left endpoints, the i th sample point is

$$x_{i-1} = (i-1)\Delta x = \frac{i-1}{n}.$$

Evaluate the function at each sample point:

$$f(x_{i-1}) = 3 - 2\left(\frac{i-1}{n}\right).$$

The Riemann sum is

$$\sum_{i=1}^n f(x_{i-1}) \Delta x = \frac{1}{n} \sum_{i=1}^n \left(3 - \frac{2(i-1)}{n}\right).$$

Separate the sum:

$$\frac{1}{n} \left(\sum_{i=1}^n 3 - \frac{2}{n} \sum_{i=1}^n (i-1) \right).$$

Using

$$\sum_{i=1}^n 1 = n, \quad \sum_{i=1}^n (i-1) = \frac{n(n-1)}{2},$$

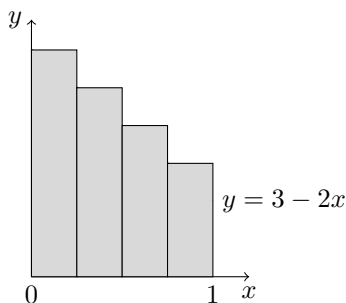
we obtain

$$\frac{1}{n} \left(3n - \frac{2}{n} \cdot \frac{n(n-1)}{2} \right) = 3 - (n-1)\frac{1}{n}.$$

Thus,

$$\int_0^1 (3-2x) dx = \lim_{n \rightarrow \infty} \left(3 - \frac{n-1}{n} \right) = 2.$$

2



Question 8

Let

$$f(x) = \frac{1}{x}$$

on the interval $[1, e]$.

Using the definition of the definite integral, express

$$\int_1^e \frac{1}{x} dx$$

as the limit of a Riemann sum using right endpoints, and evaluate the limit.

Question 8 - Solution

Partition the interval $[1, e]$ into n equal subintervals. The width of each subinterval is

$$\Delta x = \frac{e - 1}{n}.$$

Using right endpoints, the i th sample point is

$$x_i = 1 + i\Delta x = 1 + i\frac{e - 1}{n}.$$

Evaluate the function at each sample point:

$$f(x_i) = \frac{1}{1 + i\frac{e-1}{n}}.$$

The Riemann sum is

$$\sum_{i=1}^n f(x_i) \Delta x = \frac{e - 1}{n} \sum_{i=1}^n \frac{1}{1 + i\frac{e-1}{n}}.$$

Thus,

$$\int_1^e \frac{1}{x} dx = \lim_{n \rightarrow \infty} \frac{e - 1}{n} \sum_{i=1}^n \frac{1}{1 + i\frac{e-1}{n}}.$$

Evaluating the definite integral,

$$\int_1^e \frac{1}{x} dx = [\ln x]_1^e = 1.$$

1

Question 9

Let

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq 1, \\ 2, & 1 < x \leq 2. \end{cases}$$

Using the definition of the definite integral, express

$$\int_0^2 f(x) \, dx$$

as the limit of a Riemann sum, and evaluate the limit.

Question 9 - Solution

Partition the interval $[0, 2]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{2}{n}.$$

Using right endpoints, the i th sample point is

$$x_i = \frac{2i}{n}.$$

The Riemann sum is

$$\sum_{i=1}^n f(x_i) \Delta x = \frac{2}{n} \sum_{i=1}^n f\left(\frac{2i}{n}\right).$$

Split the sum according to where the function changes behavior. The condition $x_i \leq 1$ corresponds to

$$\frac{2i}{n} \leq 1 \quad \Rightarrow \quad i \leq \frac{n}{2}.$$

Thus,

$$\sum_{i=1}^n f\left(\frac{2i}{n}\right) = \sum_{i=1}^{n/2} 2\left(\frac{2i}{n}\right) + \sum_{i=n/2+1}^n 2.$$

Compute each sum:

$$\sum_{i=1}^{n/2} \frac{4i}{n} = \frac{4}{n} \cdot \frac{(n/2)(n/2+1)}{2} = \frac{n}{2} + 1,$$

$$\sum_{i=n/2+1}^n 2 = 2 \cdot \frac{n}{2} = n.$$

Therefore,

$$\sum_{i=1}^n f\left(\frac{2i}{n}\right) = \frac{n}{2} + 1 + n = \frac{3n}{2} + 1.$$

Multiply by Δx :

$$\frac{2}{n} \left(\frac{3n}{2} + 1 \right) = 3 + \frac{2}{n}.$$

Taking the limit,

$$\int_0^2 f(x) dx = \lim_{n \rightarrow \infty} \left(3 + \frac{2}{n} \right) = 3.$$

Question 10

Let

$$f(x) = \sin x$$

on the interval $[0, \pi]$.

Using the definition of the definite integral, express

$$\int_0^{\pi} \sin x \, dx$$

as the limit of a Riemann sum using midpoints, and evaluate the limit.

Question 10 - Solution

Partition the interval $[0, \pi]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{\pi}{n}.$$

The midpoint of the i th subinterval is

$$x_i^* = \left(i - \frac{1}{2}\right) \Delta x = \left(i - \frac{1}{2}\right) \frac{\pi}{n}.$$

Evaluate the function at each midpoint:

$$f(x_i^*) = \sin\left(\left(i - \frac{1}{2}\right) \frac{\pi}{n}\right).$$

The Riemann sum is

$$\sum_{i=1}^n f(x_i^*) \Delta x = \frac{\pi}{n} \sum_{i=1}^n \sin\left(\left(i - \frac{1}{2}\right) \frac{\pi}{n}\right).$$

Thus,

$$\int_0^{\pi} \sin x \, dx = \lim_{n \rightarrow \infty} \frac{\pi}{n} \sum_{i=1}^n \sin\left(\left(i - \frac{1}{2}\right) \frac{\pi}{n}\right).$$

Evaluating the definite integral directly,

$$\int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi} = (-\cos \pi) - (-\cos 0) = 2.$$

2

Question 1

Evaluate the definite integral

$$\int_0^1 (x^2 e^x - 2x e^x) \, dx.$$

Question 1 - Solution

Combine the terms:

$$\int_0^1 e^x(x^2 - 2x) dx.$$

Use integration by parts. Let

$$u = x^2 - 2x, \quad dv = e^x dx.$$

Then

$$du = (2x - 2) dx, \quad v = e^x.$$

Apply integration by parts:

$$\int e^x(x^2 - 2x) dx = (x^2 - 2x)e^x - \int e^x(2x - 2) dx.$$

Apply integration by parts again to the remaining integral. Let

$$u = 2x - 2, \quad dv = e^x dx.$$

Then

$$du = 2 dx, \quad v = e^x.$$

Thus,

$$\int e^x(2x - 2) dx = (2x - 2)e^x - \int 2e^x dx = (2x - 2)e^x - 2e^x.$$

Substitute back:

$$\int e^x(x^2 - 2x) dx = (x^2 - 2x)e^x - [(2x - 2)e^x - 2e^x].$$

Simplify:

$$(x^2 - 4x + 4)e^x.$$

Now evaluate from 0 to 1:

$$\int_0^1 e^x(x^2 - 2x) dx = [(x^2 - 4x + 4)e^x]_0^1.$$

Compute:

$$(1 - 4 + 4)e - (0 - 0 + 4) = e - 4.$$

$$\boxed{e - 4}$$

Question 2

Evaluate the definite integral

$$\int_0^{\pi/2} (\sin x + \sin^2 x) \, dx.$$

Question 2 - Solution

Split the integral:

$$\int_0^{\pi/2} \sin x \, dx + \int_0^{\pi/2} \sin^2 x \, dx.$$

Evaluate the first integral:

$$\int_0^{\pi/2} \sin x \, dx = [-\cos x]_0^{\pi/2} = 1.$$

For the second integral, use the identity

$$\sin^2 x = \frac{1 - \cos(2x)}{2}.$$

Then

$$\int_0^{\pi/2} \sin^2 x \, dx = \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2x) \, dx.$$

Integrate:

$$\frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi/2} = \frac{1}{2} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{4}.$$

Combine results:

$$1 + \frac{\pi}{4}.$$

$$\boxed{1 + \frac{\pi}{4}}$$

Question 3

Evaluate the definite integral

$$\int_{-1}^1 x^2 \sqrt{1-x^2} \, dx.$$

Question 3 - Solution

First notice a symmetry property. The function

$$x^2\sqrt{1-x^2}$$

is an *even function* because replacing x by $-x$ does not change its value. Therefore,

$$\int_{-1}^1 x^2\sqrt{1-x^2} dx = 2 \int_0^1 x^2\sqrt{1-x^2} dx.$$

Now focus on the integral from 0 to 1.

Use a trigonometric substitution. Let

$$x = \sin \theta, \quad dx = \cos \theta d\theta.$$

When $x = 0$, $\theta = 0$, and when $x = 1$, $\theta = \frac{\pi}{2}$.

Substitute into the integral:

$$\int_0^1 x^2\sqrt{1-x^2} dx = \int_0^{\pi/2} (\sin^2 \theta)(\cos \theta)(\cos \theta) d\theta = \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta.$$

Use the identity

$$\sin^2 \theta \cos^2 \theta = \frac{1}{4} \sin^2(2\theta).$$

Then

$$\int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta = \frac{1}{4} \int_0^{\pi/2} \sin^2(2\theta) d\theta.$$

Apply the identity

$$\sin^2 u = \frac{1 - \cos(2u)}{2}.$$

Thus,

$$\frac{1}{4} \int_0^{\pi/2} \frac{1 - \cos(4\theta)}{2} d\theta = \frac{1}{8} \left[\theta - \frac{1}{4} \sin(4\theta) \right]_0^{\pi/2}.$$

Evaluate:

$$\frac{1}{8} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{16}.$$

Finally, multiply by 2 to account for the symmetry:

$$\int_{-1}^1 x^2\sqrt{1-x^2} dx = 2 \cdot \frac{\pi}{16} = \boxed{\frac{\pi}{8}}.$$

Question 4

Evaluate the definite integral

$$\int_0^1 \frac{x}{\sqrt{1+x^2}} dx.$$

Question 4 - Solution

This integral is well-suited for a direct substitution because the derivative of $1+x^2$ appears in the numerator.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x \, dx \quad \Rightarrow \quad x \, dx = \frac{1}{2} \, du.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

Substitute into the integral:

$$\int_0^1 \frac{x}{\sqrt{1+x^2}} \, dx = \frac{1}{2} \int_1^2 u^{-1/2} \, du.$$

Integrate:

$$\frac{1}{2} \int u^{-1/2} \, du = \frac{1}{2} \cdot 2u^{1/2} = \sqrt{u}.$$

Apply the limits:

$$\left. \sqrt{u} \right|_1^2 = \sqrt{2} - 1.$$

$$\boxed{\sqrt{2} - 1}$$

Question 5

Evaluate the definite integral

$$\int_0^{\pi} x \sin x \, dx.$$

Question 5 - Solution

This integral suggests integration by parts because it is a product of x and $\sin x$.

Let

$$u = x, \quad dv = \sin x \, dx.$$

Then

$$du = dx, \quad v = -\cos x.$$

Apply integration by parts:

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x.$$

Now evaluate from 0 to π :

$$[-x \cos x + \sin x]_0^\pi.$$

Compute each endpoint:

$$-\pi \cos \pi + \sin \pi = \pi,$$

$$-0 \cdot \cos 0 + \sin 0 = 0.$$

Subtract:

$$\pi - 0 = \pi.$$

$$\boxed{\pi}$$

Question 6

Evaluate the definite integral

$$\int_{-2}^2 \frac{1}{1+x^2} dx.$$

Question 6 - Solution

First observe that the function

$$\frac{1}{1+x^2}$$

is an even function, since

$$\frac{1}{1+(-x)^2} = \frac{1}{1+x^2}.$$

Therefore, the integral over the symmetric interval $[-2, 2]$ can be written as

$$\int_{-2}^2 \frac{1}{1+x^2} dx = 2 \int_0^2 \frac{1}{1+x^2} dx.$$

Now evaluate the integral. Recall that

$$\int \frac{1}{1+x^2} dx = \arctan x.$$

Thus,

$$2 \int_0^2 \frac{1}{1+x^2} dx = 2 [\arctan x]_0^2.$$

Evaluate the bounds:

$$\arctan(2) - \arctan(0) = \arctan(2).$$

Therefore,

$$\int_{-2}^2 \frac{1}{1+x^2} dx = \boxed{2 \arctan(2)}.$$

Question 7

Evaluate the definite integral

$$\int_0^1 (3x^2 + 2)\sqrt{x} \, dx.$$

Question 7 - Solution

Begin by rewriting the integrand using powers of x :

$$(3x^2 + 2)\sqrt{x} = 3x^{5/2} + 2x^{1/2}.$$

Now integrate term by term:

$$\int 3x^{5/2} dx = 3 \cdot \frac{x^{7/2}}{7/2} = \frac{6}{7}x^{7/2},$$

$$\int 2x^{1/2} dx = 2 \cdot \frac{x^{3/2}}{3/2} = \frac{4}{3}x^{3/2}.$$

Combine the results:

$$\int (3x^2 + 2)\sqrt{x} dx = \frac{6}{7}x^{7/2} + \frac{4}{3}x^{3/2}.$$

Evaluate from 0 to 1:

$$\left[\frac{6}{7}x^{7/2} + \frac{4}{3}x^{3/2} \right]_0^1 = \frac{6}{7} + \frac{4}{3}.$$

Find a common denominator:

$$\frac{6}{7} + \frac{4}{3} = \frac{18 + 28}{21} = \frac{46}{21}.$$

$$\boxed{\frac{46}{21}}$$

Question 8

Evaluate the definite integral

$$\int_0^{\ln 2} e^x (1 + e^x) dx.$$

Question 8 - Solution

Begin by expanding the integrand:

$$e^x(1 + e^x) = e^x + e^{2x}.$$

Now integrate term by term:

$$\int e^x dx = e^x, \quad \int e^{2x} dx = \frac{1}{2}e^{2x}.$$

Thus,

$$\int e^x(1 + e^x) dx = e^x + \frac{1}{2}e^{2x}.$$

Evaluate from 0 to $\ln 2$:

$$\left[e^x + \frac{1}{2}e^{2x} \right]_0^{\ln 2}.$$

Compute the upper bound:

$$e^{\ln 2} + \frac{1}{2}e^{2 \ln 2} = 2 + \frac{1}{2} \cdot 4 = 4.$$

Compute the lower bound:

$$e^0 + \frac{1}{2}e^0 = 1 + \frac{1}{2} = \frac{3}{2}.$$

Subtract:

$$4 - \frac{3}{2} = \frac{5}{2}.$$

$$\boxed{\frac{5}{2}}$$

Question 9

Evaluate the definite integral

$$\int_1^4 \left(\frac{2}{\sqrt{x}} - \frac{1}{x} \right) dx.$$

Question 9 - Solution

Rewrite the integrand using exponents:

$$\frac{2}{\sqrt{x}} - \frac{1}{x} = 2x^{-1/2} - x^{-1}.$$

Integrate term by term:

$$\int 2x^{-1/2} dx = 2 \cdot \frac{x^{1/2}}{1/2} = 4\sqrt{x},$$

$$\int x^{-1} dx = \ln x.$$

Thus,

$$\int \left(\frac{2}{\sqrt{x}} - \frac{1}{x} \right) dx = 4\sqrt{x} - \ln x.$$

Evaluate from 1 to 4:

$$\left[4\sqrt{x} - \ln x \right]_1^4.$$

Compute the upper bound:

$$4\sqrt{4} - \ln 4 = 8 - \ln 4.$$

Compute the lower bound:

$$4\sqrt{1} - \ln 1 = 4.$$

Subtract:

$$(8 - \ln 4) - 4 = 4 - \ln 4.$$

$$\boxed{4 - \ln 4}$$

Question 10

Evaluate the definite integral

$$\int_0^1 \frac{2x}{1+x^2} dx.$$

Question 10 - Solution

This integral suggests a substitution because the derivative of the denominator appears in the numerator.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x \, dx.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

Substitute into the integral:

$$\int_0^1 \frac{2x}{1+x^2} \, dx = \int_1^2 \frac{1}{u} \, du.$$

Integrate:

$$\int \frac{1}{u} \, du = \ln u.$$

Apply the limits:

$$\ln u \Big|_1^2 = \ln 2.$$

$$\boxed{\ln 2}$$

Question 1

Evaluate the definite integral

$$\int_0^1 \frac{x}{\sqrt{1+x^2}} dx.$$

Question 1 - Solution

Use substitution to simplify the integrand.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x \, dx \quad \Rightarrow \quad x \, dx = \frac{1}{2} \, du.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

Substitute into the integral:

$$\int_0^1 \frac{x}{\sqrt{1+x^2}} \, dx = \frac{1}{2} \int_1^2 u^{-1/2} \, du.$$

Integrate:

$$\frac{1}{2} \int u^{-1/2} \, du = \frac{1}{2} \cdot 2u^{1/2} = \sqrt{u}.$$

Apply the new limits:

$$\left. \sqrt{u} \right|_1^2 = \sqrt{2} - 1.$$

$\sqrt{2} - 1$

Question 2

Evaluate the definite integral

$$\int_0^{\pi/2} \sin x \sqrt{1 + \cos x} \, dx.$$

Question 2 - Solution

This integral suggests a substitution involving $\cos x$ because its derivative appears in the integrand.

Let

$$u = 1 + \cos x.$$

Then

$$du = -\sin x \, dx \quad \Rightarrow \quad -du = \sin x \, dx.$$

Change the limits of integration. When $x = 0$, $\cos 0 = 1$, so $u = 2$. When $x = \frac{\pi}{2}$, $\cos(\frac{\pi}{2}) = 0$, so $u = 1$.

Substitute into the integral:

$$\int_0^{\pi/2} \sin x \sqrt{1 + \cos x} \, dx = - \int_2^1 \sqrt{u} \, du = \int_1^2 \sqrt{u} \, du.$$

Integrate:

$$\int \sqrt{u} \, du = \frac{2}{3} u^{3/2}.$$

Apply the limits:

$$\left. \frac{2}{3} u^{3/2} \right|_1^2 = \frac{2}{3} (2^{3/2} - 1).$$

$$\boxed{\frac{2}{3} (2^{3/2} - 1)}$$

Question 3

Evaluate the definite integral

$$\int_1^2 \frac{\ln x}{x} dx.$$

Question 3 - Solution

This integral is well suited for substitution because the derivative of $\ln x$ appears in the denominator.

Let

$$u = \ln x.$$

Then

$$du = \frac{1}{x} dx.$$

Change the limits of integration. When $x = 1$, $u = \ln 1 = 0$. When $x = 2$, $u = \ln 2$.

Substitute into the integral:

$$\int_1^2 \frac{\ln x}{x} dx = \int_0^{\ln 2} u du.$$

Integrate:

$$\int u du = \frac{u^2}{2}.$$

Apply the limits:

$$\frac{u^2}{2} \Big|_0^{\ln 2} = \frac{(\ln 2)^2}{2}.$$

$$\boxed{\frac{(\ln 2)^2}{2}}$$

Question 4

Evaluate the definite integral

$$\int_0^1 \frac{x^3}{\sqrt{1+x^4}} dx.$$

Question 4 - Solution

This integral suggests a substitution involving the expression inside the square root.

Let

$$u = 1 + x^4.$$

Then

$$du = 4x^3 dx \quad \Rightarrow \quad x^3 dx = \frac{1}{4} du.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

Substitute into the integral:

$$\int_0^1 \frac{x^3}{\sqrt{1+x^4}} dx = \frac{1}{4} \int_1^2 u^{-1/2} du.$$

Integrate:

$$\frac{1}{4} \int u^{-1/2} du = \frac{1}{4} \cdot 2u^{1/2} = \frac{1}{2} \sqrt{u}.$$

Apply the limits:

$$\frac{1}{2} \sqrt{u} \Big|_1^2 = \frac{1}{2} (\sqrt{2} - 1).$$

$$\boxed{\frac{\sqrt{2} - 1}{2}}$$

Question 5

Evaluate the definite integral

$$\int_0^{\pi/2} \frac{\cos x}{1 + \sin x} dx.$$

Question 5 - Solution

This integral is well suited for substitution because the derivative of $\sin x$ appears in the numerator.

Let

$$u = 1 + \sin x.$$

Then

$$du = \cos x \, dx.$$

Change the limits of integration. When $x = 0$, $\sin 0 = 0$, so $u = 1$. When $x = \frac{\pi}{2}$, $\sin(\frac{\pi}{2}) = 1$, so $u = 2$.

Substitute into the integral:

$$\int_0^{\pi/2} \frac{\cos x}{1 + \sin x} \, dx = \int_1^2 \frac{1}{u} \, du.$$

Integrate:

$$\int \frac{1}{u} \, du = \ln u.$$

Apply the limits:

$$\ln u \Big|_1^2 = \ln 2.$$

$$\boxed{\ln 2}$$

Question 6

Evaluate the definite integral

$$\int_0^{\sqrt{3}} \frac{x}{1+x^2} dx.$$

Question 6 - Solution

The integrand suggests a substitution involving $1 + x^2$.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x \, dx \quad \Rightarrow \quad x \, dx = \frac{1}{2} \, du.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = \sqrt{3}$, $u = 4$.

Substitute into the integral:

$$\int_0^{\sqrt{3}} \frac{x}{1+x^2} \, dx = \frac{1}{2} \int_1^4 \frac{1}{u} \, du.$$

Integrate:

$$\frac{1}{2} \int \frac{1}{u} \, du = \frac{1}{2} \ln u.$$

Apply the limits:

$$\frac{1}{2} \ln u \Big|_1^4 = \frac{1}{2} \ln 4 = \ln 2.$$

$$\boxed{\ln 2}$$

Question 7

Evaluate the definite integral

$$\int_1^e \frac{1}{x} \sqrt{1 + \ln x} \, dx.$$

Question 7 - Solution

The integrand suggests a substitution involving $\ln x$.

Let

$$u = 1 + \ln x.$$

Then

$$du = \frac{1}{x} dx.$$

Change the limits of integration. When $x = 1$, $u = 1$. When $x = e$, $u = 2$.

Substitute into the integral:

$$\int_1^e \frac{1}{x} \sqrt{1 + \ln x} dx = \int_1^2 \sqrt{u} du.$$

Integrate:

$$\int \sqrt{u} du = \frac{2}{3} u^{3/2}.$$

Apply the limits:

$$\left. \frac{2}{3} u^{3/2} \right|_1^2 = \frac{2}{3} (2^{3/2} - 1).$$

$$\boxed{\frac{2}{3} (2^{3/2} - 1)}$$

Question 8

Evaluate the definite integral

$$\int_0^1 \frac{2x}{(1+x^2)^2} dx.$$

Question 8 - Solution

The integrand suggests a substitution involving $1 + x^2$.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x \, dx.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

Substitute into the integral:

$$\int_0^1 \frac{2x}{(1+x^2)^2} \, dx = \int_1^2 u^{-2} \, du.$$

Integrate:

$$\int u^{-2} \, du = -u^{-1}.$$

Apply the limits:

$$-u^{-1} \Big|_1^2 = -\frac{1}{2} - (-1) = \frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$

Question 9

Evaluate the definite integral

$$\int_0^1 \frac{\sqrt{1-x}}{1+x} dx.$$

Question 9 - Solution

This integral becomes simpler if we eliminate the square root by substitution.

Let

$$u = \sqrt{1-x}.$$

Then

$$x = 1 - u^2, \quad dx = -2u \, du.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 0$.

Substitute into the integral:

$$\int_0^1 \frac{\sqrt{1-x}}{1+x} dx = \int_1^0 \frac{u}{1+(1-u^2)} (-2u) du.$$

Simplify:

$$\int_1^0 \frac{-2u^2}{2-u^2} du = \int_0^1 \frac{2u^2}{2-u^2} du.$$

Rewrite the integrand:

$$\frac{2u^2}{2-u^2} = -2 + \frac{4}{2-u^2}.$$

Thus,

$$\int_0^1 \left(-2 + \frac{4}{2-u^2} \right) du.$$

Integrate term by term:

$$\begin{aligned} \int -2 \, du &= -2u, \\ \int \frac{4}{2-u^2} \, du &= 2 \int \frac{1}{1-(u/\sqrt{2})^2} \, du. \end{aligned}$$

Use the standard result

$$\int \frac{1}{1-a^2} da = \operatorname{artanh}(a).$$

Thus,

$$2 \int \frac{1}{1-(u/\sqrt{2})^2} du = 2\sqrt{2} \operatorname{artanh}\left(\frac{u}{\sqrt{2}}\right).$$

Evaluate from 0 to 1:

$$\left. -2u + 2\sqrt{2} \operatorname{artanh}\left(\frac{u}{\sqrt{2}}\right) \right|_0^1.$$

Final answer:

$$-2 + 2\sqrt{2} \operatorname{artanh}\left(\frac{1}{\sqrt{2}}\right)$$

Question 10

Evaluate the definite integral

$$\int_0^1 \sqrt{1-x^2} \arcsin x \, dx.$$

Question 10 - Solution

This integral is interesting because it mixes an inverse trigonometric function with an algebraic expression, suggesting a trigonometric substitution.

Let

$$x = \sin \theta.$$

Then

$$dx = \cos \theta \, d\theta, \quad \sqrt{1-x^2} = \cos \theta.$$

Change the limits of integration. When $x = 0$, $\theta = 0$. When $x = 1$, $\theta = \frac{\pi}{2}$.

Substitute into the integral:

$$\int_0^1 \sqrt{1-x^2} \arcsin x \, dx = \int_0^{\pi/2} \theta \cos^2 \theta \, d\theta.$$

Use the identity

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}.$$

Then

$$\int_0^{\pi/2} \theta \cos^2 \theta \, d\theta = \frac{1}{2} \int_0^{\pi/2} \theta \, d\theta + \frac{1}{2} \int_0^{\pi/2} \theta \cos(2\theta) \, d\theta.$$

The first integral is

$$\frac{1}{2} \int_0^{\pi/2} \theta \, d\theta = \frac{1}{2} \cdot \frac{(\pi/2)^2}{2} = \frac{\pi^2}{16}.$$

For the second integral, use integration by parts. Let

$$u = \theta, \quad dv = \cos(2\theta) \, d\theta.$$

Then

$$du = d\theta, \quad v = \frac{1}{2} \sin(2\theta).$$

Thus,

$$\frac{1}{2} \int_0^{\pi/2} \theta \cos(2\theta) \, d\theta = \frac{1}{2} \left[\frac{1}{2} \theta \sin(2\theta) - \frac{1}{2} \int \sin(2\theta) \, d\theta \right]_0^{\pi/2}.$$

Since $\sin(\pi) = 0$ and $\sin(0) = 0$, the boundary term vanishes. Also,

$$\int \sin(2\theta) \, d\theta = -\frac{1}{2} \cos(2\theta).$$

Therefore,

$$\frac{1}{2} \int_0^{\pi/2} \theta \cos(2\theta) \, d\theta = \frac{1}{8} [\cos(2\theta)]_0^{\pi/2} = \frac{1}{8} (-1 - 1) = -\frac{1}{4}.$$

Combine results:

$$\frac{\pi^2}{16} - \frac{1}{4}.$$

$$\boxed{\frac{\pi^2}{16} - \frac{1}{4}}$$