

Question 1

The position of an object moving along a straight line is given by:

$$s(t) = 3t^3 - 5t^2 + 2t \quad (\text{in meters, where } t \text{ is in seconds})$$

- (a) Find the velocity function $v(t)$.
- (b) Find the instantaneous velocity at $t = 2$ seconds.
- (c) Find the average velocity between $t = 1$ and $t = 2$.

Question 1 - Solution

We are given:

$$s(t) = 3t^3 - 5t^2 + 2t$$

(a) Velocity is the derivative of position:

$$v(t) = s'(t) = 9t^2 - 10t + 2$$

(b) Instantaneous velocity at $t = 2$:

$$v(2) = 9(4) - 10(2) + 2 = 36 - 20 + 2 = 18$$

Answer: $v(2) = 18 \text{ m/s}$

(c) Average velocity from $t = 1$ to $t = 2$:

$$\frac{s(2) - s(1)}{2 - 1} = s(2) - s(1)$$

Compute $s(2)$:

$$s(2) = 3(8) - 5(4) + 2(2) = 24 - 20 + 4 = 8$$

Compute $s(1)$:

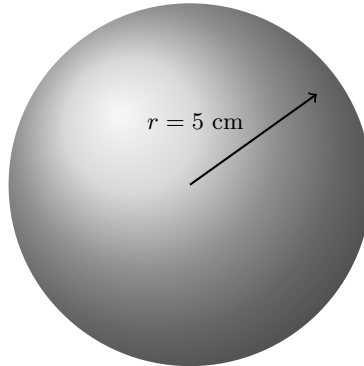
$$s(1) = 3(1) - 5(1) + 2(1) = 3 - 5 + 2 = 0$$

$$\text{Average velocity} = \frac{8 - 0}{1} = 8$$

Answer: 8 m/s

Question 2**Problem:**

A spherical balloon is being inflated so that its volume increases at a rate of $100 \text{ cm}^3/\text{sec}$. How fast is the radius increasing when the radius is 5 cm ?



Inflating sphere

Question 2 - Solution

We are given the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

Differentiate both sides with respect to time t :

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

We are told that:

$$\frac{dV}{dt} = 100, \quad r = 5$$

Substitute into the equation:

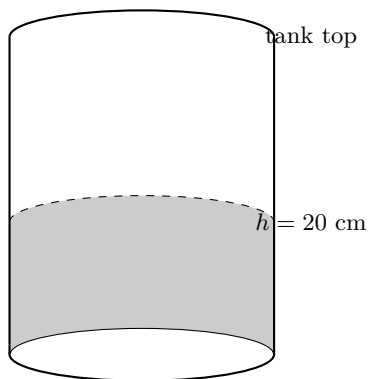
$$100 = 4\pi(5)^2 \cdot \frac{dr}{dt} \Rightarrow 100 = 100\pi \cdot \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{\pi}$$

Answer:

$\frac{dr}{dt} = \frac{1}{\pi} \text{ cm/sec}$
--

Question 3**Problem:**

Water is being poured into a cylindrical tank at a rate of $500 \text{ cm}^3/\text{s}$. The tank has a radius of 10 cm. How fast is the water level rising when the water is 20 cm deep?



Question 3 - Solution

The volume of a cylinder is:

$$V = \pi r^2 h$$

Given: $\frac{dV}{dt} = 500$, $r = 10$, $h = 20$

We differentiate with respect to time:

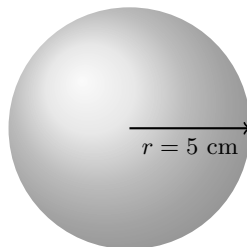
$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} \Rightarrow 500 = \pi (10)^2 \frac{dh}{dt} \Rightarrow 500 = 100\pi \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{500}{100\pi} = \frac{5}{\pi}$$

Answer:

$\frac{dh}{dt} = \frac{5}{\pi} \text{ cm/sec}$
--

Question 4**Problem:**

A spherical balloon is being inflated so that its volume increases at a rate of $100 \text{ cm}^3/\text{s}$. How fast is the radius increasing when the radius is 5 cm ?



Question 4 - Solution

We use the formula for the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

Differentiate both sides with respect to t :

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Substitute:

$$\frac{dV}{dt} = 100, \quad r = 5$$

$$100 = 4\pi(5)^2 \frac{dr}{dt} \Rightarrow 100 = 100\pi \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{100}{100\pi} = \frac{1}{\pi}$$

Answer:

$\frac{dr}{dt} = \frac{1}{\pi} \text{ cm/sec}$
--

Question 5

Lighthouse Problem:

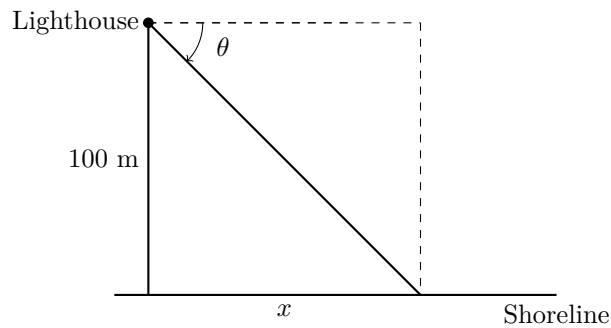
A lighthouse is located at the top of a cliff that is 100 meters high. The shoreline is straight and lies in a vertical plane. The beam of light rotates in this plane at a constant rate of

$$\frac{d\theta}{dt} = 0.05 \text{ rad/s.}$$

Let x be the distance along the shoreline from the base of the cliff to the point where the light hits the shore.

How fast is the spot of light moving along the shoreline when

$$\theta = \frac{\pi}{4} ?$$



Question 5 - Solution

Let:

- θ be the angle between the light beam and the horizontal shoreline.
- x be the distance along the shoreline to the spot of light.

From the right triangle formed by the beam, the cliff, and the shoreline:

$$\tan(\theta) = \frac{x}{100} \quad \Rightarrow \quad x = 100 \tan(\theta)$$

Differentiate both sides with respect to time t :

$$\frac{dx}{dt} = 100 \sec^2(\theta) \frac{d\theta}{dt}$$

Substitute the given values:

$$\theta = \frac{\pi}{4}, \quad \frac{d\theta}{dt} = 0.05 \text{ rad/s}, \quad \sec^2\left(\frac{\pi}{4}\right) = 2$$

$$\frac{dx}{dt} = 100(2)(0.05) = 10 \text{ m/s}$$

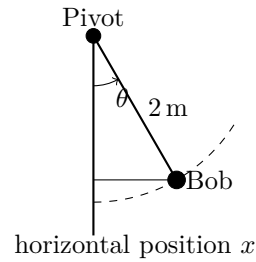
Answer:

$$\frac{dx}{dt} = 10 \text{ m/s}$$

Question 6

Problem:

A pendulum of length 2 meters swings back and forth in a vertical plane. At a certain instant, the angle θ between the pendulum and the vertical is $\frac{\pi}{6}$, and the angle is increasing at a rate of 0.4 rad/s . How fast is the pendulum bob moving horizontally at that instant?



Question 6 - Solution

Let:

- θ be the angle between the pendulum and vertical
- x be the horizontal displacement of the bob from the vertical

Using trigonometry:

$$x = L \sin(\theta) = 2 \sin(\theta)$$

Differentiate both sides with respect to time t :

$$\frac{dx}{dt} = 2 \cos(\theta) \cdot \frac{d\theta}{dt}$$

Substitute:

$$\theta = \frac{\pi}{6}, \quad \frac{d\theta}{dt} = 0.4 \text{ rad/s}, \quad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\frac{dx}{dt} = 2 \cdot \frac{\sqrt{3}}{2} \cdot 0.4 = \sqrt{3} \cdot 0.4 \approx 0.6928 \text{ m/s}$$

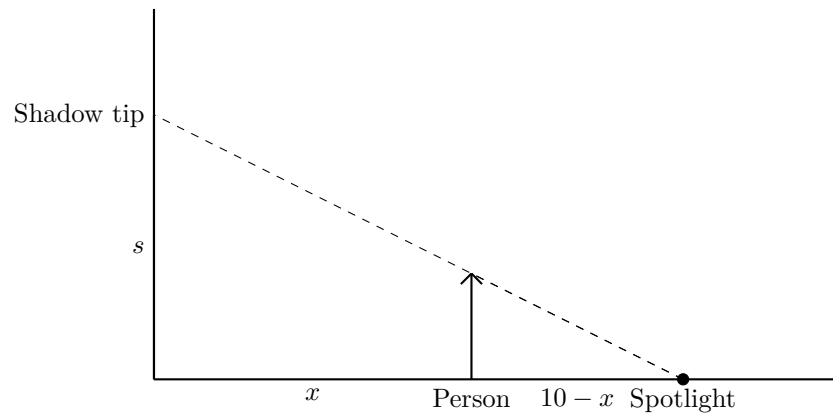
Answer:

$$\frac{dx}{dt} \approx 0.693 \text{ m/s}$$

Question 7

Problem:

A spotlight is located on the ground 10 meters from a vertical wall. A person 2 meters tall walks away from the wall at a speed of 1.5 m/s. How fast is the length of their shadow on the wall increasing when the person is 6 meters from the wall?



Question 7 - Solution

Let:

- x : distance from person to wall (in m)
- s : length of the shadow on the wall (in m)
- Person's height: 2 m
- Spotlight is 10 m from the wall

Using similar triangles (person is between wall and spotlight, so spotlight-person distance is $10 - x$):

$$\frac{2}{10 - x} = \frac{s}{10} \quad \Rightarrow \quad s = \frac{20}{10 - x}$$

Differentiate with respect to t :

$$\frac{ds}{dt} = 20 \cdot \frac{d}{dt} ((10 - x)^{-1}) = 20(10 - x)^{-2} \cdot \frac{dx}{dt}$$

Substitute $x = 6$ and $\frac{dx}{dt} = 1.5$:

$$\frac{ds}{dt} = \frac{20(1.5)}{(10 - 6)^2} = \frac{30}{16} = \frac{15}{8}$$

Answer:

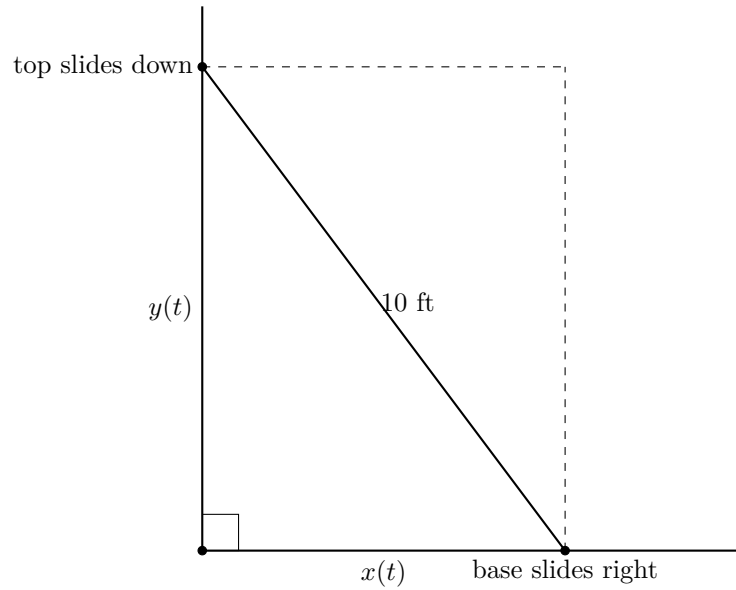
$\frac{ds}{dt} = \frac{15}{8} \text{ m/s}$
--

 (The shadow is increasing at this rate.)

Question 8

Problem:

A 10-foot ladder is leaning against a vertical wall. The bottom of the ladder is sliding away from the wall at a rate of 2 ft/s. How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 feet from the wall?



Question 8 - Solution

Let:

- $x(t)$: distance from the wall to the base of the ladder
- $y(t)$: height of the top of the ladder on the wall
- Given: $\frac{dx}{dt} = 2 \text{ ft/s}$, $x = 6$

Using the Pythagorean theorem:

$$x^2 + y^2 = 10^2 = 100$$

Differentiate both sides with respect to time t :

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow x \frac{dx}{dt} + y \frac{dy}{dt} = 0$$

We know:

$$x = 6, \quad \frac{dx}{dt} = 2$$

Solve for y :

$$36 + y^2 = 100 \Rightarrow y^2 = 64 \Rightarrow y = 8$$

Substitute into the differentiated equation:

$$6(2) + 8 \frac{dy}{dt} = 0 \Rightarrow 12 + 8 \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{12}{8} = -\frac{3}{2}$$

Answer:

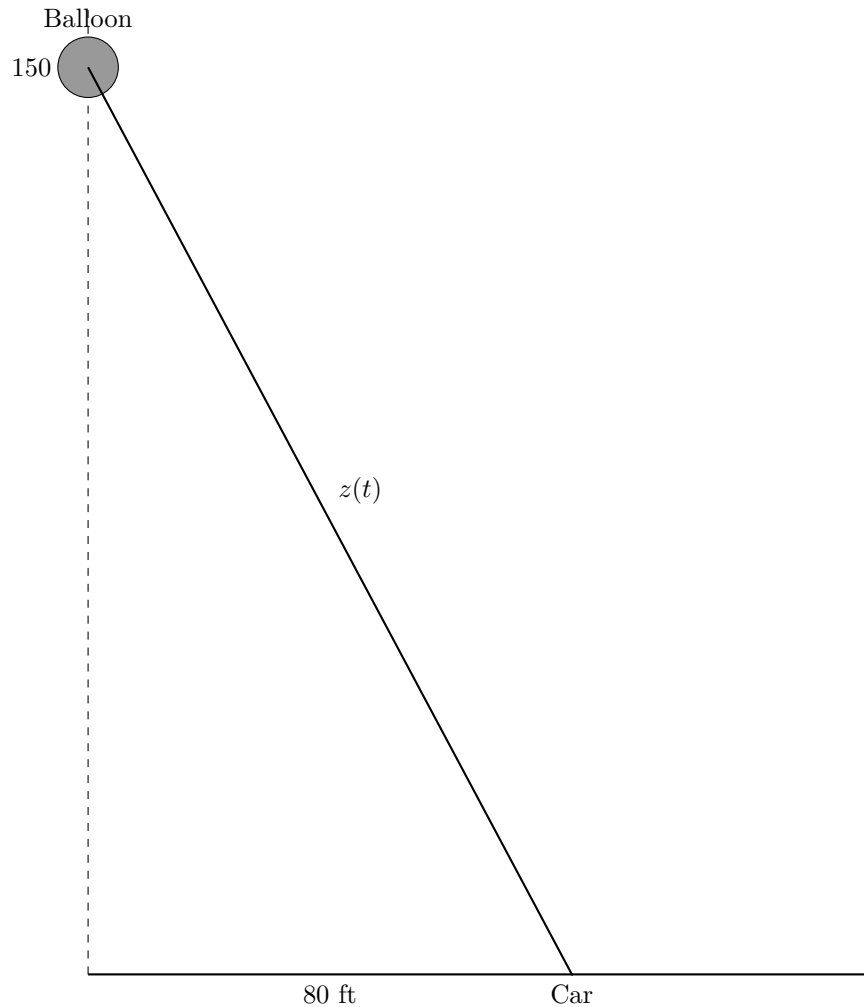
$$\boxed{\frac{dy}{dt} = -\frac{3}{2} \text{ ft/s}}$$

The negative sign indicates that the top of the ladder is sliding downward.

Question 9

Problem:

A balloon is rising vertically at a constant rate of 5 ft/s, and at the same time, a car drives directly away from the point on the ground beneath the balloon at 60 ft/s. How fast is the distance between the car and the balloon increasing when the car is 80 feet from the balloon's launch point and the balloon is 150 feet high?



Question 9 - Solution

Let:

- $x(t)$: distance from car to balloon launch point (horizontal) — $\frac{dx}{dt} = 60$
- $y(t)$: height of balloon — $\frac{dy}{dt} = 5$
- $z(t)$: distance between car and balloon (hypotenuse)

We apply the Pythagorean theorem:

$$z^2 = x^2 + y^2$$

Differentiate both sides:

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow z \frac{dz}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt}$$

Plug in values:

$$x = 80, \quad y = 150 \Rightarrow z = \sqrt{80^2 + 150^2} = \sqrt{6400 + 22500} = \sqrt{28900} = 170$$

Now plug in:

$$170 \cdot \frac{dz}{dt} = 80 \cdot 60 + 150 \cdot 5 \Rightarrow 170 \cdot \frac{dz}{dt} = 4800 + 750 = 5550 \Rightarrow \frac{dz}{dt} = \frac{5550}{170} = 32.65 \text{ ft/s}$$

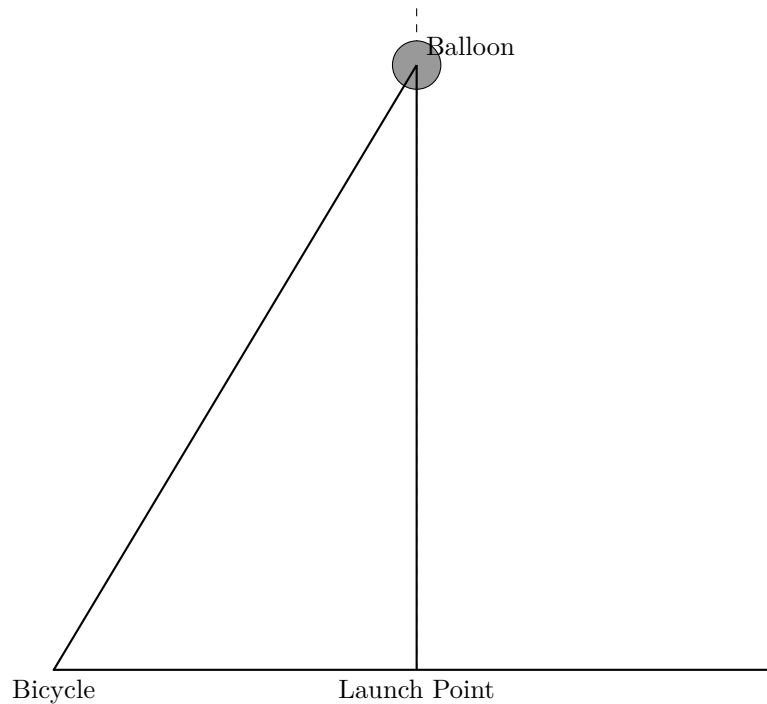
Answer:

$\frac{dz}{dt} = 32.65 \text{ ft/s}$

Question 10

Problem:

A hot air balloon is descending vertically at a rate of 2 ft/s, while a bicyclist passes directly beneath it, traveling in a straight line at a speed of 15 ft/s. How fast is the distance between the bicyclist and the balloon changing when the bicyclist is 60 feet from the point directly below the balloon and the balloon is 100 feet above the ground?



Question 10 - Solution

Let:

- $x(t)$: horizontal distance of bicycle from point below the balloon — $\frac{dx}{dt} = 15$
- $y(t)$: vertical distance of balloon from ground — $\frac{dy}{dt} = -2$
- $z(t)$: distance between balloon and bicycle

Using the Pythagorean Theorem:

$$z^2 = x^2 + y^2$$

Differentiate both sides with respect to time t :

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow z \frac{dz}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt}$$

Plug in known values:

$$x = 60, \quad y = 100, \quad z = \sqrt{60^2 + 100^2} = \sqrt{3600 + 10000} = \sqrt{13600} = 116.6$$

$$z \frac{dz}{dt} = 60(15) + 100(-2) = 900 - 200 = 700 \Rightarrow \frac{dz}{dt} = \frac{700}{116.6} \approx 6.01 \text{ ft/s}$$

Answer:

$$\boxed{\frac{dz}{dt} \approx 6.01 \text{ ft/s}}$$

Question 1

Let

$$f(x) = x^3 - 6x^2 + 9x + 1.$$

- (a) Find all critical points of $f(x)$.
- (b) Classify each critical point as a local minimum, local maximum, or neither using the First Derivative Test.

Question 1 - Solution

We are given

$$f(x) = x^3 - 6x^2 + 9x + 1.$$

(a) Find $f'(x)$:

$$f'(x) = 3x^2 - 12x + 9.$$

Set $f'(x) = 0$ to find critical points:

$$3x^2 - 12x + 9 = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x - 1)(x - 3) = 0.$$

Critical points: $x = 1, 3$

(b) First Derivative Test:

- Choose test values around $x = 1$ and $x = 3$.
- Use $f'(x) = 3x^2 - 12x + 9$.

Evaluate the sign of $f'(x)$:

- On $(-\infty, 1)$, pick $x = 0$:

$$f'(0) = 9 > 0.$$

- On $(1, 3)$, pick $x = 2$:

$$f'(2) = 3(4) - 24 + 9 = -3 < 0.$$

- On $(3, \infty)$, pick $x = 4$:

$$f'(4) = 48 - 48 + 9 = 9 > 0.$$

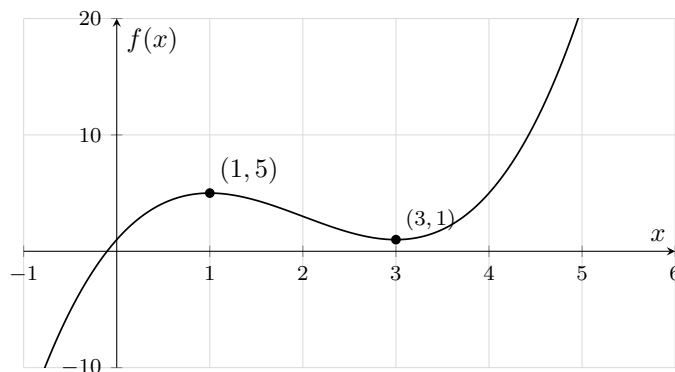
Conclusion:

- At $x = 1$, $f'(x)$ changes from positive to negative, so f has a local maximum.
- At $x = 3$, $f'(x)$ changes from negative to positive, so f has a local minimum.

Answer:

- Local maximum at $x = 1$ with $f(1) = 5$
- Local minimum at $x = 3$ with $f(3) = 1$

Graph of $f(x)$ with critical points marked



Question 2

A company models its profit $P(x)$, in thousands of dollars, as a function of the number of units x (in hundreds) it produces and sells:

$$P(x) = -2x^3 + 15x^2 + 36x.$$

- (a) Find the critical points of the function.
- (b) Use the First Derivative Test to classify each critical point as a local maximum, minimum, or neither.
- (c) How many units should be produced to maximize profit?

Question 2 - Solution

We are given

$$P(x) = -2x^3 + 15x^2 + 36x.$$

(a) Find $P'(x)$:

$$P'(x) = -6x^2 + 30x + 36.$$

Set $P'(x) = 0$ to find critical points:

$$-6x^2 + 30x + 36 = 0 \Rightarrow x^2 - 5x - 6 = 0 \Rightarrow (x - 6)(x + 1) = 0.$$

Critical points: $x = -1, 6$

(b) First Derivative Test:

Evaluate the sign of $P'(x)$ on intervals determined by the critical points.

- On $(-\infty, -1)$, pick $x = -2$:

$$P'(-2) = -6(4) + 30(-2) + 36 = -24 - 60 + 36 = -48 < 0.$$

- On $(-1, 6)$, pick $x = 0$:

$$P'(0) = 36 > 0.$$

- On $(6, \infty)$, pick $x = 7$:

$$P'(7) = -6(49) + 30(7) + 36 = -294 + 210 + 36 = -48 < 0.$$

Conclusion:

- At $x = -1$, $P'(x)$ changes from negative to positive, so P has a local minimum.
- At $x = 6$, $P'(x)$ changes from positive to negative, so P has a local maximum.

(c) Maximizing profit:

Since $x = 6$ is a local maximum, the profit is maximized at $x = 6$ (hundreds of units), which is

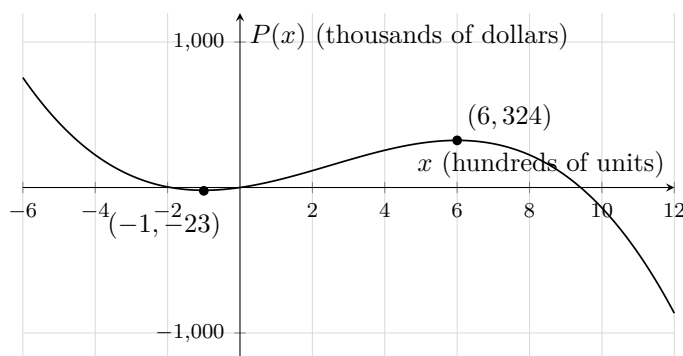
$$6 \times 100 = \boxed{600 \text{ units}}.$$

Compute the maximum profit:

$$P(6) = -2(6)^3 + 15(6)^2 + 36(6) = -432 + 540 + 216 = \boxed{324}.$$

This represents $\boxed{\$324,000}$ since P is in thousands of dollars.

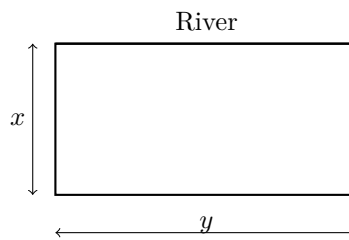
Graph of $P(x)$ with critical points marked



Question 3**Problem:**

A rectangular garden is to be built along a straight river. No fence is needed along the river, but fencing is required on the other three sides. If 120 meters of fencing is available, find the dimensions of the garden that maximize the enclosed area.

- (a) Define the objective function and constraint.
- (b) Use calculus to find the critical point(s).
- (c) Determine the dimensions that maximize the area.



Question 3 - Solution

Let x be the length perpendicular to the river, and y the length parallel to the river.

(a) Constraint:

Only three sides need fencing (two x 's and one y):

$$2x + y = 120 \quad \Rightarrow \quad y = 120 - 2x$$

Objective: Maximize the area:

$$A(x) = x \cdot y = x(120 - 2x) = 120x - 2x^2$$

(b) Find critical points:

Take the derivative:

$$A'(x) = 120 - 4x$$

Set $A'(x) = 0$:

$$120 - 4x = 0 \Rightarrow x = 30$$

(c) Check dimensions:

$$y = 120 - 2(30) = 60$$

Answer: The garden has dimensions $\boxed{30 \text{ m} \times 60 \text{ m}}$, giving a maximum area.

Question 4**Problem**

A company models its revenue $R(x)$, in thousands of dollars, as a function of the number of items x (in hundreds) it produces and sells:

$$R(x) = -x^3 + 12x^2 + 9x.$$

- (a) Find all critical points of $R(x)$.
- (b) Use the First Derivative Test to classify each critical point as a local maximum, local minimum, or neither.
- (c) How many items should be produced to maximize revenue?

Question 4 - Solution

We are given

$$R(x) = -x^3 + 12x^2 + 9x.$$

(a) Find the derivative:

$$R'(x) = -3x^2 + 24x + 9.$$

Set $R'(x) = 0$:

$$-3x^2 + 24x + 9 = 0 \Rightarrow x^2 - 8x - 3 = 0.$$

Using the quadratic formula:

$$x = \frac{8 \pm \sqrt{64 + 12}}{2} = \frac{8 \pm \sqrt{76}}{2} = 4 \pm \sqrt{19}.$$

Critical points: $x = 4 - \sqrt{19}$, $4 + \sqrt{19}$

(b) First Derivative Test

Since $R'(x)$ is a downward-opening parabola, it is negative outside its roots and positive between them.

- At $x = 4 - \sqrt{19}$, $R'(x)$ changes from negative to positive, so $R(x)$ has a local minimum.
- At $x = 4 + \sqrt{19}$, $R'(x)$ changes from positive to negative, so $R(x)$ has a local maximum.

(c) Maximizing revenue

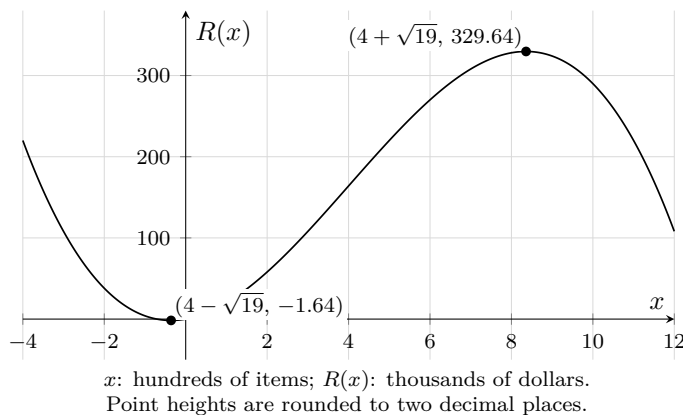
Revenue is maximized at

$$x = 4 + \sqrt{19} \approx 8.36.$$

Since x is measured in hundreds of items, this corresponds to

836 items.

Graph of $R(x)$ with critical points marked



Question 5**Problem**

Let

$$f(x) = x^4 - 4x^3 + 6x^2.$$

- (a) Find the critical points of $f(x)$.
- (b) Classify each critical point as a local maximum, local minimum, or neither.

Question 5 - Solution

We are given

$$f(x) = x^4 - 4x^3 + 6x^2.$$

(a) Find the critical points.

Differentiate:

$$f'(x) = 4x^3 - 12x^2 + 12x.$$

Factor:

$$f'(x) = 4x(x^2 - 3x + 3).$$

Set $f'(x) = 0$:

$$4x(x^2 - 3x + 3) = 0.$$

This gives $x = 0$, or $x^2 - 3x + 3 = 0$. The quadratic has discriminant

$$(-3)^2 - 4(1)(3) = 9 - 12 = -3 < 0,$$

so it has no real solutions. Therefore,

Only critical point: $x = 0$.

(b) Classify the critical point.

Compute the second derivative:

$$f''(x) = 12x^2 - 24x + 12.$$

Evaluate at $x = 0$:

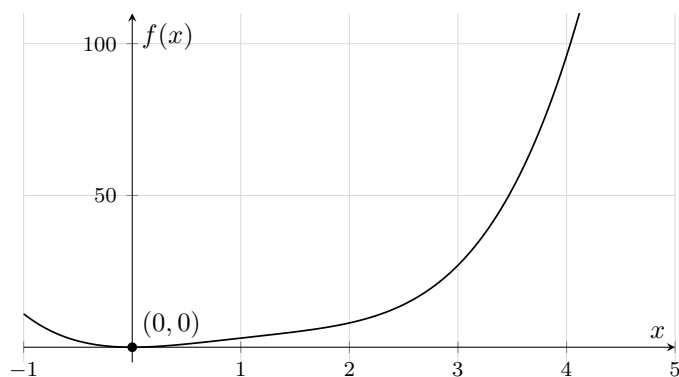
$$f''(0) = 12 > 0,$$

so $f(x)$ has a local minimum at $x = 0$. Also $f(0) = 0$, so the minimum point is $(0, 0)$.

Conclusion

Local minimum at $x = 0$ (point $(0, 0)$)

Graph of $f(x)$ with the critical point marked



Question 6**Problem:**Let $f(x) = \frac{x^3}{3} - 2x^2 + x + 5$.

- (a) Find all critical points of $f(x)$.
- (b) Use the First Derivative Test to classify each critical point as a local maximum, local minimum, or neither.

Question 6 - Solution

We are given:

$$f(x) = \frac{x^3}{3} - 2x^2 + x + 5$$

(a) **Find the critical points:**

Take the first derivative:

$$f'(x) = x^2 - 4x + 1$$

Set $f'(x) = 0$:

$$x^2 - 4x + 1 = 0 \Rightarrow x = 2 \pm \sqrt{3}$$

Critical points: $x = 2 + \sqrt{3}$, $x = 2 - \sqrt{3}$

(b) **First Derivative Test:**

Evaluate $f'(x)$ on intervals around the critical points:

Interval $(-\infty, 2 - \sqrt{3})$: pick $x = 0$, $f'(0) = 1 > 0$

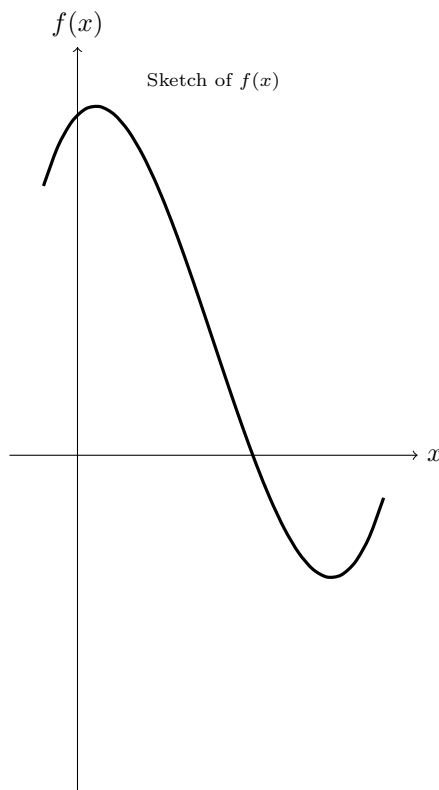
Interval $(2 - \sqrt{3}, 2 + \sqrt{3})$: pick $x = 2$, $f'(2) = -3 < 0$

Interval $(2 + \sqrt{3}, \infty)$: pick $x = 4$, $f'(4) = 1 > 0$

Conclusion:

$f'(x)$ changes from positive to negative at $x = 2 - \sqrt{3} \rightarrow$ local maximum $f'(x)$ changes from negative to positive at $x = 2 + \sqrt{3} \rightarrow$ local minimum

Local max at $x = 2 - \sqrt{3}$
Local min at $x = 2 + \sqrt{3}$



Question 7**Problem:**

A company's profit function (in thousands of dollars) from selling x hundred units of a product is given by:

$$P(x) = -x^3 + 6x^2 + 60$$

- (a) Find all critical points of $P(x)$.
- (b) Determine whether each critical point is a local maximum or minimum using the First Derivative Test.
- (c) Interpret your result in the context of the company's production.

Question 7 - Solution

(a) Find critical points:

Take the derivative:

$$P'(x) = -3x^2 + 12x$$

Set $P'(x) = 0$:

$$-3x^2 + 12x = 0 \Rightarrow x(-3x + 12) = 0 \Rightarrow x = 0 \text{ or } x = 4$$

Critical points: $x = 0$, $x = 4$

(b) First Derivative Test:

Check intervals: For $x < 0$: not relevant (negative production)Between $0 < x < 4$: pick $x = 2$, $P'(2) = -3(4) + 12(2) = -12 + 24 = 12 > 0$ For $x > 4$: pick $x = 5$, $P'(5) = -3(25) + 12(5) = -75 + 60 = -15 < 0$

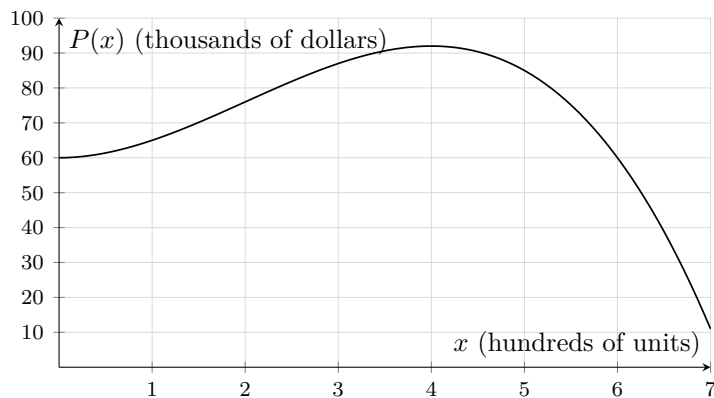
Conclusion: $P(x)$ increases before $x = 4$, decreases after $\rightarrow$ local max at $x = 4$
 $x = 0$ is a stationary point, but no increase before it (since domain is $x \geq 0$)

(c) Interpretation:

Producing 400 units ($x = 4$) yields the maximum profit. Profit decreases beyond 400 units, and no benefit is gained producing less than 400.

Maximum profit occurs when the company produces 400 units.

Profit Function Graph:



Question 8**Problem:**

A ball is launched vertically upward, and its height in meters at time t seconds is modeled by:

$$h(t) = -5t^2 + 20t + 1$$

- (a) Find all critical points of the function $h(t)$.
- (b) Use the First Derivative Test to classify each critical point.
- (c) What is the maximum height reached by the ball and at what time?

Question 8 - Solution

(a) Find critical points:

Take the derivative:

$$h'(t) = -10t + 20$$

Set $h'(t) = 0$:

$$-10t + 20 = 0 \Rightarrow t = 2$$

Critical point: $t = 2$

(b) First Derivative Test:

For $t < 2$: pick $t = 1$, $h'(1) = -10(1) + 20 = 10 > 0$ For $t > 2$: pick $t = 3$, $h'(3) = -10(3) + 20 = -10 < 0$ Since $h'(t)$ changes from positive to negative at $t = 2$, this is a local maximum.

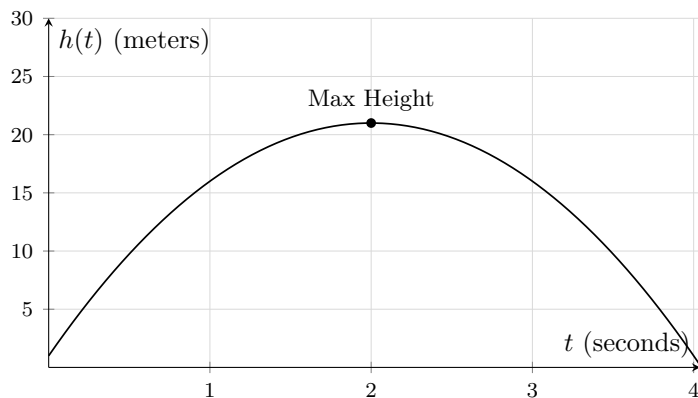
(c) Maximum Height:

Substitute $t = 2$ into $h(t)$:

$$h(2) = -5(4) + 20(2) + 1 = -20 + 40 + 1 = 21$$

Answer: The maximum height is 21 meters, reached at $t = 2$ seconds.

Graph of the Trajectory:



Question 9**Problem:**

Let the function be defined as:

$$f(x) = x^3 - 3x^2 - 9x + 5$$

- (a) Find all critical points of $f(x)$.
- (b) Use the First Derivative Test to classify each critical point as a local maximum or local minimum.
- (c) Determine the function value at each critical point.

Question 9 - Solution

(a) Find critical points:

Differentiate:

$$f'(x) = 3x^2 - 6x - 9$$

Set $f'(x) = 0$:

$$3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x - 3)(x + 1) = 0 \Rightarrow x = -1, 3$$

Critical points: $x = -1, x = 3$

(b) First Derivative Test:

Interval $(-\infty, -1)$: Choose $x = -2$, $f'(-2) = 3(4) + 12 - 9 = 15 > 0$ Interval $(-1, 3)$: Choose $x = 0$, $f'(0) = -9 < 0$ Interval $(3, \infty)$: Choose $x = 4$, $f'(4) = 3(16) - 24 - 9 = 15 > 0$ So: At $x = -1$, f' changes from positive to negative $\rightarrow$ local maximum At $x = 3$, f' changes from negative to positive $\rightarrow$ local minimum

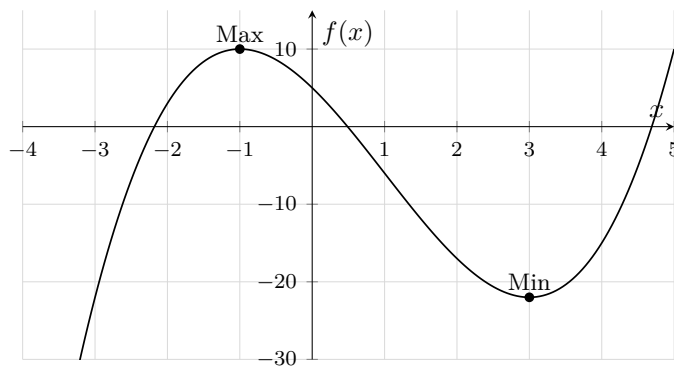
(c) Function values:

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = -1 - 3 + 9 + 5 = 10$$

$$f(3) = 27 - 27 - 27 + 5 = -22$$

Local maximum: $f(-1) = 10$ Local minimum: $f(3) = -22$

Graph of the function:



Question 10**Problem:**

Consider the function:

$$f(x) = x^4 - 4x^3 + 6x^2 + 1$$

- (a) Find all critical points of $f(x)$.
- (b) Classify each critical point using the First Derivative Test.
- (c) Determine the function value at each critical point.

Question 10 - Solution

(a) Find critical points:

First, compute the derivative:

$$f'(x) = 4x^3 - 12x^2 + 12x$$

Factor:

$$f'(x) = 4x(x^2 - 3x + 3)$$

Set derivative to zero:

$$4x(x^2 - 3x + 3) = 0 \Rightarrow x = 0 \quad (\text{real root})$$

Solve $x^2 - 3x + 3 = 0$:

$$\text{Discriminant: } D = (-3)^2 - 4(1)(3) = 9 - 12 = -3 \Rightarrow \text{no real roots}$$

Critical point: $x = 0$

(b) **First Derivative Test:** Test sign of $f'(x)$ around $x = 0$:

For $x < 0$: try $x = -1$

$$f'(-1) = 4(-1)^3 - 12(-1)^2 + 12(-1) = -4 - 12 - 12 = -28 < 0$$

For $x > 0$: try $x = 1$

$$f'(1) = 4(1)^3 - 12(1)^2 + 12(1) = 4 - 12 + 12 = 4 > 0$$

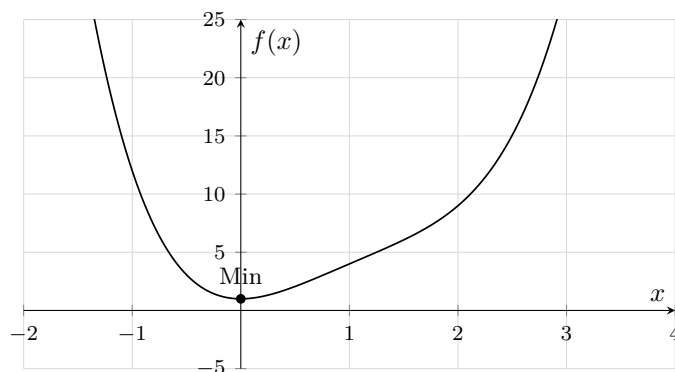
Since $f'(x)$ changes from negative to positive at $x = 0$, there is a local minimum.

(c) **Function value at critical point:**

$$f(0) = 0^4 - 4(0)^3 + 6(0)^2 + 1 = 1$$

Local minimum at: $(0, 1)$

Graph of the function:



Question 1

Let:

$$f(x) = 2x^3 - 9x^2 + 12x + 1$$

- (a) Find the critical points of $f(x)$.
- (b) Use the second derivative test to determine whether each critical point is a local minimum, maximum, or neither.

Question 1 - Solution

We are given:

$$f(x) = 2x^3 - 9x^2 + 12x + 1$$

(a) First derivative:

$$f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$$

Critical points: $x = 1, 2$

(b) Use the second derivative:

$$f''(x) = \frac{d}{dx}(f'(x)) = 12x - 18$$

Evaluate at each critical point:

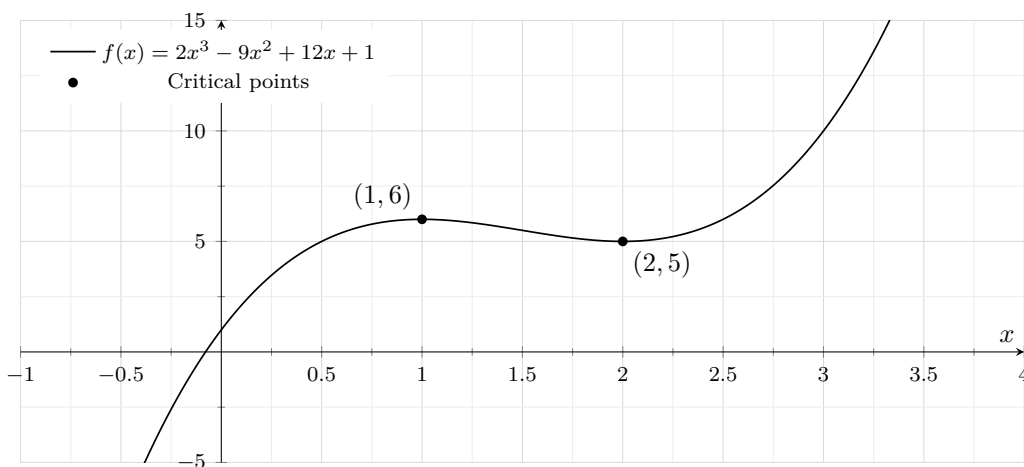
$$f''(1) = 12(1) - 18 = -6 \Rightarrow \text{local maximum at } x = 1$$

$$f''(2) = 12(2) - 18 = 6 \Rightarrow \text{local minimum at } x = 2$$

Answer:

Local maximum at $x = 1$

Local minimum at $x = 2$

Graph of $f(x)$ 

Question 2**Problem:**

Let $f(x) = x^3 - 6x^2 + 9x + 2$ on the closed interval $[0, 4]$.

- (a) Find the critical points of $f(x)$ in the interval.
- (b) Determine the absolute maximum and minimum values of $f(x)$ on the interval $[0, 4]$.

Question 2 - Solution**(a) Find critical points:**

First, compute the derivative:

$$f'(x) = 3x^2 - 12x + 9$$

Set the derivative to zero:

$$3x^2 - 12x + 9 = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x - 1)(x - 3) = 0 \Rightarrow x = 1, x = 3$$

Both critical points are in the interval $[0, 4]$.

(b) Evaluate function at critical points and endpoints:

$$f(0) = 0 - 0 + 0 + 2 = 2$$

$$f(1) = 1 - 6 + 9 + 2 = 6$$

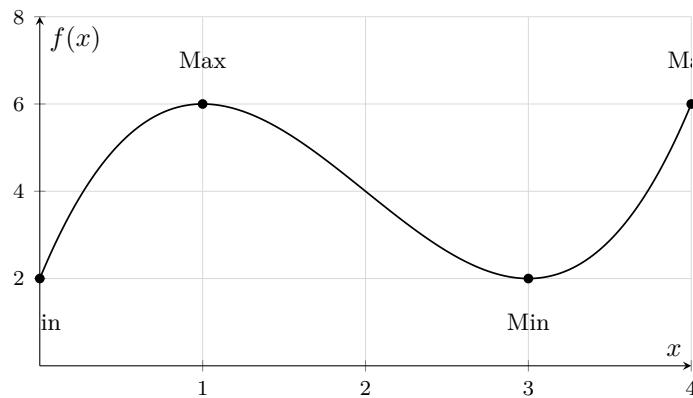
$$f(3) = 27 - 54 + 27 + 2 = 2$$

$$f(4) = 64 - 96 + 36 + 2 = 6$$

Conclusion:

- Maximum value: $\boxed{6}$ at $x = 1$ and $x = 4$ - Minimum value: $\boxed{2}$ at $x = 0$ and $x = 3$

Graph of the function on $[0, 4]$:



Question 3**Problem:**

Consider the function $f(x) = x^4 - 4x^3 + 10$ on the interval $[0, 4]$.

- (a) Find the critical points of $f(x)$ on the interval.
- (b) Determine the absolute maximum and minimum values of $f(x)$ on the interval $[0, 4]$.

Question 3 - Solution**(a) Find critical points:**

First, compute the derivative:

$$f'(x) = 4x^3 - 12x^2$$

Factor:

$$f'(x) = 4x^2(x - 3) \Rightarrow f'(x) = 0 \text{ when } x = 0 \text{ or } x = 3$$

Both are within the interval $[0, 4]$, so check both.

(b) Evaluate function at critical points and endpoints:

$$f(0) = 0 - 0 + 10 = 10$$

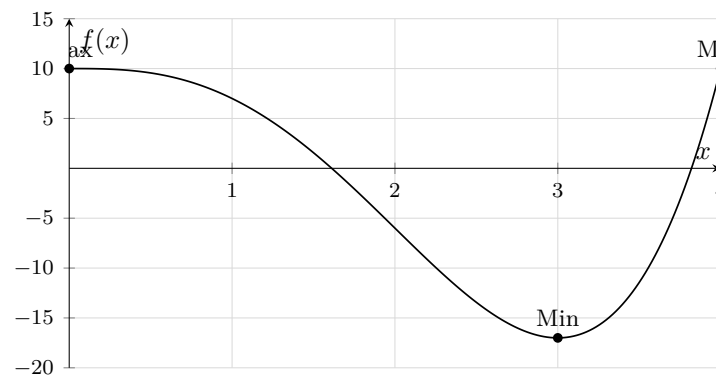
$$f(3) = 81 - 108 + 10 = -17$$

$$f(4) = 256 - 256 + 10 = 10$$

Conclusion:

- Minimum value: -17 at $x = 3$ - Maximum value: 10 at $x = 0$ and $x = 4$

Graph of the function on $[0, 4]$:



Question 4**Problem:**

Let $f(x) = \frac{x}{x^2+1}$, and consider the interval $[-2, 2]$.

- (a) Find the critical points of $f(x)$ on the interval.
- (b) Determine the absolute maximum and minimum values of $f(x)$ on $[-2, 2]$.

Question 4 - Solution**(a) Find the critical points:**

We find $f'(x)$ using the quotient rule:

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

Set the numerator equal to zero:

$$1 - x^2 = 0 \Rightarrow x = \pm 1$$

Both $x = -1$ and $x = 1$ are in the interval $[-2, 2]$.

(b) Evaluate at critical points and endpoints:

$$f(-2) = \frac{-2}{4 + 1} = \frac{-2}{5}$$

$$f(-1) = \frac{-1}{1 + 1} = \frac{-1}{2}$$

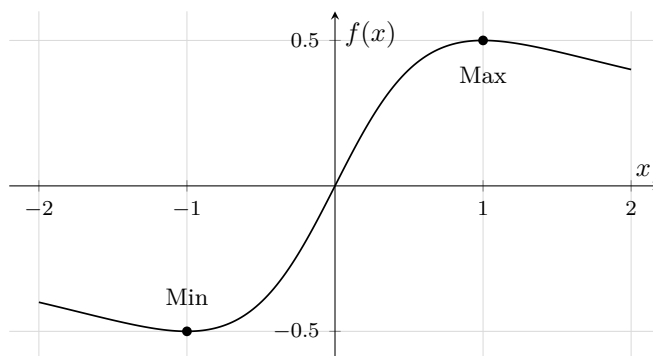
$$f(1) = \frac{1}{1 + 1} = \frac{1}{2}$$

$$f(2) = \frac{2}{4 + 1} = \frac{2}{5}$$

Conclusion:

- Maximum value: $\boxed{\frac{1}{2}}$ at $x = 1$ - Minimum value: $\boxed{-\frac{1}{2}}$ at $x = -1$

Graph of $f(x) = \frac{x}{x^2+1}$ on $[-2, 2]$:



Question 5**Problem:**

Let $f(x) = x^3 - 3x + 1$ on the interval $[-3, 3]$.

- (a) Find all critical points of $f(x)$ on the interval.
- (b) Determine the absolute maximum and minimum values of $f(x)$ on $[-3, 3]$.

Question 5 - Solution**(a) Find critical points:**

Take the derivative:

$$f'(x) = 3x^2 - 3$$

Set derivative equal to 0:

$$3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

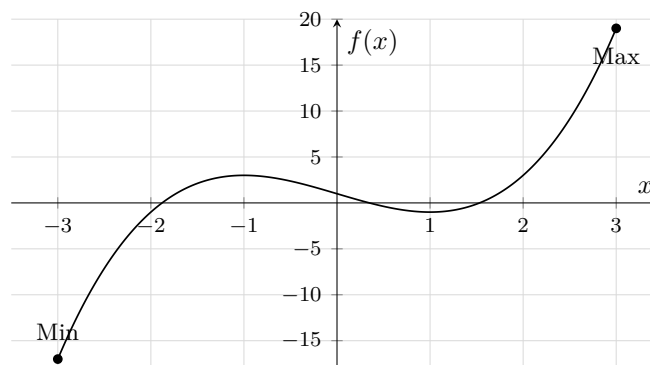
Critical points: $x = -1, x = 1$ **(b) Evaluate at critical points and endpoints:**

$$f(-3) = (-3)^3 - 3(-3) + 1 = -27 + 9 + 1 = -17$$

$$f(-1) = (-1)^3 - 3(-1) + 1 = -1 + 3 + 1 = 3$$

$$f(1) = 1^3 - 3(1) + 1 = 1 - 3 + 1 = -1$$

$$f(3) = 27 - 9 + 1 = 19$$

Conclusion:- Absolute maximum: $\boxed{19}$ at $x = 3$ - Absolute minimum: $\boxed{-17}$ at $x = -3$ **Graph of** $f(x) = x^3 - 3x + 1$ on $[-3, 3]$:

Question 6**Problem:**

Consider the function $f(x) = x^4 - 4x^2$ on the interval $[-3, 3]$.

- (a) Find all critical points of $f(x)$ in the interval.
- (b) Determine the absolute maximum and minimum values of $f(x)$ on $[-3, 3]$.

Question 6 - Solution

(a) Find critical points:

Differentiate:

$$f'(x) = 4x^3 - 8x$$

Set derivative to zero:

$$4x^3 - 8x = 0 \Rightarrow 4x(x^2 - 2) = 0 \Rightarrow x = 0, \pm\sqrt{2}$$

Critical points in $[-3, 3]$: $x = -\sqrt{2}, 0, \sqrt{2}$ (b) Evaluate $f(x)$ at critical points and endpoints:

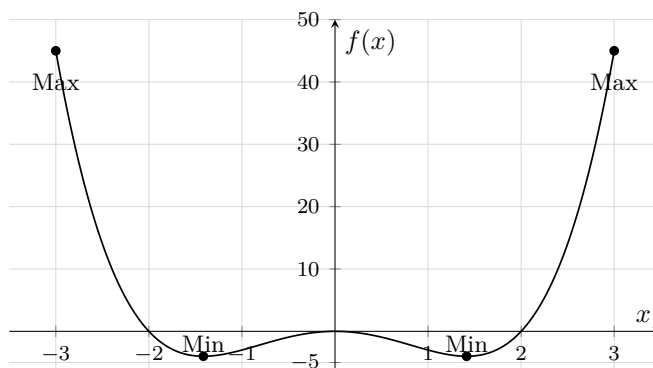
$$f(-3) = (-3)^4 - 4(-3)^2 = 81 - 36 = 45$$

$$f(-\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2 = 4 - 8 = -4$$

$$f(0) = 0^4 - 4(0)^2 = 0$$

$$f(\sqrt{2}) = 4 - 8 = -4$$

$$f(3) = 81 - 36 = 45$$

Conclusion:- Absolute maximum: $\boxed{45}$ at $x = -3$ and $x = 3$ - Absolute minimum: $\boxed{-4}$ at $x = \pm\sqrt{2}$ **Graph of** $f(x) = x^4 - 4x^2$ on $[-3, 3]$:

Question 7**Problem:**

Let $f(x) = x^3 - 3x^2 + 1$ on the interval $[0, 4]$.

- (a) Find all critical points of $f(x)$ in the interval.
- (b) Determine the absolute maximum and minimum values of $f(x)$ on the given interval.

Question 7 - Solution

(a) Find critical points:

Differentiate:

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

Set $f'(x) = 0$ to find critical points:

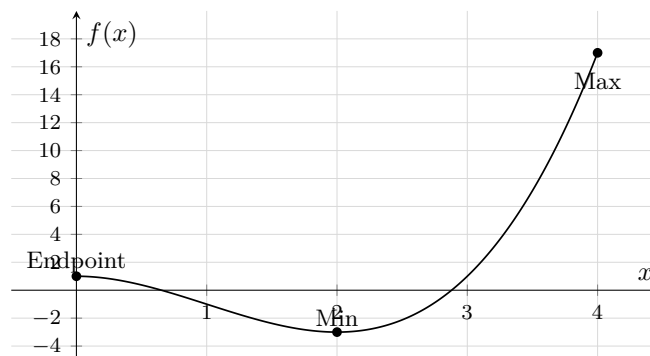
$$3x(x - 2) = 0 \Rightarrow x = 0, x = 2$$

(b) Evaluate function at critical points and endpoints:

$$f(0) = 0^3 - 3(0)^2 + 1 = 1$$

$$f(2) = 8 - 12 + 1 = -3$$

$$f(4) = 64 - 48 + 1 = 17$$

Conclusion:- Absolute maximum: $\boxed{17}$ at $x = 4$ - Absolute minimum: $\boxed{-3}$ at $x = 2$ **Graph of** $f(x) = x^3 - 3x^2 + 1$ on $[0, 4]$:

Question 8**Problem:**

Let $f(x) = \frac{x}{x^2+1}$ on the interval $[-3, 3]$.

- (a) Find all critical points of $f(x)$ in the interval.
- (b) Determine the absolute maximum and minimum values of $f(x)$ on the given interval.

Question 8 - Solution

(a) Find critical points:

Differentiate using the quotient rule:

$$f(x) = \frac{x}{x^2 + 1}, \quad f'(x) = \frac{(1)(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

Set $f'(x) = 0$:

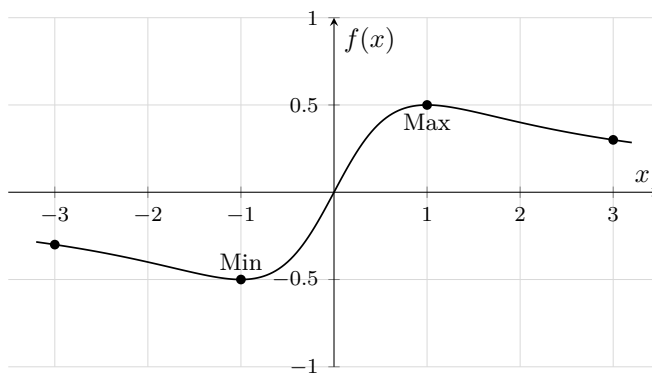
$$1 - x^2 = 0 \Rightarrow x = \pm 1$$

(b) Evaluate function at endpoints and critical points:

$$f(-3) = \frac{-3}{9+1} = -\frac{3}{10} \quad f(-1) = \frac{-1}{2} = -\frac{1}{2} \quad f(1) = \frac{1}{2} \quad f(3) = \frac{3}{10}$$

Conclusion:

- Absolute maximum: $\boxed{\frac{1}{2}}$ at $x = 1$ - Absolute minimum: $\boxed{-\frac{1}{2}}$ at $x = -1$

Graph of $f(x) = \frac{x}{x^2+1}$ on $[-3, 3]$:

Question 9**Problem:**

A farmer has 240 meters of fencing to enclose a rectangular field. One side of the rectangle lies along a river and does not require fencing.

- (a) Express the area of the field as a function of one variable.
- (b) Find the dimensions of the field that maximize the area.
- (c) What is the maximum area?

Question 9 - Solution**(a) Express the area as a function:**

Let x be the length of the side perpendicular to the river. Since only three sides are fenced (two x 's and one along the length), the total fencing is:

$$2x + y = 240 \Rightarrow y = 240 - 2x$$

Area:

$$A(x) = x \cdot y = x(240 - 2x) = 240x - 2x^2$$

(b) Maximize the area:

Differentiate:

$$A'(x) = 240 - 4x$$

Set derivative to zero:

$$240 - 4x = 0 \Rightarrow x = 60$$

Check that this gives a maximum using second derivative:

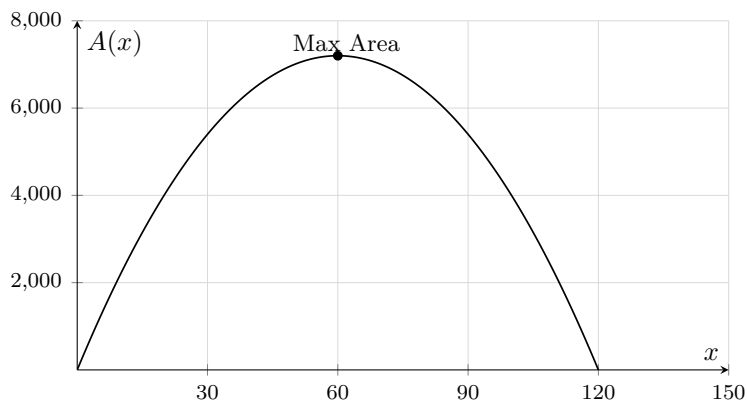
$$A''(x) = -4 < 0 \Rightarrow \text{Maximum}$$

Find y :

$$y = 240 - 2(60) = 120$$

Dimensions: 60 m $\times$ 120 m**(c) Maximum area:**

$$A = 60 \cdot 120 = \text{7200 m}^2$$

Graph of $A(x) = 240x - 2x^2$:

Question 10**Problem:**

A box with a square base and open top must have a volume of 500 cm^3 . Find the dimensions of the box that minimize the surface area.

- (a) Express the surface area as a function of one variable.
- (b) Find the dimensions that minimize the surface area.
- (c) What is the minimum surface area?

Question 10 - Solution

Let x be the side length of the square base, and h be the height of the box.

(a) Surface area as a function of one variable:

Volume constraint:

$$x^2 h = 500 \Rightarrow h = \frac{500}{x^2}$$

Surface area (no top):

$$A(x) = x^2 + 4xh = x^2 + 4x \cdot \frac{500}{x^2} = x^2 + \frac{2000}{x}$$

(b) Minimize the surface area:

Differentiate:

$$A'(x) = 2x - \frac{2000}{x^2}$$

Set derivative to 0:

$$2x - \frac{2000}{x^2} = 0 \Rightarrow 2x^3 = 2000 \Rightarrow x^3 = 1000 \Rightarrow x = 10$$

Find h :

$$h = \frac{500}{10^2} = 5$$

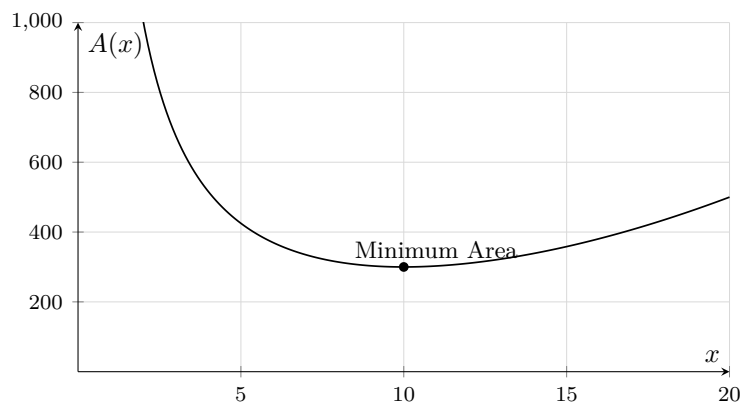
Dimensions:

$$x = 10 \text{ cm}, h = 5 \text{ cm}$$

(c) Minimum surface area:

$$A = 10^2 + \frac{2000}{10} = 100 + 200 = \boxed{300 \text{ cm}^2}$$

Graph of $A(x) = x^2 + \frac{2000}{x}$:



Question 1

Let:

$$f(x) = x^3 - 6x^2 + 9x + 2$$

Find the absolute maximum and minimum values of $f(x)$ on the closed interval $[0, 4]$.

Question 1 - Solution

We are given:

$$f(x) = x^3 - 6x^2 + 9x + 2$$

Step 1: Find critical points

$$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0 \Rightarrow x = 1, 3$$

Step 2: Evaluate at endpoints and critical points in $[0, 4]$

$$f(0) = 2$$

$$f(1) = 6$$

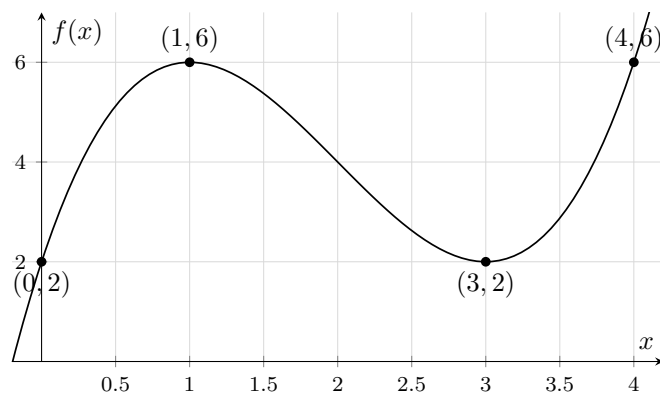
$$f(3) = 2$$

$$f(4) = 6$$

Absolute Maximum: $\boxed{6}$ at $x = 1$ and $x = 4$

Absolute Minimum: $\boxed{2}$ at $x = 0$ and $x = 3$

Graph of the function on $[0, 4]$



Question 2**Problem:**

Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1$$

on the interval $[0, 4]$.

- (a) Identify all critical points of $f(x)$ in the interval.
- (b) Evaluate $f(x)$ at the endpoints and critical points.
- (c) Determine the absolute maximum and minimum values.

Question 2 - Solution

We are given:

$$f(x) = x^3 - 3x^2 + 1$$

(a) **Find critical points:**

Take the derivative:

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

Set $f'(x) = 0$:

$$3x(x - 2) = 0 \Rightarrow x = 0, x = 2$$

Both are in the interval $[0, 4]$.

(b) **Evaluate function at critical points and endpoints:**

$$f(0) = 0^3 - 3(0)^2 + 1 = 1$$

$$f(2) = 8 - 12 + 1 = -3$$

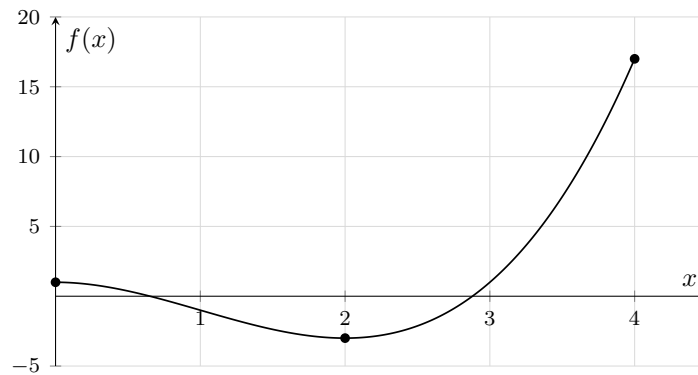
$$f(4) = 64 - 48 + 1 = 17$$

(c) **Absolute extrema:**

• Absolute minimum: $f(2) = -3$

• Absolute maximum: $f(4) = 17$

Graph of $f(x) = x^3 - 3x^2 + 1$ on $[0, 4]$:



Question 3**Problem:**

Find the absolute maximum and minimum values of the function

$$f(x) = \frac{x}{x^2 + 1}$$

on the interval $[-2, 2]$.

- (a) Find the critical points of $f(x)$ in the interval.
- (b) Evaluate $f(x)$ at the endpoints and at critical points.
- (c) Determine the absolute maximum and minimum values.

Question 3 - Solution

We are given:

$$f(x) = \frac{x}{x^2 + 1}$$

(a) Find critical points:

Use quotient rule:

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

Set $f'(x) = 0$:

$$\frac{1 - x^2}{(x^2 + 1)^2} = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1$$

So, critical points: $x = -1, x = 1$

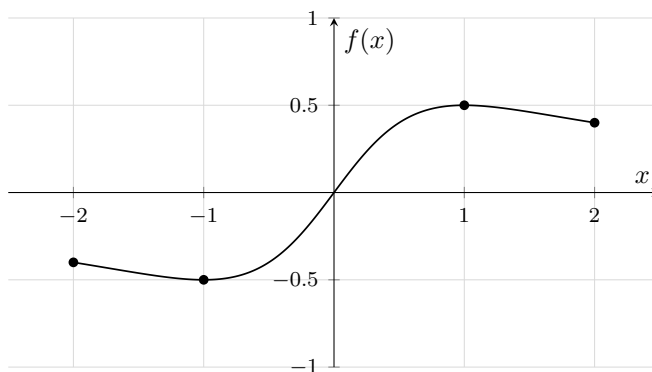
(b) Evaluate at endpoints and critical points:

$$f(-2) = \frac{-2}{4 + 1} = -\frac{2}{5}, \quad f(-1) = \frac{-1}{1 + 1} = -\frac{1}{2}$$
$$f(1) = \frac{1}{2}, \quad f(2) = \frac{2}{5}$$

(c) Conclusion:

- Absolute minimum: $f(-1) = -\frac{1}{2}$
- Absolute maximum: $f(1) = \frac{1}{2}$

Graph of $f(x) = \frac{x}{x^2 + 1}$ on $[-2, 2]$:



Question 4**Problem:**

Find the absolute maximum and minimum values of the function

$$f(x) = x^2 \ln(x)$$

on the interval $[1, 4]$.

- (a) Find the critical points of $f(x)$ in the interval.
- (b) Evaluate $f(x)$ at the endpoints and critical points.
- (c) Determine the absolute maximum and minimum values.

Question 4 - Solution

We are given:

$$f(x) = x^2 \ln(x)$$

(a) Find critical points:

Use product rule:

$$f'(x) = \frac{d}{dx}[x^2] \cdot \ln(x) + x^2 \cdot \frac{d}{dx}[\ln(x)] = 2x \ln(x) + x$$

Set derivative to zero:

$$f'(x) = 2x \ln(x) + x = x(2 \ln(x) + 1) = 0 \Rightarrow 2 \ln(x) + 1 = 0 \Rightarrow \ln(x) = -\frac{1}{2} \Rightarrow x = e^{-1/2}$$

Since $e^{-1/2} \approx 0.606 \notin [1, 4]$, there are no critical points in the interval.

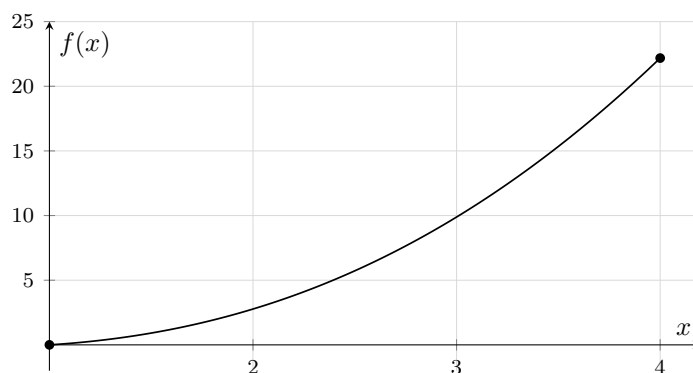
(b) Evaluate at endpoints:

$$f(1) = 1^2 \ln(1) = 0, \quad f(4) = 16 \ln(4) = 16 \cdot \ln(2^2) = 16 \cdot 2 \ln(2) \approx 16 \cdot 1.386 = 22.18$$

(c) Conclusion:

- Absolute minimum: $f(1) = 0$
- Absolute maximum: $f(4) \approx 22.18$

Graph of $f(x) = x^2 \ln(x)$ on $[1, 4]$:



Question 5**Problem:**

Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x + 1$$

on the interval $[-2, 2]$.

- (a) Find the critical points of $f(x)$ in the interval.
- (b) Evaluate $f(x)$ at the endpoints and critical points.
- (c) Identify the absolute maximum and minimum values.

Question 5 - Solution

We are given:

$$f(x) = x^3 - 3x + 1$$

(a) Find critical points:

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x - 1)(x + 1)$$

Set derivative to zero:

$$f'(x) = 0 \Rightarrow x = \pm 1$$

Critical points: $x = -1, x = 1$

(b) Evaluate function:

$$f(-2) = (-2)^3 - 3(-2) + 1 = -8 + 6 + 1 = -1$$

$$f(-1) = (-1)^3 - 3(-1) + 1 = -1 + 3 + 1 = 3$$

$$f(1) = 1 - 3 + 1 = -1$$

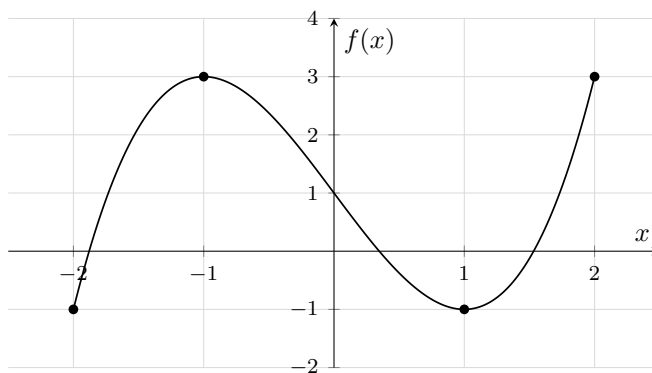
$$f(2) = 8 - 6 + 1 = 3$$

(c) Conclusion:

• Absolute maximum: $f(-1) = f(2) = 3$

• Absolute minimum: $f(-2) = f(1) = -1$

Graph of $f(x) = x^3 - 3x + 1$ on $[-2, 2]$:



Question 6**Problem:**

Let $f(x) = x^4 - 4x^2$.

- (a) Find the critical points of $f(x)$ in the interval $[-3, 3]$.
- (b) Evaluate the function at the critical points and endpoints.
- (c) Identify the absolute maximum and minimum values of $f(x)$ on the given interval.

Question 6 - Solution

We are given:

$$f(x) = x^4 - 4x^2$$

(a) Find critical points:

$$f'(x) = 4x^3 - 8x = 4x(x^2 - 2)$$

Set derivative to zero:

$$f'(x) = 0 \Rightarrow x = 0, \pm\sqrt{2}$$

(b) Evaluate at critical points and endpoints:

$$f(-3) = (-3)^4 - 4(-3)^2 = 81 - 36 = 45$$

$$f(-\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2 = 4 - 8 = -4$$

$$f(0) = 0^4 - 4(0)^2 = 0$$

$$f(\sqrt{2}) = \text{same as } f(-\sqrt{2}) = -4$$

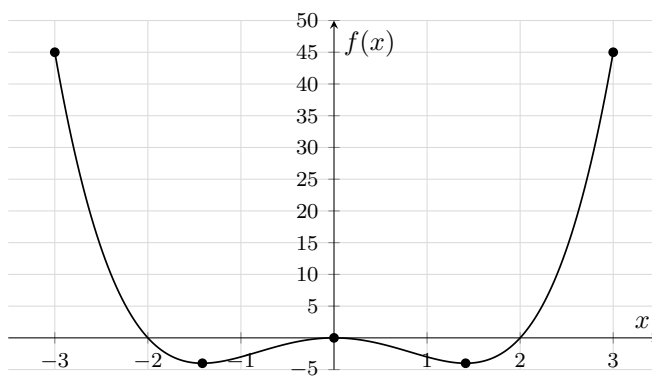
$$f(3) = 81 - 36 = 45$$

(c) Conclusion:

- Absolute maximum: $f(-3) = f(3) = 45$

- Absolute minimum: $f(\pm\sqrt{2}) = -4$

Graph of $f(x) = x^4 - 4x^2$ on $[-3, 3]$:



Question 7**Problem:**

A company models its daily profit (in hundreds of dollars) as a function of the number of units produced, using the function:

$$P(x) = -x^3 + 6x^2 + 15$$

where x is the number of units (in hundreds) produced per day, and $x \in [0, 5]$.

- (a) Find all critical points of $P(x)$ on the interval.
- (b) Determine the absolute maximum and minimum profit on the interval $[0, 5]$.

Question 7 - Solution

We are given:

$$P(x) = -x^3 + 6x^2 + 15$$

(a) Find critical points:

Differentiate:

$$P'(x) = -3x^2 + 12x = -3x(x - 4)$$

Set derivative equal to zero:

$$-3x(x - 4) = 0 \Rightarrow x = 0, x = 4$$

These are critical points in the interval $[0, 5]$.

(b) Evaluate endpoints and critical points:

$$P(0) = -0 + 0 + 15 = 15$$

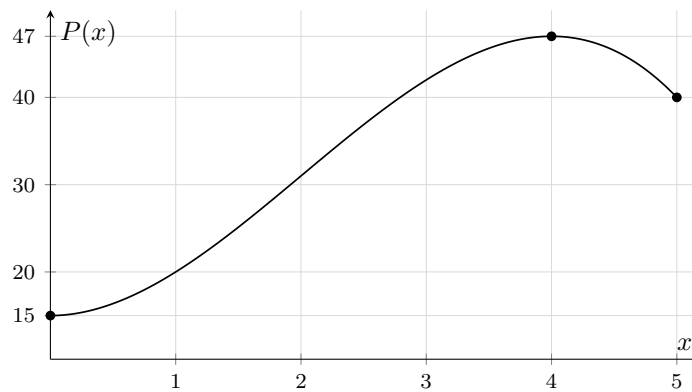
$$P(4) = -64 + 96 + 15 = 47$$

$$P(5) = -125 + 150 + 15 = 40$$

Conclusion:

- Absolute Maximum: $P(4) = 47$
- Absolute Minimum: $P(0) = 15$

Graph of $P(x) = -x^3 + 6x^2 + 15$ on $[0, 5]$:



Question 8**Problem:**

Let $f(x) = \frac{x}{x^2+1}$.

- (a) Find all critical points of $f(x)$ on the interval $[-3, 3]$.
- (b) Evaluate $f(x)$ at the critical points and endpoints.
- (c) Determine the absolute maximum and minimum values of $f(x)$ on the given interval.

Question 8 - Solution

We are given:

$$f(x) = \frac{x}{x^2 + 1}$$

(a) Find critical points:

Use the quotient rule:

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{-x^2 + 1}{(x^2 + 1)^2}$$

Set numerator to 0:

$$-x^2 + 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

(b) Evaluate at endpoints and critical points:

$$f(-3) = \frac{-3}{9 + 1} = -\frac{3}{10}$$

$$f(-1) = \frac{-1}{1 + 1} = -\frac{1}{2}$$

$$f(1) = \frac{1}{1 + 1} = \frac{1}{2}$$

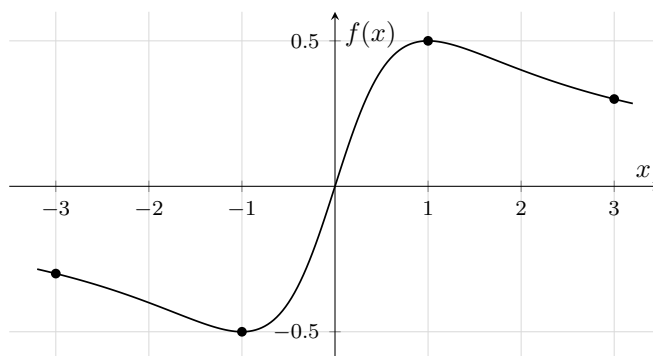
$$f(3) = \frac{3}{9 + 1} = \frac{3}{10}$$

(c) Conclusion:

• Absolute maximum: $f(1) = \frac{1}{2}$

• Absolute minimum: $f(-1) = -\frac{1}{2}$

Graph of $f(x) = \frac{x}{x^2 + 1}$ on $[-3, 3]$:



Question 9**Problem:**

Let $f(x) = x^3 - 3x^2 + 1$.

- (a) Find the critical points of $f(x)$ on the interval $[0, 3]$.
- (b) Evaluate the function at the critical points and endpoints.
- (c) Determine the absolute maximum and minimum values of $f(x)$ on the interval.

Question 9 - Solution

We are given:

$$f(x) = x^3 - 3x^2 + 1$$

(a) Find critical points:

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

Set derivative to zero:

$$3x(x - 2) = 0 \Rightarrow x = 0, x = 2$$

(b) Evaluate $f(x)$ at critical points and endpoints:

$$f(0) = 0 - 0 + 1 = 1$$

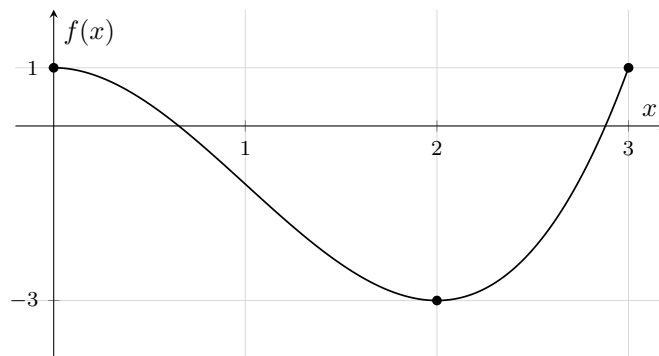
$$f(2) = 8 - 12 + 1 = -3$$

$$f(3) = 27 - 27 + 1 = 1$$

(c) Absolute extrema:

- Absolute minimum: $f(2) = -3$
- Absolute maximum: $f(0) = f(3) = 1$

Graph of $f(x) = x^3 - 3x^2 + 1$ on $[0, 3]$:



Question 10**Problem:**

Let $f(x) = \frac{x}{x^2+1}$.

- (a) Find the critical points of $f(x)$ on the interval $[-2, 2]$.
- (b) Evaluate the function at the critical points and endpoints.
- (c) Determine the absolute maximum and minimum values of $f(x)$ on the interval.

Question 10 - Solution

We are given:

$$f(x) = \frac{x}{x^2 + 1}$$

(a) Find critical points:

Use quotient rule:

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

Set $f'(x) = 0$:

$$1 - x^2 = 0 \Rightarrow x = \pm 1$$

(b) Evaluate $f(x)$:

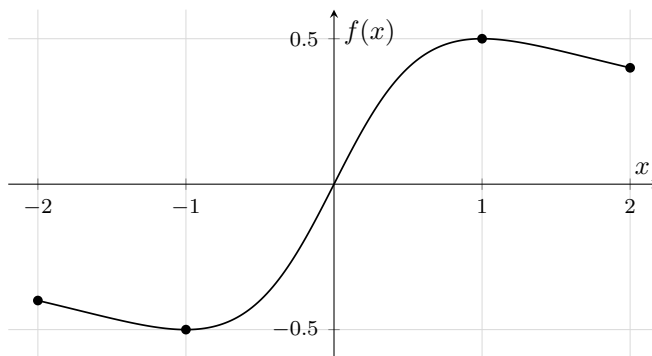
$$f(-2) = \frac{-2}{4+1} = -\frac{2}{5}, \quad f(-1) = \frac{-1}{1+1} = -\frac{1}{2}, \quad f(1) = \frac{1}{1+1} = \frac{1}{2}, \quad f(2) = \frac{2}{4+1} = \frac{2}{5}$$

(c) Absolute extrema:

• Absolute maximum: $f(1) = \frac{1}{2}$

• Absolute minimum: $f(-1) = -\frac{1}{2}$

Graph of $f(x) = \frac{x}{x^2+1}$ on $[-2, 2]$:



Question 1

Let:

$$f(x) = x^4 - 4x^3 + 6x^2$$

- (a) Find the intervals where $f(x)$ is concave up or concave down.
- (b) Determine the inflection points of $f(x)$, if any.

Question 1 - Solution

We are given:

$$f(x) = x^4 - 4x^3 + 6x^2$$

Step 1: Find the second derivative

$$f'(x) = 4x^3 - 12x^2 + 12x$$

$$f''(x) = 12x^2 - 24x + 12 = 12(x - 1)^2$$

(a) Since $f''(x) = 12(x - 1)^2 \geq 0$, the second derivative is never negative.

For all $x \neq 1$, $f''(x) > 0$, so the function is concave up.

At $x = 1$, the second derivative is zero, but the sign of $f''(x)$ does not change.

Answer

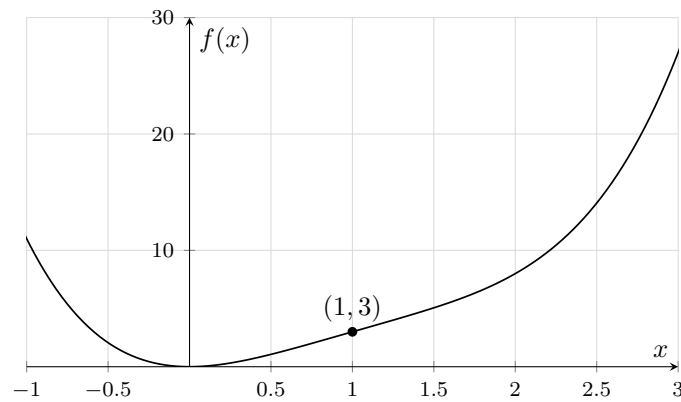
Concave up on

Concave down:

(b) Because the concavity does not change at $x = 1$, there is no inflection point.

Answer

Graph of the function



Question 2**Problem**

Given the function

$$f(x) = x^3 - 3x^2 - 9x + 5$$

- (a) Find the intervals where the function is increasing and decreasing.
- (b) Identify and classify all local extrema.

Question 2 - Solution

We are given

$$f(x) = x^3 - 3x^2 - 9x + 5$$

(a) **First derivative**

$$f'(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$$

The critical points are

$$x = -1 \quad \text{and} \quad x = 3$$

We test the sign of $f'(x)$.

For $x < -1$, $f'(x) > 0$. For $-1 < x < 3$, $f'(x) < 0$. For $x > 3$, $f'(x) > 0$.

Therefore, the function is increasing on $(-\infty, -1) \cup (3, \infty)$

and decreasing on $(-1, 3)$.

(b) **Local extrema**

Since the function changes from increasing to decreasing at $x = -1$, there is a local maximum:

$$f(-1) = 10$$

Since the function changes from decreasing to increasing at $x = 3$, there is a local minimum:

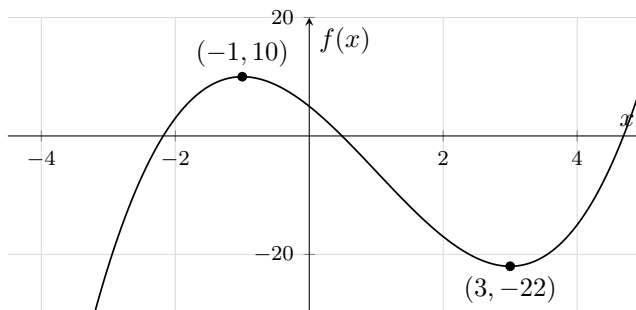
$$f(3) = -22$$

Answer

Local maximum at $(-1, 10)$

Local minimum at $(3, -22)$

Graph of $f(x)$



Question 3**Problem**

Consider the function

$$f(x) = \frac{x^2 - 4x}{x + 1}$$

- (a) Find the intervals where $f(x)$ is increasing or decreasing.
- (b) Identify any local maximum or minimum points.

Question 3 - Solution

We are given

$$f(x) = \frac{x^2 - 4x}{x + 1} = \frac{x(x - 4)}{x + 1}, \quad x \neq -1$$

(a) First derivative

Using the quotient rule,

$$f'(x) = \frac{(x + 1)(2x - 4) - (x^2 - 4x)}{(x + 1)^2} = \frac{x^2 + 2x - 4}{(x + 1)^2}.$$

Setting the numerator equal to zero gives

$$x^2 + 2x - 4 = 0 \Rightarrow x = -1 \pm \sqrt{5}.$$

Since $(x + 1)^2 > 0$ for $x \neq -1$, the sign of $f'(x)$ is determined by the numerator.

Answer

The function is increasing on

$$(-\infty, -1 - \sqrt{5}) \cup (-1 + \sqrt{5}, \infty).$$

The function is decreasing on

$$(-1 - \sqrt{5}, -1) \cup (-1, -1 + \sqrt{5}).$$

(b) Local extrema

At $x = -1 - \sqrt{5}$, the derivative changes from positive to negative, so there is a local maximum.

At $x = -1 + \sqrt{5}$, the derivative changes from negative to positive, so there is a local minimum.

$$f(-1 - \sqrt{5}) = -6 - 2\sqrt{5}, \quad f(-1 + \sqrt{5}) = -6 + 2\sqrt{5}.$$

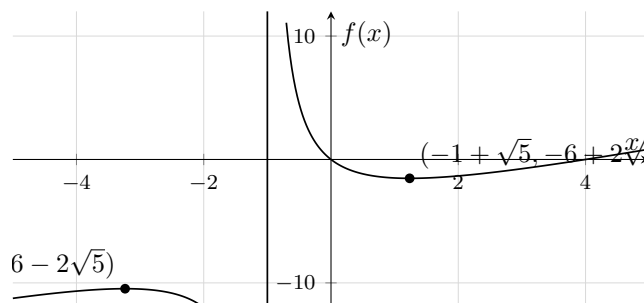
Answer

Local maximum at

$$(-1 - \sqrt{5}, -6 - 2\sqrt{5})$$

Local minimum at

$$(-1 + \sqrt{5}, -6 + 2\sqrt{5})$$

Graph of $f(x)$ 

Question 4**Problem:**

Let

$$f(x) = x^3 - 3x^2 - 9x + 5$$

- (a) Find the intervals where $f(x)$ is increasing or decreasing.
- (b) Identify any local maximum and minimum values.

Question 4 - Solution

We are given:

$$f(x) = x^3 - 3x^2 - 9x + 5$$

(a) Find the first derivative:

$$f'(x) = 3x^2 - 6x - 9$$

Set $f'(x) = 0$ to find critical points:

$$3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0 \Rightarrow x = -1, 3$$

Test intervals for sign of $f'(x)$:

For $x < -1$: pick $x = -2$: $f'(-2) = 3(-2)^2 - 6(-2) - 9 = 12 + 12 - 9 = 15 > 0$

For $-1 < x < 3$: pick $x = 0$: $f'(0) = -9 < 0$

For $x > 3$: pick $x = 4$: $f'(4) = 3(16) - 6(4) - 9 = 48 - 24 - 9 = 15 > 0$

Conclusion:

$f(x)$ is increasing on $(-\infty, -1) \cup (3, \infty)$

$f(x)$ is decreasing on $(-1, 3)$

(b) Local Extrema:

At $x = -1$: changes from increasing to decreasing $\rightarrow$ local maximum At $x = 3$: changes from decreasing to increasing $\rightarrow$ local minimum

Evaluate $f(x)$ at critical points:

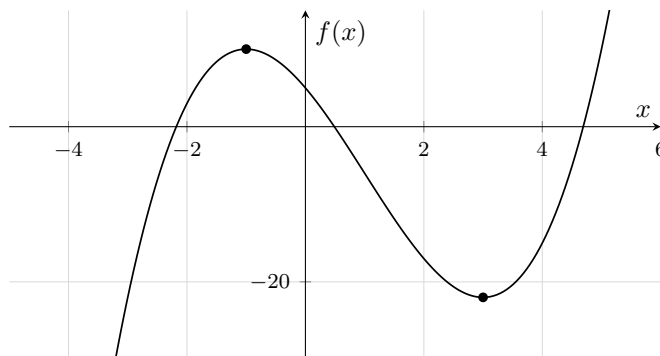
$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = -1 - 3 + 9 + 5 = 10$$

$$f(3) = 27 - 27 - 27 + 5 = -22$$

Local Maximum: $(-1, 10)$

Local Minimum: $(3, -22)$

Graph of $f(x) = x^3 - 3x^2 - 9x + 5$:



Question 5**Problem:**

Let

$$f(x) = x^4 - 4x^3 + 6$$

- (a) Find the intervals where $f(x)$ is increasing or decreasing.
- (b) Identify any local maximum and minimum values.

Question 5 - Solution

We are given:

$$f(x) = x^4 - 4x^3 + 6$$

(a) **First Derivative:**

$$f'(x) = 4x^3 - 12x^2$$

$$f'(x) = 4x^2(x - 3)$$

Set $f'(x) = 0$:

$$4x^2(x - 3) = 0 \Rightarrow x = 0, 3$$

Test intervals:

For $x < 0$: pick $x = -1$: $f'(-1) = 4(-1)^2(-1 - 3) = 4(1)(-4) = -16 < 0$

For $0 < x < 3$: pick $x = 1$: $f'(1) = 4(1)^2(1 - 3) = 4(-2) = -8 < 0$

For $x > 3$: pick $x = 4$: $f'(4) = 4(16)(1) = 64 > 0$

Conclusion:

$f(x)$ is decreasing on $(-\infty, 3)$

$f(x)$ is increasing on $(3, \infty)$

(b) **Local Extrema:**

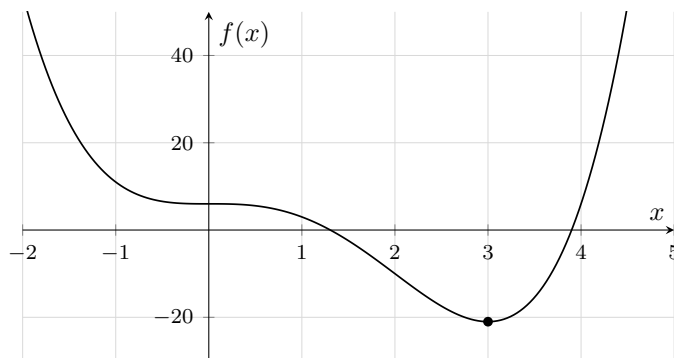
$f'(x)$ does not change sign around $x = 0$ (negative on both sides) $\rightarrow$ no extremum at $x = 0$ At $x = 3$:
from decreasing to increasing $\rightarrow$ local minimum

Evaluate:

$$f(3) = 3^4 - 4(3)^3 + 6 = 81 - 108 + 6 = -21$$

Local Minimum: $(3, -21)$

Graph of $f(x) = x^4 - 4x^3 + 6$:



Question 6**Problem:**

Let

$$f(x) = \frac{x^2}{x+1}$$

- (a) Find the critical points of $f(x)$ and determine where $f(x)$ is increasing or decreasing.
- (b) Identify any local maximum or minimum values.

Question 6 - Solution

We are given:

$$f(x) = \frac{x^2}{x+1}$$

(a) First Derivative:

Use the quotient rule:

$$f'(x) = \frac{(2x)(x+1) - x^2(1)}{(x+1)^2} = \frac{2x(x+1) - x^2}{(x+1)^2} = \frac{2x^2 + 2x - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}$$

Set $f'(x) = 0$:

$$\frac{x(x+2)}{(x+1)^2} = 0 \Rightarrow x = 0, -2$$

Also, note $f(x)$ is undefined at $x = -1$

Test intervals:

$(-\infty, -2)$: pick $x = -3$: numerator is positive, denominator is positive $\rightarrow f' > 0$

$(-2, -1)$: pick $x = -1.5$: numerator is negative, denominator is positive $\rightarrow f' < 0$

$(-1, 0)$: pick $x = -0.5$: numerator is negative, denominator is positive $\rightarrow f' < 0$

$(0, \infty)$: pick $x = 1$: numerator is positive, denominator is positive $\rightarrow f' > 0$

Conclusion:

$f(x)$ increases on $(-\infty, -2) \cup (0, \infty)$

$f(x)$ decreases on $(-2, -1) \cup (-1, 0)$

(b) Local Extrema:

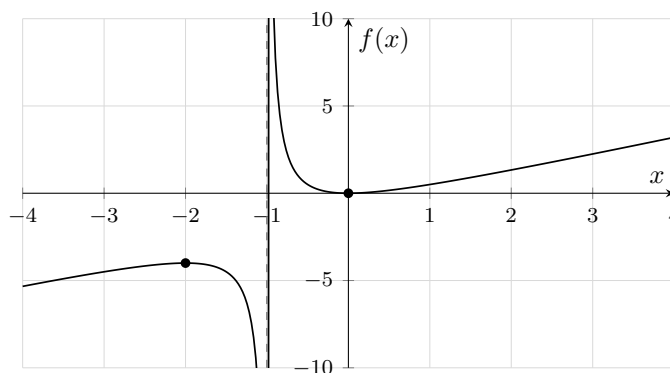
At $x = -2$: from increasing to decreasing $\rightarrow$ local max $f(-2) = \frac{(-2)^2}{-2+1} = \frac{4}{-1} = -4$

At $x = 0$: from decreasing to increasing $\rightarrow$ local min $f(0) = 0$

Answer:

Local maximum at $(-2, -4)$, Local minimum at $(0, 0)$

Graph of $f(x) = \frac{x^2}{x+1}$:



Question 7**Problem:**

Let

$$f(x) = x^3 - 3x^2 - 9x + 5$$

- (a) Find the critical points of $f(x)$ and determine the intervals where $f(x)$ is increasing or decreasing.
- (b) Identify all local maximum and minimum values.

Question 7 - Solution

We are given:

$$f(x) = x^3 - 3x^2 - 9x + 5$$

(a) **First Derivative:**

$$f'(x) = 3x^2 - 6x - 9$$

Set $f'(x) = 0$ to find critical points:

$$3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x - 3)(x + 1) = 0 \Rightarrow x = -1, 3$$

Test Intervals:

$(-\infty, -1)$: pick $x = -2$: $f'(-2) = 3(-2)^2 - 6(-2) - 9 = 12 + 12 - 9 = 15 > 0$

$(-1, 3)$: pick $x = 0$: $f'(0) = -9 < 0$

$(3, \infty)$: pick $x = 4$: $f'(4) = 3(16) - 6(4) - 9 = 48 - 24 - 9 = 15 > 0$

Conclusion:

$f(x)$ is increasing on $(-\infty, -1) \cup (3, \infty)$

$f(x)$ is decreasing on $(-1, 3)$

(b) **Local Extrema:**

At $x = -1$: changes from increasing to decreasing $\rightarrow$ local maximum

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = -1 - 3 + 9 + 5 = 10$$

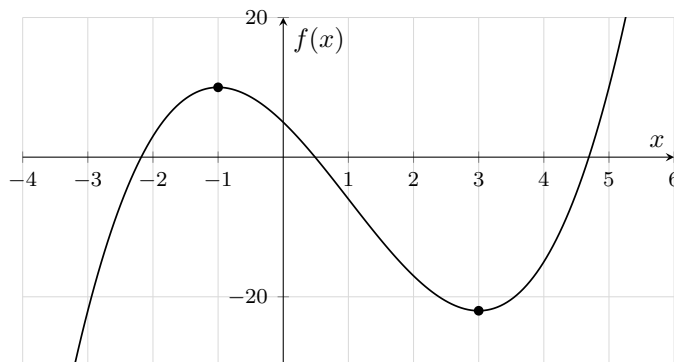
At $x = 3$: changes from decreasing to increasing $\rightarrow$ local minimum

$$f(3) = 27 - 27 - 27 + 5 = -22$$

Answer:

Local maximum at $(-1, 10)$, Local minimum at $(3, -22)$

Graph of $f(x) = x^3 - 3x^2 - 9x + 5$:



Question 8**Problem:**

Let

$$f(x) = \frac{x^2}{x+2}$$

- (a) Find the domain of $f(x)$ and determine all critical points.
- (b) Identify intervals where $f(x)$ is increasing or decreasing.
- (c) Classify each critical point as a local maximum, local minimum, or neither.

Question 8 - Solution

We are given:

$$f(x) = \frac{x^2}{x+2}$$

(a) Domain and Critical Points:

The function is undefined at $x = -2$, so the domain is:

$$(-\infty, -2) \cup (-2, \infty)$$

Find the derivative using the quotient rule:

$$f'(x) = \frac{(2x)(x+2) - x^2(1)}{(x+2)^2} = \frac{2x(x+2) - x^2}{(x+2)^2} = \frac{2x^2 + 4x - x^2}{(x+2)^2} = \frac{x^2 + 4x}{(x+2)^2}$$

Set the derivative to zero:

$$f'(x) = 0 \Rightarrow x^2 + 4x = 0 \Rightarrow x(x+4) = 0 \Rightarrow x = 0, x = -4$$

Critical Points: $x = -4$ and $x = 0$

(b) Test Intervals for Increasing/Decreasing:

Test sign of $f'(x) = \frac{x(x+4)}{(x+2)^2}$

Interval $(-\infty, -4)$: pick $x = -5$: $f'(-5) = \frac{-5(-1)}{(-3)^2} = \frac{5}{9} > 0$

Interval $(-4, -2)$: pick $x = -3$: $f'(-3) = \frac{-3(1)}{(1)^2} = -3 < 0$

Interval $(-2, 0)$: pick $x = -1$: $f'(-1) = \frac{-1(3)}{(1)^2} = -3 < 0$

Interval $(0, \infty)$: pick $x = 1$: $f'(1) = \frac{1(5)}{(3)^2} = \frac{5}{9} > 0$

Increasing: $(-\infty, -4) \cup (0, \infty)$ **Decreasing:** $(-4, -2) \cup (-2, 0)$

(c) Classify Critical Points:

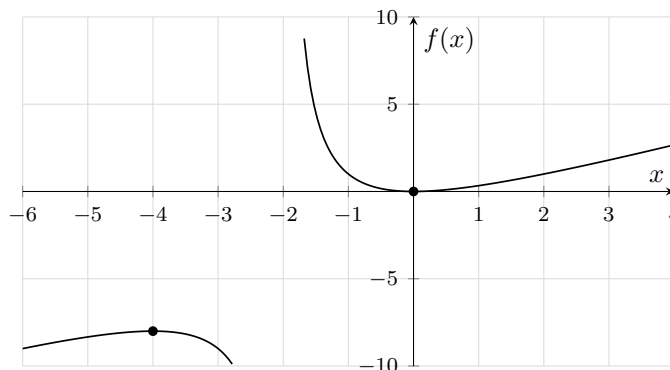
At $x = -4$: changes from increasing to decreasing $\rightarrow$ local max $f(-4) = \frac{16}{-2} = -8$

At $x = 0$: changes from decreasing to increasing $\rightarrow$ local min $f(0) = 0$

Conclusion:

Local maximum at $(-4, -8)$, Local minimum at $(0, 0)$

Graph of $f(x) = \frac{x^2}{x+2}$:



Question 9**Problem:**

Let

$$f(x) = x^3 - 6x^2 + 9x + 1$$

- (a) Find all critical points of $f(x)$.
- (b) Determine the intervals where $f(x)$ is increasing or decreasing.
- (c) Identify and classify all local extrema.

Question 9 - Solution

We are given:

$$f(x) = x^3 - 6x^2 + 9x + 1$$

(a) Critical Points:

Compute the first derivative:

$$f'(x) = 3x^2 - 12x + 9$$

Set derivative equal to 0:

$$3x^2 - 12x + 9 = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow x = 1, x = 3$$

Critical points: $x = 1, x = 3$

(b) Increasing/Decreasing Intervals:

Test sign of $f'(x)$ in each interval:

$(-\infty, 1)$: pick $x = 0$, $f'(0) = 9 > 0$

$(1, 3)$: pick $x = 2$, $f'(2) = 3(4) - 12(2) + 9 = 12 - 24 + 9 = -3 < 0$

$(3, \infty)$: pick $x = 4$, $f'(4) = 3(16) - 48 + 9 = 48 - 48 + 9 = 9 > 0$

Increasing: $(-\infty, 1) \cup (3, \infty)$ **Decreasing:** $(1, 3)$

(c) Local Extrema:

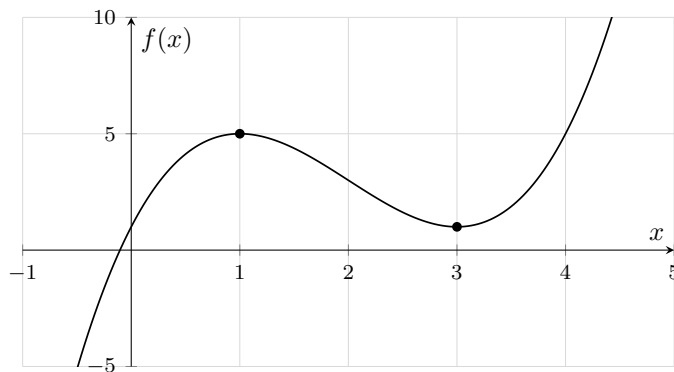
At $x = 1$: increasing to decreasing $\rightarrow$ local max $f(1) = 1 - 6 + 9 + 1 = 5$

At $x = 3$: decreasing to increasing $\rightarrow$ local min $f(3) = 27 - 54 + 27 + 1 = 1$

Conclusion:

Local maximum at $(1, 5)$, Local minimum at $(3, 1)$

Graph of $f(x) = x^3 - 6x^2 + 9x + 1$:



Question 10**Problem:**

Let

$$f(x) = \frac{x}{x^2 + 1}$$

- (a) Find the critical points of $f(x)$.
- (b) Determine where the function is increasing and decreasing.
- (c) Identify and classify any local extrema.

Question 10 - Solution

We are given:

$$f(x) = \frac{x}{x^2 + 1}$$

(a) Critical Points:

Use the quotient rule:

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{-x^2 + 1}{(x^2 + 1)^2}$$

Set numerator equal to zero:

$$-x^2 + 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

Critical points: $x = -1, x = 1$

(b) Increasing/Decreasing Intervals:

Analyze the sign of $f'(x) = \frac{1-x^2}{(x^2+1)^2}$

Denominator is always positive. So sign depends on numerator $1 - x^2$:

$x < -1$: $f'(x) < 0$ $-1 < x < 1$: $f'(x) > 0$ $x > 1$: $f'(x) < 0$

Increasing: $(-1, 1)$ **Decreasing:** $(-\infty, -1) \cup (1, \infty)$

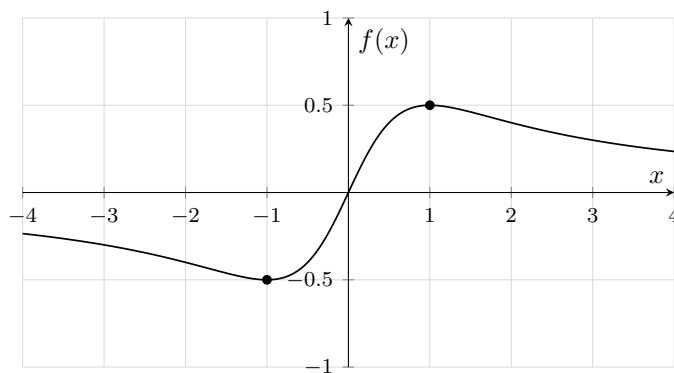
(c) Local Extrema:

$f'(x)$ changes from negative to positive at $x = -1 \rightarrow$ local min $f(-1) = \frac{-1}{1+1} = -\frac{1}{2}$

$f'(x)$ changes from positive to negative at $x = 1 \rightarrow$ local max $f(1) = \frac{1}{1+1} = \frac{1}{2}$

Conclusion: Local minimum at $(-1, -\frac{1}{2})$, Local maximum at $(1, \frac{1}{2})$

Graph of $f(x) = \frac{x}{x^2+1}$:



Question 1

Let

$$f(x) = x^3 - 3x^2 - 9x + 5$$

- (a) Find the intervals where $f(x)$ is increasing or decreasing.
- (b) Identify all local maxima and minima.
- (c) Determine intervals of concavity and any inflection points.

Question 1 - Solution

We are given

$$f(x) = x^3 - 3x^2 - 9x + 5$$

(a) First derivative

$$f'(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$$

The critical points are $x = -1$ and $x = 3$.

For $x < -1$, the derivative is positive. For $-1 < x < 3$, the derivative is negative. For $x > 3$, the derivative is positive.

Therefore, the function is increasing on

$$(-\infty, -1) \cup (3, \infty)$$

and decreasing on

$$(-1, 3).$$

(b) Local extrema

At $x = -1$, the function changes from increasing to decreasing, so there is a local maximum.

$$f(-1) = 10$$

At $x = 3$, the function changes from decreasing to increasing, so there is a local minimum.

$$f(3) = -22$$

Local maximum at $(-1, 10)$

Local minimum at $(3, -22)$

(c) Second derivative

$$f''(x) = 6x - 6$$

Setting $f''(x) = 0$ gives $x = 1$.

For $x < 1$, the second derivative is negative, so the graph is concave down. For $x > 1$, the second derivative is positive, so the graph is concave up.

The inflection point occurs at

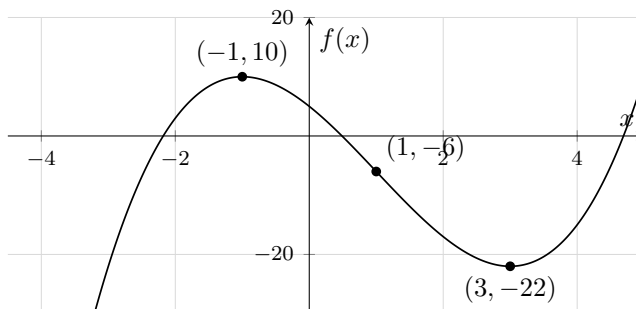
$$f(1) = -6$$

Concave down on $(-\infty, 1)$

Concave up on $(1, \infty)$

Inflection point at $(1, -6)$

Graph of $f(x)$



Question 2

Let

$$f(x) = x^4 - 4x^3$$

- (a) Find the intervals where the graph of f is concave up and concave down.
- (b) Identify any inflection points.

Question 2 - Solution

We are given

$$f(x) = x^4 - 4x^3$$

First derivative

$$f'(x) = 4x^3 - 12x^2$$

Second derivative

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

(a) Concavity

Setting $f''(x) = 0$ gives the critical values

$$x = 0 \quad \text{and} \quad x = 2.$$

For $x < 0$, the second derivative is positive, so the graph is concave up. For $0 < x < 2$, the second derivative is negative, so the graph is concave down. For $x > 2$, the second derivative is positive, so the graph is concave up.

Concave up on

$$(-\infty, 0) \cup (2, \infty)$$

Concave down on

$$(0, 2)$$

(b) Inflection points

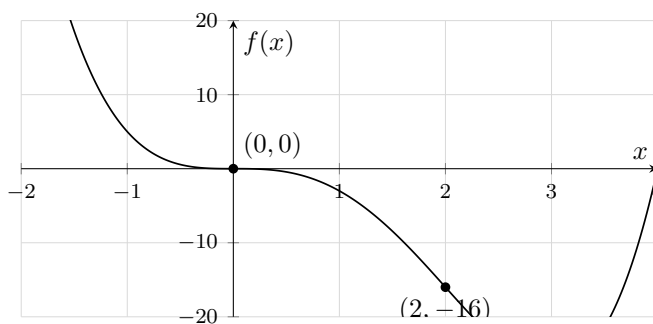
Because the concavity changes at both $x = 0$ and $x = 2$, inflection points occur at these values.

$$f(0) = 0 \quad f(2) = 16 - 32 = -16$$

Inflection points at

$$(0, 0) \quad \text{and} \quad (2, -16)$$

Graph of $f(x)$



Question 3

Let

$$f(x) = \frac{x^2}{x^2 + 1}$$

- (a) Determine the intervals where the graph of f is concave up and concave down.
- (b) Identify all inflection points.

Question 3 - Solution

We are given

$$f(x) = \frac{x^2}{x^2 + 1}$$

First derivative

Using the quotient rule,

$$f'(x) = \frac{(2x)(x^2 + 1) - x^2(2x)}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}.$$

Second derivative

Differentiating $f'(x)$ gives

$$f''(x) = \frac{2(1 - 3x^2)}{(x^2 + 1)^3}.$$

(a) Concavity

Setting $f''(x) = 0$ yields

$$1 - 3x^2 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}}.$$

For $|x| < \frac{1}{\sqrt{3}}$, the second derivative is positive, so the graph is concave up.

For $|x| > \frac{1}{\sqrt{3}}$, the second derivative is negative, so the graph is concave down.

Concave up on

$$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

Concave down on

$$\left(-\infty, -\frac{1}{\sqrt{3}} \right) \cup \left(\frac{1}{\sqrt{3}}, \infty \right)$$

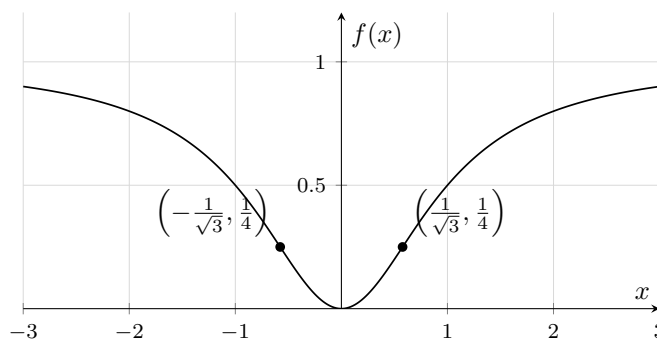
(b) Inflection points

Because the concavity changes at $x = \pm \frac{1}{\sqrt{3}}$, inflection points occur there.

$$f\left(\pm \frac{1}{\sqrt{3}}\right) = \frac{1}{4}$$

Inflection points at

$$\left(-\frac{1}{\sqrt{3}}, \frac{1}{4} \right) \quad \text{and} \quad \left(\frac{1}{\sqrt{3}}, \frac{1}{4} \right)$$

Graph of $f(x)$ 

Question 4

Let

$$f(x) = x^3 - 3x$$

- (a) Determine the intervals where the graph of f is concave up and concave down.
- (b) Identify all inflection points.

Question 4 - Solution

We are given

$$f(x) = x^3 - 3x$$

First derivative

$$f'(x) = 3x^2 - 3$$

Second derivative

$$f''(x) = 6x$$

(a) Concavity

Setting $f''(x) = 0$ gives

$$x = 0.$$

For $x < 0$, the second derivative is negative, so the graph is concave down.

For $x > 0$, the second derivative is positive, so the graph is concave up.

Concave down on

$$(-\infty, 0)$$

Concave up on

$$(0, \infty)$$

(b) Inflection point

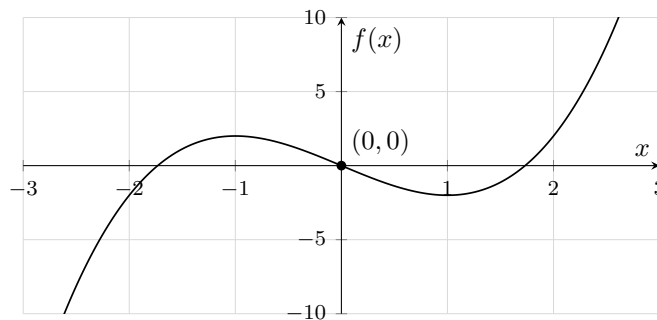
Because the concavity changes at $x = 0$, an inflection point occurs there.

$$f(0) = 0$$

Inflection point at

$$(0, 0)$$

Graph of $f(x)$



Question 5

Let

$$f(x) = \frac{x}{x^2 + 1}$$

- (a) Determine the intervals where the graph of f is concave up and concave down.
- (b) Identify all inflection points.

Question 5 - Solution

We are given

$$f(x) = \frac{x}{x^2 + 1}$$

First derivative

Using the quotient rule,

$$f'(x) = \frac{(x^2 + 1) - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}.$$

Second derivative

Differentiating $f'(x)$,

$$f''(x) = \frac{-2x(3 - x^2)}{(x^2 + 1)^3}.$$

(a) Concavity

Setting $f''(x) = 0$ gives

$$x = -\sqrt{3}, \quad x = 0, \quad x = \sqrt{3}.$$

For $x < -\sqrt{3}$, the second derivative is negative, so the graph is concave down.

For $-\sqrt{3} < x < 0$, the second derivative is positive, so the graph is concave up.

For $0 < x < \sqrt{3}$, the second derivative is negative, so the graph is concave down.

For $x > \sqrt{3}$, the second derivative is positive, so the graph is concave up.

Concave up on

$$(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$$

Concave down on

$$(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$$

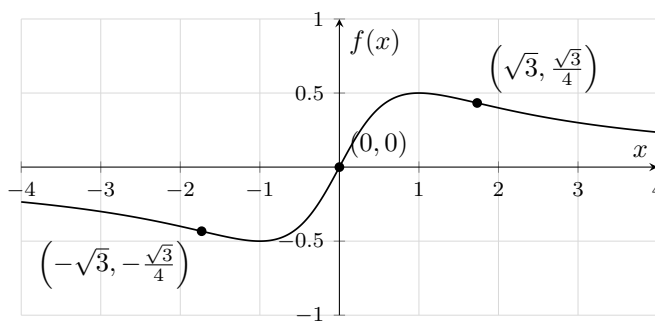
(b) Inflection points

Because the concavity changes at each critical value, inflection points occur at all three values.

$$f(0) = 0, \quad f(\pm\sqrt{3}) = \pm\frac{\sqrt{3}}{4}$$

Inflection points at

$$\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), \quad (0, 0), \quad \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$$

Graph of $f(x)$ 

Question 6

Let

$$f(x) = x^4 - 4x^3 + 10$$

- (a) Determine the intervals where the graph of f is concave up and concave down.
- (b) Identify all inflection points.

Question 6 - Solution

We are given

$$f(x) = x^4 - 4x^3 + 10$$

First derivative

$$f'(x) = 4x^3 - 12x^2$$

Second derivative

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

(a) Concavity

Setting $f''(x) = 0$ gives

$$x = 0 \quad \text{and} \quad x = 2.$$

For $x < 0$, the second derivative is positive, so the graph is concave up.

For $0 < x < 2$, the second derivative is negative, so the graph is concave down.

For $x > 2$, the second derivative is positive, so the graph is concave up.

Concave up on

$$(-\infty, 0) \cup (2, \infty)$$

Concave down on

$$(0, 2)$$

(b) Inflection points

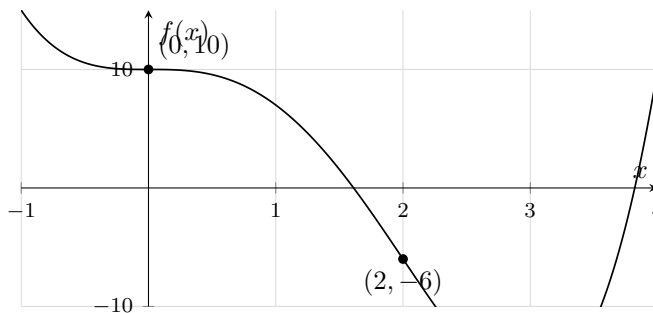
Because the concavity changes at both values, inflection points occur at $x = 0$ and $x = 2$.

$$f(0) = 10, \quad f(2) = -6$$

Inflection points at

$$(0, 10) \quad \text{and} \quad (2, -6)$$

Graph of $f(x)$



Question 7

Let

$$f(x) = x^3 - 6x^2 + 12x - 5$$

- (a) Determine the intervals where the graph of f is concave up and concave down.
- (b) Find all inflection points.

Question 7 - Solution

We are given

$$f(x) = x^3 - 6x^2 + 12x - 5$$

First derivative

$$f'(x) = 3x^2 - 12x + 12$$

Second derivative

$$f''(x) = 6x - 12$$

(a) Concavity

Setting $f''(x) = 0$ gives

$$x = 2.$$

For $x < 2$, the second derivative is negative, so the graph is concave down.

For $x > 2$, the second derivative is positive, so the graph is concave up.

Concave down on

$$(-\infty, 2)$$

Concave up on

$$(2, \infty)$$

(b) Inflection point

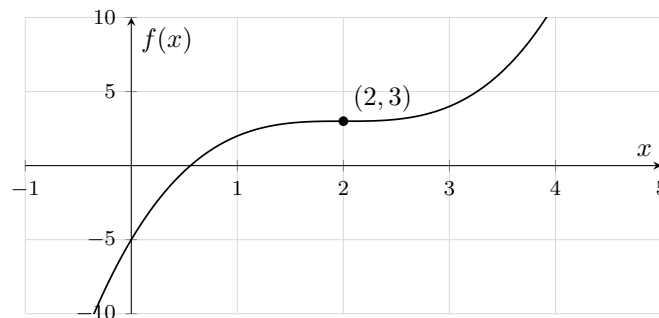
Because the concavity changes at $x = 2$, an inflection point occurs there.

$$f(2) = 3$$

Inflection point at

$$(2, 3)$$

Graph of $f(x)$



Question 8

Let

$$f(x) = \frac{x^2}{x^2 + 1}$$

- (a) Determine the intervals where the graph of f is concave up and concave down.
- (b) Find all inflection points, if any.

Question 8 - Solution

We are given

$$f(x) = \frac{x^2}{x^2 + 1}$$

First derivative

Using the quotient rule,

$$f'(x) = \frac{(2x)(x^2 + 1) - x^2(2x)}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}.$$

Second derivative

Differentiating $f'(x)$,

$$f''(x) = \frac{2(1 - 3x^2)}{(x^2 + 1)^3}.$$

(a) Concavity

Setting $f''(x) = 0$ gives

$$1 - 3x^2 = 0 \quad \Rightarrow \quad x = \pm \frac{1}{\sqrt{3}}.$$

For $|x| < \frac{1}{\sqrt{3}}$, the second derivative is positive, so the graph is concave up.

For $|x| > \frac{1}{\sqrt{3}}$, the second derivative is negative, so the graph is concave down.

Concave up on

$$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

Concave down on

$$\left(-\infty, -\frac{1}{\sqrt{3}} \right) \cup \left(\frac{1}{\sqrt{3}}, \infty \right)$$

(b) Inflection points

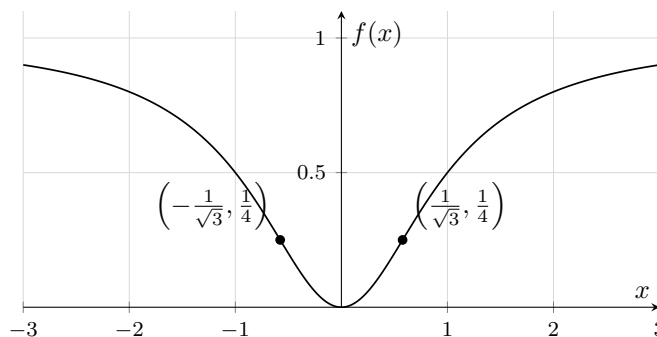
Because the concavity changes at both values, inflection points occur at

$$x = \pm \frac{1}{\sqrt{3}}.$$

$$f\left(\pm \frac{1}{\sqrt{3}}\right) = \frac{1}{4}$$

Inflection points at

$$\left(-\frac{1}{\sqrt{3}}, \frac{1}{4} \right) \quad \text{and} \quad \left(\frac{1}{\sqrt{3}}, \frac{1}{4} \right)$$

Graph of $f(x)$ 

Question 9

Let

$$f(x) = x^3 - 3x$$

- (a) Determine the intervals where the graph of f is concave up and concave down.
- (b) Find all inflection points.

Question 9 - Solution

We are given

$$f(x) = x^3 - 3x$$

First derivative

$$f'(x) = 3x^2 - 3$$

Second derivative

$$f''(x) = 6x$$

(a) Concavity

Setting $f''(x) = 0$ gives

$$x = 0.$$

For $x < 0$, the second derivative is negative, so the graph is concave down.

For $x > 0$, the second derivative is positive, so the graph is concave up.

Concave down on

$$(-\infty, 0)$$

Concave up on

$$(0, \infty)$$

(b) Inflection point

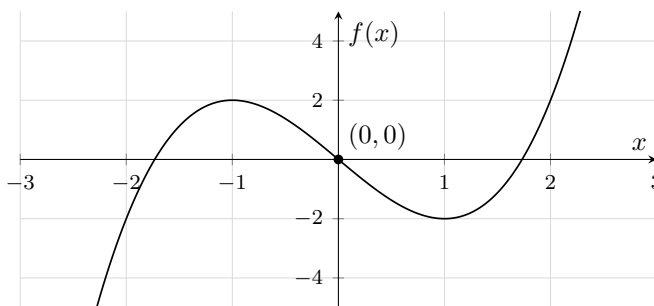
Because the concavity changes at $x = 0$, an inflection point occurs there.

$$f(0) = 0$$

Inflection point at

$$(0, 0)$$

Graph of $f(x)$



Question 10

Let

$$f(x) = \frac{x}{x^2 + 1}$$

- (a) Determine the intervals where f is concave up and concave down.
- (b) Find the inflection points of f , if any.

Question 10 - Solution

We are given

$$f(x) = \frac{x}{x^2 + 1}$$

First derivative

Using the quotient rule,

$$f'(x) = \frac{(x^2 + 1) - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}.$$

Second derivative

Differentiating $f'(x)$,

$$f''(x) = \frac{-2x(3 - x^2)}{(x^2 + 1)^3}.$$

(a) Concavity

Setting $f''(x) = 0$ gives

$$x = -\sqrt{3}, \quad x = 0, \quad x = \sqrt{3}.$$

For $x < -\sqrt{3}$, the second derivative is negative, so the graph is concave down.

For $-\sqrt{3} < x < 0$, the second derivative is positive, so the graph is concave up.

For $0 < x < \sqrt{3}$, the second derivative is negative, so the graph is concave down.

For $x > \sqrt{3}$, the second derivative is positive, so the graph is concave up.

Concave up on

$$(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$$

Concave down on

$$(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$$

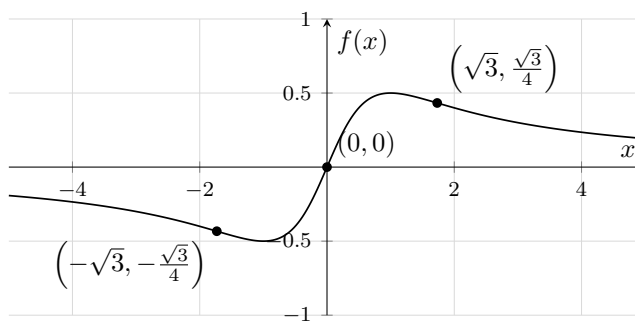
(b) Inflection points

Because the concavity changes at all three values, inflection points occur at each one.

$$f(0) = 0, \quad f(\pm\sqrt{3}) = \pm\frac{\sqrt{3}}{4}$$

Inflection points at

$$\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), \quad (0, 0), \quad \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$$

Graph of $f(x)$ 

Question 1

Let

$$f(x) = x^2 + 2x$$

- (a) Verify that $f(x)$ satisfies the hypotheses of the Mean Value Theorem on the interval $[1, 4]$.
- (b) Find all numbers $c \in (1, 4)$ that satisfy the conclusion of the Mean Value Theorem.

Question 1 - Solution

We are given

$$f(x) = x^2 + 2x$$

(a) Verifying the hypotheses

The function $f(x)$ is a polynomial, so it is continuous on the closed interval $[1, 4]$ and differentiable on the open interval $(1, 4)$.

Therefore, the hypotheses of the Mean Value Theorem are satisfied.

(b) Applying the Mean Value Theorem

First compute the derivative:

$$f'(x) = 2x + 2$$

Next compute the average rate of change of f on $[1, 4]$:

$$\frac{f(4) - f(1)}{4 - 1} = \frac{(16 + 8) - (1 + 2)}{3} = \frac{21}{3} = 7$$

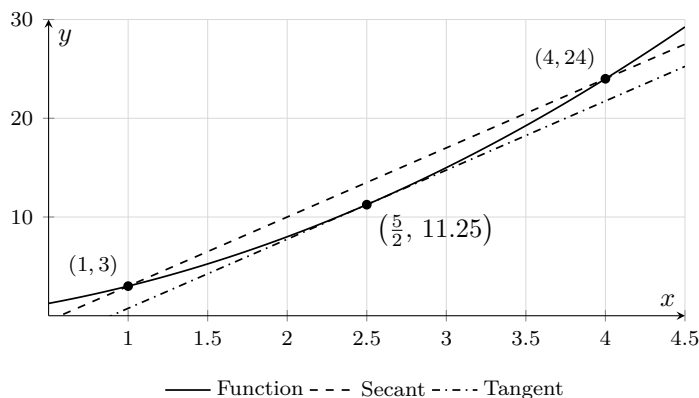
Set the derivative equal to this value:

$$2c + 2 = 7 \Rightarrow 2c = 5 \Rightarrow c = \frac{5}{2}$$

Thus, the number that satisfies the conclusion of the Mean Value Theorem is

$$c = \frac{5}{2}$$

Graph of $f(x)$, the secant line, and the tangent line at c



Question 2**Problem:**

Let

$$f(x) = \ln(x), \quad \text{for } x \in [1, 4]$$

- (a) Verify that f satisfies the hypotheses of the Mean Value Theorem on the interval.
- (b) Find all values of $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}$$

Question 2 - Solution

We are given:

$$f(x) = \ln(x)$$

(a) **Check MVT hypotheses:**

1. $f(x)$ is continuous on $[1, 4]$: Yes, $\ln(x)$ is continuous for $x > 0$, so continuous on $[1, 4]$
 2. $f(x)$ is differentiable on $(1, 4)$: Yes, $\ln(x)$ is differentiable for $x > 0$, so differentiable on $(1, 4)$
- The Mean Value Theorem applies.

(b) **Find the value(s) of c :**

Compute average rate of change:

$$\frac{f(4) - f(1)}{4 - 1} = \frac{\ln(4) - \ln(1)}{3} = \frac{\ln(4)}{3} = \frac{2\ln(2)}{3}$$

Now compute $f'(x)$:

$$f'(x) = \frac{1}{x}$$

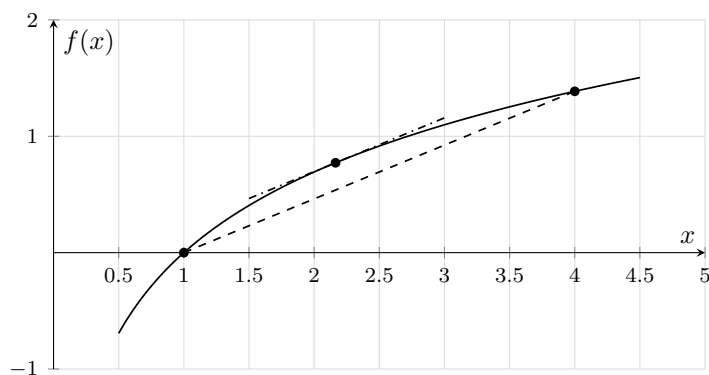
Set:

$$f'(c) = \frac{1}{c} = \frac{2\ln(2)}{3} \Rightarrow c = \frac{3}{2\ln(2)}$$

Answer:

$$c = \frac{3}{2\ln(2)} \approx 2.164$$

Graph of $f(x) = \ln(x)$ on $[1, 4]$:



Question 2**Problem:**

Let

$$f(x) = x^3 - 3x^2 + 2x, \quad \text{on the interval } [1, 3]$$

- (a) Verify that f satisfies the hypotheses of the Mean Value Theorem on $[1, 3]$.
- (b) Find all values of $c \in (1, 3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

Question 2 - Solution

We are given:

$$f(x) = x^3 - 3x^2 + 2x$$

(a) **Hypotheses of the MVT:**

- $f(x)$ is a polynomial continuous on $[1, 3]$ - $f(x)$ is differentiable on $(1, 3)$
Therefore, MVT applies.

(b) **Apply the MVT:**

Compute:

$$f(3) = 27 - 27 + 6 = 6, \quad f(1) = 1 - 3 + 2 = 0$$

$$\text{Average rate of change} = \frac{6 - 0}{3 - 1} = \frac{6}{2} = 3$$

Compute derivative:

$$f'(x) = 3x^2 - 6x + 2$$

Set $f'(c) = 3$:

$$3c^2 - 6c + 2 = 3 \quad \Rightarrow \quad 3c^2 - 6c - 1 = 0$$

Solve:

$$c = \frac{6 \pm \sqrt{(-6)^2 - 4(3)(-1)}}{2(3)} = \frac{6 \pm \sqrt{36 + 12}}{6} = \frac{6 \pm \sqrt{48}}{6} = \frac{6 \pm 4\sqrt{3}}{6} = \frac{3 \pm 2\sqrt{3}}{3}$$

Answer:

$$\boxed{c = \frac{3 \pm 2\sqrt{3}}{3}} \quad \text{Only the value(s) in } (1, 3) \text{ are valid.}$$

Since:

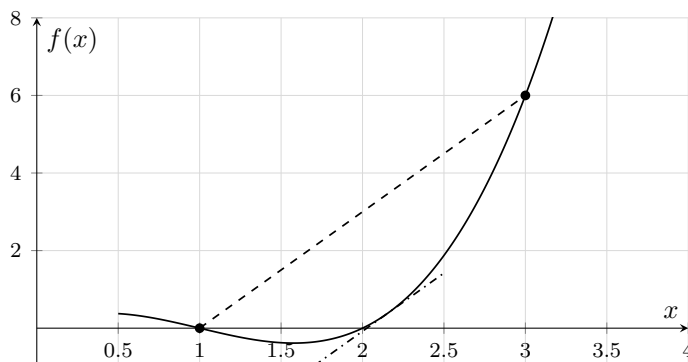
$$\frac{3 - 2\sqrt{3}}{3} \approx 0.845 < 1 \quad (\text{not valid})$$

$$\frac{3 + 2\sqrt{3}}{3} \approx 2.155 \in (1, 3)$$

Final answer:

$$\boxed{c = \frac{3 + 2\sqrt{3}}{3}}$$

Graph of $f(x) = x^3 - 3x^2 + 2x$ on $[1, 3]$:



Question 4**Problem:**

Let

$$f(x) = \ln(x^2 + 1) \quad \text{on the interval } [-1, 2].$$

- (a) Verify that f satisfies the hypotheses of the Mean Value Theorem on $[-1, 2]$.
- (b) Find all values of $c \in (-1, 2)$ such that

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}.$$

Question 4 - Solution

The function $f(x) = \ln(1 + x^2)$ is continuous on $[-1, 2]$ and differentiable on $(-1, 2)$, so the Mean Value Theorem applies.

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{\ln(5/2)}{3}, \quad f'(c) = \frac{2c}{1 + c^2}.$$

Let $L = \ln(5/2)$. Then $Lc^2 - 6c + L = 0$, so

$$c = \frac{3 \pm \sqrt{9 - L^2}}{L}.$$

The plus root exceeds 2. The only admissible value is

$$c = \frac{3 - \sqrt{9 - [\ln(5/2)]^2}}{\ln(5/2)} \approx 0.156453.$$

Question 5**Problem:**

Let

$$f(x) = \frac{x}{x+2} \quad \text{on the interval } [1, 4].$$

- (a) Verify that $f(x)$ satisfies the hypotheses of the Mean Value Theorem on $[1, 4]$.
- (b) Find all values $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

Question 5 - Solution

We are given:

$$f(x) = \frac{x}{x+2}$$

(a) **Check MVT conditions:**

- $f(x)$ is a rational function with denominator $x+2 \neq 0$ on $[1, 4]$ continuous on $[1, 4]$. - It is differentiable on $(1, 4)$ as it's smooth where defined.

MVT applies.

(b) **Compute average rate of change:**

$$f(1) = \frac{1}{3}, \quad f(4) = \frac{4}{6} = \frac{2}{3}$$

$$\frac{f(4) - f(1)}{4 - 1} = \frac{\frac{2}{3} - \frac{1}{3}}{3} = \frac{1}{9}$$

Find derivative:

Use quotient rule:

$$f'(x) = \frac{(x+2)(1) - x(1)}{(x+2)^2} = \frac{x+2-x}{(x+2)^2} = \frac{2}{(x+2)^2}$$

Set:

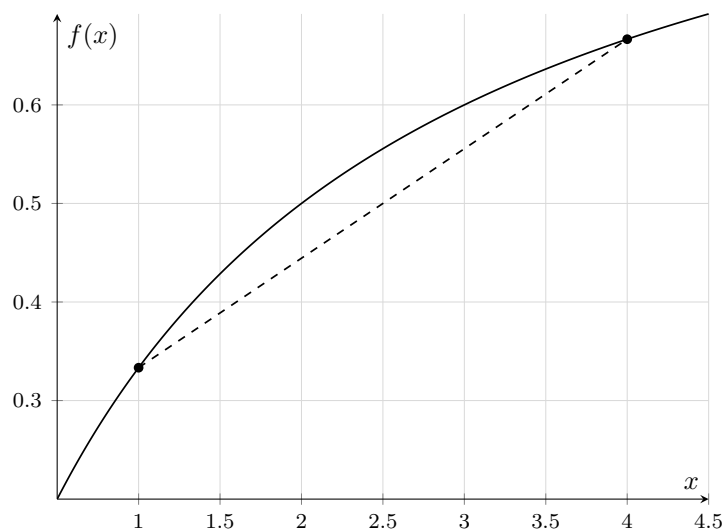
$$\frac{2}{(c+2)^2} = \frac{1}{9} \Rightarrow (c+2)^2 = 18 \Rightarrow c+2 = \pm\sqrt{18} = \pm 3\sqrt{2} \Rightarrow c = -2 \pm 3\sqrt{2}$$

Now, only one of these is in the interval $(1, 4)$:

$$c = -2 + 3\sqrt{2} \approx -2 + 4.24 = 2.24$$

$$\boxed{c = -2 + 3\sqrt{2}} \text{ is in } (1, 4)$$

Graph of $f(x)$ and secant line:



Question 6**Problem:**

Let

$$f(x) = \ln(x^2 + 1)$$

on the interval $[-1, 2]$.

- (a) Verify that $f(x)$ satisfies the hypotheses of the Mean Value Theorem on the interval.
- (b) Find all values of $c \in (-1, 2)$ such that

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}.$$

Question 6 - Solution

The function $f(x) = \ln(1 + x^2)$ is continuous on $[-1, 2]$ and differentiable on $(-1, 2)$, so the Mean Value Theorem applies.

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{\ln(5/2)}{3}, \quad f'(c) = \frac{2c}{1 + c^2}.$$

Let $L = \ln(5/2)$. Then $Lc^2 - 6c + L = 0$, so

$$c = \frac{3 \pm \sqrt{9 - L^2}}{L}.$$

The plus root exceeds 2. The only admissible value is

$$c = \frac{3 - \sqrt{9 - [\ln(5/2)]^2}}{\ln(5/2)} \approx 0.156453.$$

Question 7**Problem:**

Let

$$f(x) = \frac{x}{x+1}$$

on the interval $[1, 4]$.

- (a) Show that the hypotheses of the Mean Value Theorem are satisfied.
- (b) Find all values of $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

Question 7 - Solution

We are given:

$$f(x) = \frac{x}{x+1}$$

(a) **MVT Conditions:**

- $f(x)$ is continuous on $[1, 4]$: The function is a rational function with no discontinuities on this interval.
- $f(x)$ is differentiable on $(1, 4)$: Differentiable as a quotient of differentiable functions, and denominator never zero.

MVT applies.

(b) **Compute the average rate of change:**

$$\begin{aligned} f(1) &= \frac{1}{2}, & f(4) &= \frac{4}{5} \\ \frac{f(4) - f(1)}{4 - 1} &= \frac{\frac{4}{5} - \frac{1}{2}}{3} = \frac{\frac{8-5}{10}}{3} = \frac{3}{10} \cdot \frac{1}{3} = \frac{1}{10} \end{aligned}$$

Find derivative:

$$f'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$$

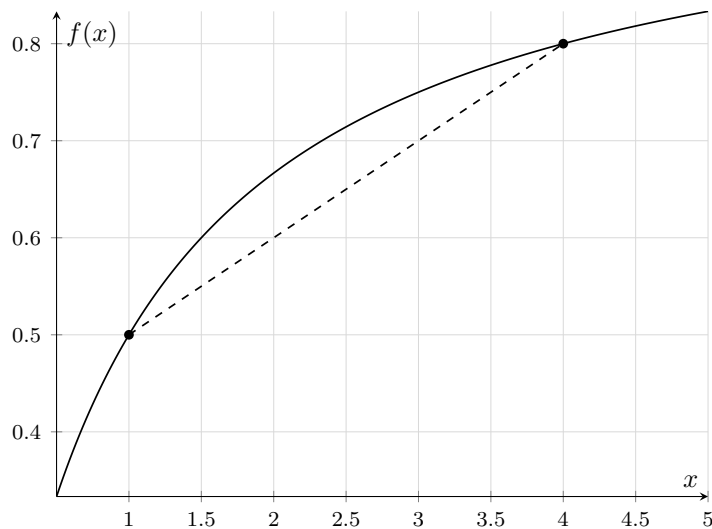
Set:

$$\frac{1}{(c+1)^2} = \frac{1}{10} \Rightarrow (c+1)^2 = 10 \Rightarrow c+1 = \sqrt{10} \Rightarrow c = \sqrt{10} - 1$$

Since $\sqrt{10} \approx 3.16$, we find $c \approx 2.16 \in (1, 4)$

$c = \sqrt{10} - 1$

Graph of $f(x)$ and Secant Line:



Question 8**Problem:**

Let

$$f(x) = \sin(x)$$

on the interval $[0, \pi]$.

- (a) Verify that $f(x)$ satisfies the hypotheses of the Mean Value Theorem.
- (b) Find all numbers $c \in (0, \pi)$ such that

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}.$$

Question 8 - Solution

We are given:

$$f(x) = \sin(x)$$

(a) MVT Conditions:

- $f(x)$ is continuous on $[0, \pi]$: Sine is continuous everywhere.
 - $f(x)$ is differentiable on $(0, \pi)$: Sine is differentiable everywhere.
- MVT applies.

(b) Apply MVT:

Average rate of change:

$$f(\pi) = \sin(\pi) = 0, \quad f(0) = \sin(0) = 0$$

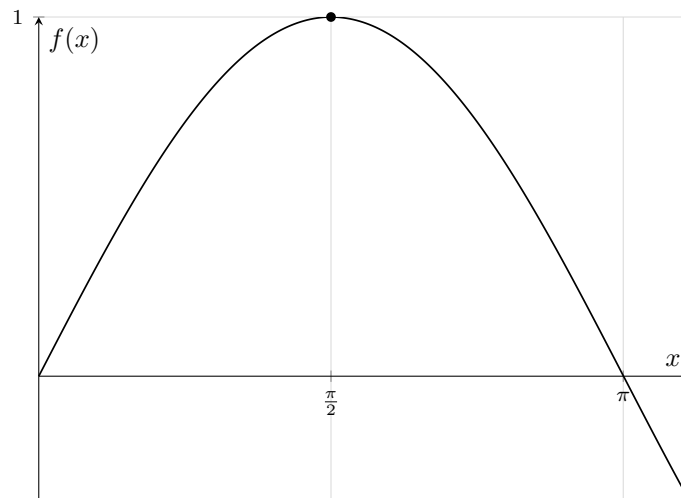
$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

We want:

$$f'(c) = \cos(c) = 0 \Rightarrow c = \frac{\pi}{2}$$

$$\boxed{c = \frac{\pi}{2}}$$

Graph of $f(x) = \sin(x)$ on $[0, \pi]$:



Question 9**Problem:**

Let

$$f(x) = \frac{1}{x}$$

on the interval $[1, 3]$.

- (a) Verify that $f(x)$ satisfies the hypotheses of the Mean Value Theorem.
- (b) Find all numbers $c \in (1, 3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}.$$

Question 9 - Solution

We are given:

$$f(x) = \frac{1}{x}$$

(a) **Conditions for MVT:**

- $f(x)$ is continuous on $[1, 3]$
 - $f(x)$ is differentiable on $(1, 3)$
- Therefore, MVT applies.

(b) **Apply MVT:**

$$f(3) = \frac{1}{3}, \quad f(1) = 1$$

$$\frac{f(3) - f(1)}{3 - 1} = \frac{\frac{1}{3} - 1}{2} = \frac{-\frac{2}{3}}{2} = -\frac{1}{3}$$

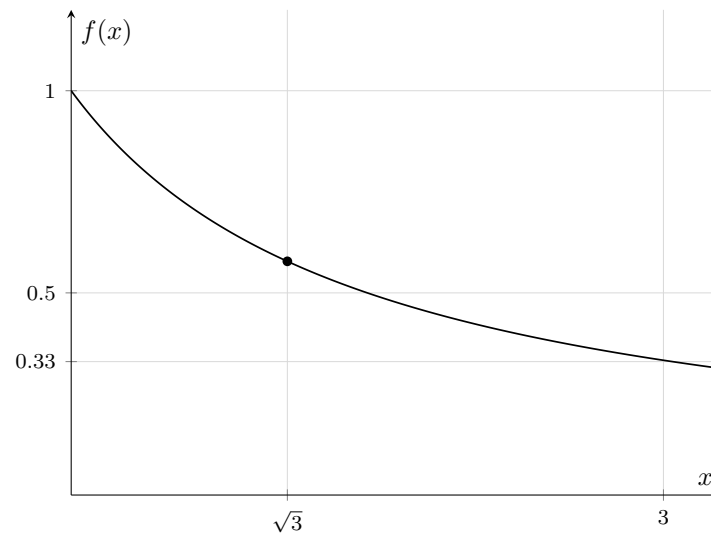
Now compute derivative:

$$f'(x) = \frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2}$$

$$f'(c) = -\frac{1}{c^2} = -\frac{1}{3} \Rightarrow c^2 = 3 \Rightarrow c = \sqrt{3}$$

$$c = \sqrt{3} \in (1, 3)$$

Graph of $f(x) = \frac{1}{x}$ on $[1, 3]$:



Question 10**Problem:**

Let

$$f(x) = \sin(x)$$

on the interval $[0, \pi]$.

- (a) Verify that $f(x)$ satisfies the conditions of the Mean Value Theorem on $[0, \pi]$.
- (b) Find all values of $c \in (0, \pi)$ such that

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}.$$

Question 10 - Solution

We are given:

$$f(x) = \sin(x)$$

(a) **Conditions for MVT:**

- $f(x)$ is continuous on $[0, \pi]$ - $f(x)$ is differentiable on $(0, \pi)$
MVT applies.

(b) **Apply MVT:**

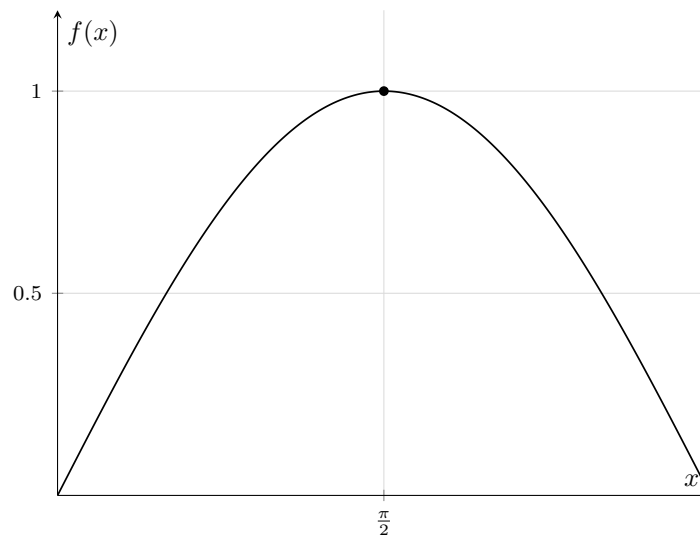
$$\begin{aligned} f(\pi) &= 0, & f(0) &= 0 \\ \frac{f(\pi) - f(0)}{\pi - 0} &= \frac{0 - 0}{\pi} = 0 \end{aligned}$$

We need:

$$f'(c) = \cos(c) = 0 \Rightarrow c = \frac{\pi}{2}$$

$$\boxed{c = \frac{\pi}{2}}$$

Graph of $f(x) = \sin(x)$ on $[0, \pi]$:

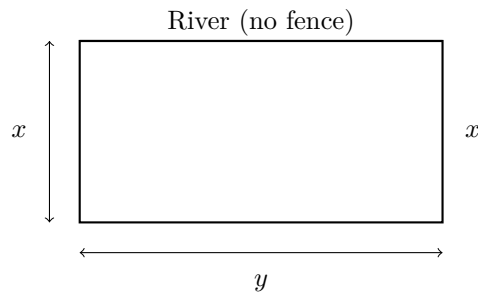


Question 1

A rectangular garden is to be fenced on three sides (two widths and one length), with the fourth side along a straight river needing no fence. The area of the garden must be 120 m^2 . What dimensions will minimize the amount of fencing used?

Clearly define variables, write the function to be minimized, and find the minimum using calculus.

Diagram:



Question 1 - Solution

Let:

- x = width (perpendicular to the river) - y = length (along the river)

Given: - Area constraint: $A = xy = 120 \Rightarrow y = \frac{120}{x}$ - Fence needed: two widths and one length
 $\Rightarrow F = 2x + y$

Substitute into fencing function:

$$F(x) = 2x + \frac{120}{x}$$

Minimize $F(x)$:

Take derivative:

$$F'(x) = 2 - \frac{120}{x^2}$$

Set $F'(x) = 0$:

$$2 = \frac{120}{x^2} \Rightarrow x^2 = 60 \Rightarrow x = \sqrt{60} = 2\sqrt{15}$$

Then:

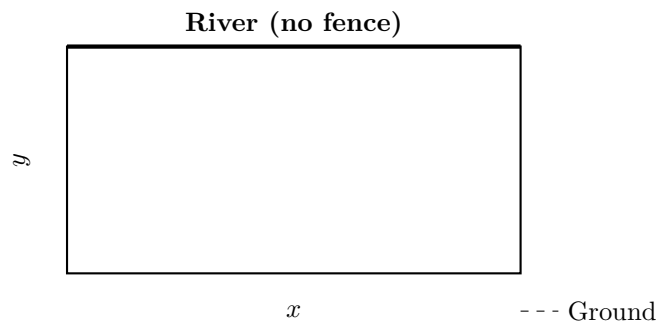
$$y = \frac{120}{x} = \frac{120}{2\sqrt{15}} = \frac{60}{\sqrt{15}} = \frac{60\sqrt{15}}{15} = 4\sqrt{15}$$

Answer: Width = $\boxed{2\sqrt{15} \text{ m}}$, Length = $\boxed{4\sqrt{15} \text{ m}}$

Question 2**Problem:**

A rectangular field is to be enclosed on three sides with fencing (the fourth side lies along a river and requires no fence). The total length of fencing available is 240 meters.

- (a) What dimensions will maximize the area of the field?
- (b) What is the maximum area?

Diagram:

Question 2 - Solution

Let: - x = length parallel to the river (no fence needed) - y = width (two sides to be fenced)

From the constraint:

$$x + 2y = 240 \quad \Rightarrow \quad x = 240 - 2y$$

The area is:

$$A = x \cdot y = (240 - 2y)y = 240y - 2y^2$$

Differentiate:

$$\frac{dA}{dy} = 240 - 4y$$

Set derivative to zero:

$$240 - 4y = 0 \quad \Rightarrow \quad y = 60$$

Back-substitute:

$$x = 240 - 2(60) = 120$$

(a) Optimal Dimensions:

$$x = 120 \text{ m}, \quad y = 60 \text{ m}$$

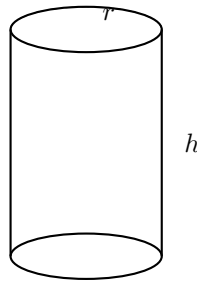
(b) Maximum Area:

$$A = 120 \times 60 = 7200 \text{ m}^2$$

Question 3**Problem:**

A cylindrical can is to be made to hold a volume of 1000 cm^3 . The material for the top and bottom costs twice as much per square centimeter as the material for the side.

- (a) Find the dimensions (radius and height) of the can that minimize the cost of the material.
- (b) What is the minimum total cost in terms of material area?

Diagram:

Question 3 - Solution

Let: - r : radius of the cylinder - h : height of the cylinder

Volume constraint:

$$V = \pi r^2 h = 1000 \quad \Rightarrow \quad h = \frac{1000}{\pi r^2}$$

Cost: - Side area = $2\pi r h$, unit cost = 1 - Top and bottom area = $2\pi r^2$, unit cost = 2

Total cost C (proportional to material area):

$$C = 2\pi r h + 2 \cdot 2\pi r^2 = 2\pi r h + 4\pi r^2$$

Substitute h :

$$C(r) = 2\pi r \cdot \frac{1000}{\pi r^2} + 4\pi r^2 = \frac{2000}{r} + 4\pi r^2$$

Differentiate:

$$C'(r) = -\frac{2000}{r^2} + 8\pi r$$

Set derivative to 0:

$$-\frac{2000}{r^2} + 8\pi r = 0 \Rightarrow 8\pi r^3 = 2000 \Rightarrow r^3 = \frac{250}{\pi}$$

$$r = \sqrt[3]{\frac{250}{\pi}}, \quad h = \frac{1000}{\pi r^2}$$

(a) Optimal Dimensions:

$$r = \sqrt[3]{\frac{250}{\pi}}, \quad h = \frac{1000}{\pi r^2}$$

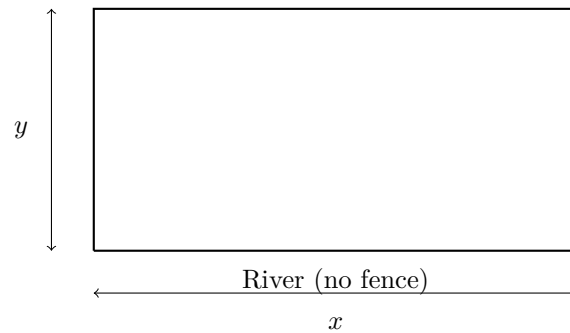
(b) Minimum Cost (in material area units):

$$C = \frac{2000}{r} + 4\pi r^2 \quad \text{where } r = \sqrt[3]{\frac{250}{\pi}}$$

Question 4**Problem:**

A farmer wants to build a rectangular fence next to a straight river. No fencing is needed along the river. The farmer has 240 meters of fencing available.

- (a) What dimensions will maximize the area of the enclosure?
- (b) What is the maximum area that can be enclosed?

Diagram:

Question 4 - Solution

Let: - x : length of the fence parallel to the river (only 1 needed) - y : width of the fence perpendicular to the river (2 needed)

Fencing constraint:

$$x + 2y = 240 \quad \Rightarrow \quad x = 240 - 2y$$

Area:

$$A = x \cdot y = (240 - 2y)y = 240y - 2y^2$$

Differentiate:

$$A'(y) = 240 - 4y$$

Set derivative to 0:

$$240 - 4y = 0 \quad \Rightarrow \quad y = 60$$

$$x = 240 - 2(60) = 120$$

(a) Optimal Dimensions:

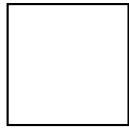
$$x = 120 \text{ m}, \quad y = 60 \text{ m}$$

(b) Maximum Area:

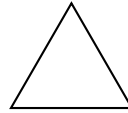
$$A = x \cdot y = 120 \cdot 60 = 7200 \text{ m}^2$$

Question 5

A square and an equilateral triangle are to be constructed from a single wire 100 cm long. The square and the triangle do not overlap. How much wire should be used for each shape in order to minimize the total enclosed area?



Square



Equilateral Triangle

Question 5 - Solution

Let:

$$x = \text{length of wire used for the square, } 0 < x < 100$$

Then:

$$100 - x = \text{length of wire used for the triangle}$$

Step 1: Expressions for Area

Square:

Let $s = \frac{x}{4}$ be the side of the square. Then area of the square is:

$$A_{\text{square}} = s^2 = \left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$$

Triangle:

Let each side of the triangle be $t = \frac{100-x}{3}$. Then area of equilateral triangle is:

$$A_{\text{triangle}} = \frac{\sqrt{3}}{4}t^2 = \frac{\sqrt{3}}{4} \left(\frac{100-x}{3}\right)^2 = \frac{\sqrt{3}}{36}(100-x)^2$$

Total Area:

$$A(x) = \frac{x^2}{16} + \frac{\sqrt{3}}{36}(100-x)^2$$

Step 2: Minimize Total Area

Differentiate:

$$A'(x) = \frac{1}{8}x - \frac{\sqrt{3}}{18}(100-x)$$

Set $A'(x) = 0$:

$$\frac{1}{8}x = \frac{\sqrt{3}}{18}(100-x) \Rightarrow 18x = 8\sqrt{3}(100-x) \Rightarrow 18x = 800\sqrt{3} - 8\sqrt{3}x \Rightarrow (18+8\sqrt{3})x = 800\sqrt{3} \Rightarrow x = \frac{800\sqrt{3}}{18+8\sqrt{3}}$$

Rationalizing denominator:

$$x = \frac{800\sqrt{3}(18-8\sqrt{3})}{(18+8\sqrt{3})(18-8\sqrt{3})} = \frac{800\sqrt{3}(18-8\sqrt{3})}{324-192} = \frac{800\sqrt{3}(18-8\sqrt{3})}{132}$$

Approximate:

$$x \approx 43.5 \Rightarrow \text{Use about } \boxed{43.5 \text{ cm for the square}}, \boxed{56.5 \text{ cm for the triangle}}$$

Step 3: Justification for Minimum

We compute the second derivative:

$$A''(x) = \frac{1}{8} + \frac{\sqrt{3}}{18} > 0$$

Since $A''(x) > 0$, this critical point gives a local minimum.

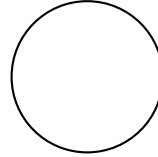
The wire distribution minimizes total area.

Question 6

A 100 cm long wire is to be cut into two pieces. One piece is bent to form a square, and the other is bent to form a circle. How should the wire be cut to minimize the total enclosed area?



Square



Circle

Question 6 - Solution

Let:

x = length used for the square, $0 < x < 100 \Rightarrow 100 - x$ = length used for the circle

Step 1: Area Expressions

Square:

$$\text{Perimeter} = x \Rightarrow \text{Side} = \frac{x}{4}, \quad A_{\text{square}} = \left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$$

Circle:

$$\text{Circumference} = 100 - x = 2\pi r \Rightarrow r = \frac{100 - x}{2\pi} \Rightarrow A_{\text{circle}} = \pi r^2 = \pi \left(\frac{100 - x}{2\pi}\right)^2 = \frac{(100 - x)^2}{4\pi}$$

Total Area:

$$A(x) = \frac{x^2}{16} + \frac{(100 - x)^2}{4\pi}$$

Step 2: Minimize Area

Differentiate:

$$A'(x) = \frac{1}{8}x - \frac{1}{2\pi}(100 - x)$$

Set $A'(x) = 0$:

$$\frac{1}{8}x = \frac{1}{2\pi}(100 - x) \Rightarrow \pi x = 4(100 - x) \Rightarrow \pi x = 400 - 4x \Rightarrow x(\pi + 4) = 400 \Rightarrow x = \frac{400}{\pi + 4}$$

Step 3: Approximate and Justify Minimum

$$x \approx \frac{400}{3.1416 + 4} = \frac{400}{7.1416} \approx 56.0$$

$$\boxed{x \approx 56.0 \text{ cm for the square}}, \quad \boxed{44.0 \text{ cm for the circle}}$$

Second derivative:

$$A''(x) = \frac{1}{8} + \frac{1}{2\pi} > 0 \Rightarrow \text{This critical point is a minimum.}$$

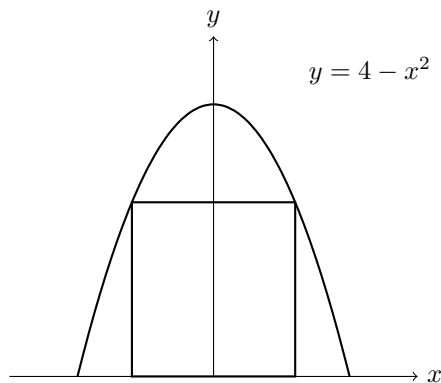
$$\boxed{\text{This wire division minimizes the total area.}}$$

Question 7

Find the dimensions of the rectangle of maximum area that can be inscribed under the parabola

$$y = 4 - x^2$$

with its base lying along the x-axis and the top two corners touching the curve.



Question 7 - Solution

Let the rectangle extend from $-x$ to x , so its width is $2x$, and its height is the y-value of the curve:

$$\text{Height} = y = 4 - x^2$$

Step 1: Area Function

$$A(x) = \text{width} \times \text{height} = 2x(4 - x^2) = 8x - 2x^3$$

Step 2: Maximize the Area

Differentiate:

$$A'(x) = 8 - 6x^2$$

Set $A'(x) = 0$:

$$8 - 6x^2 = 0 \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \frac{2}{\sqrt{3}} \approx 1.1547$$

Step 3: Dimensions of the Rectangle

$$\text{Width} = 2x = \frac{4}{\sqrt{3}} \approx 2.309$$

$$\text{Height} = 4 - x^2 = 4 - \frac{4}{3} = \frac{8}{3} \approx 2.667$$

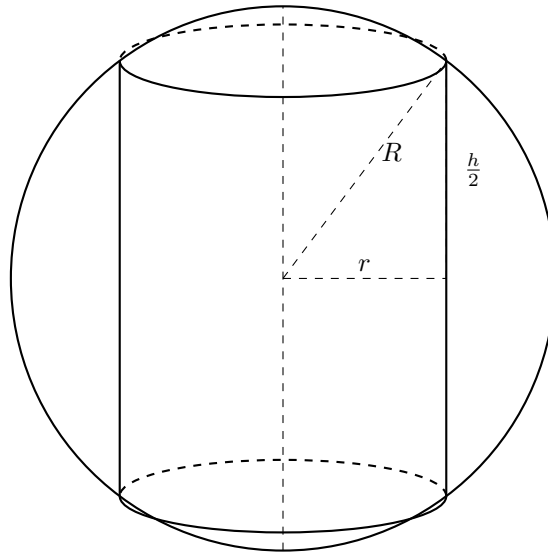
Maximum Area Rectangle Dimensions: Width ≈ 2.309 , Height ≈ 2.667

Step 4: Confirm Maximum

$$A''(x) = -12x < 0 \quad \text{at } x > 0 \Rightarrow \text{maximum confirmed.}$$

Question 8

A cylinder is inscribed in a sphere of radius R . Find the maximum possible volume of such a cylinder and determine its dimensions.



Cylinder inscribed in a sphere of radius R

Question 8 - Solution

Let the height of the cylinder be h and its radius be r . From the diagram, the diagonal of the right triangle gives the constraint:

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2 \Rightarrow h = 2\sqrt{R^2 - r^2}$$

Step 1: Volume Function

$$V = \pi r^2 h = \pi r^2 \cdot 2\sqrt{R^2 - r^2} = 2\pi r^2 \sqrt{R^2 - r^2}$$

Step 2: Maximize $V(r)$

Let:

$$V(r) = 2\pi r^2 (R^2 - r^2)^{1/2}$$

Differentiate using the product and chain rule:

$$V'(r) = 2\pi \left[2r(R^2 - r^2)^{1/2} - \frac{r^3}{(R^2 - r^2)^{1/2}} \right] = \frac{2\pi}{(R^2 - r^2)^{1/2}} (2r(R^2 - r^2) - r^3)$$

Set $V'(r) = 0$:

$$2r(R^2 - r^2) - r^3 = 0 \Rightarrow r(2R^2 - 3r^2) = 0 \Rightarrow r = 0 \text{ or } r^2 = \frac{2R^2}{3}$$

Step 3: Dimensions

$$r = \sqrt{\frac{2}{3}}R, \quad h = 2\sqrt{R^2 - r^2} = 2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}$$

Step 4: Max Volume

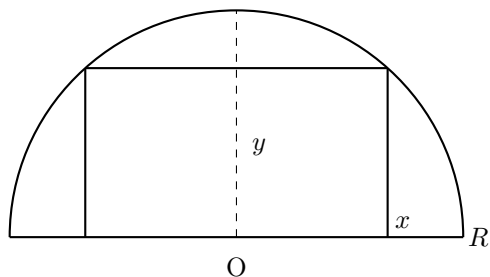
$$V = \pi r^2 h = \pi \cdot \left(\frac{2R^2}{3}\right) \cdot \frac{2R}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}$$

Radius: $r = \sqrt{\frac{2}{3}}R$
Height: $h = \frac{2R}{\sqrt{3}}$
Max Volume: $V = \frac{4\pi R^3}{3\sqrt{3}}$

Question 1

A rectangle is inscribed in a semicircle of radius R , with the rectangle's base lying along the diameter of the semicircle.

Find the dimensions of the rectangle that has the maximum area.



Rectangle inscribed in a semicircle

Question 1 - Solution

Let the coordinates of the top-right corner of the rectangle be (x, y) . Since the rectangle is symmetric and lies in a semicircle of radius R centered at the origin:

$$x^2 + y^2 = R^2 \quad (\text{semicircle equation}) \Rightarrow y = \sqrt{R^2 - x^2}$$

Step 1: Area of Rectangle

Since the base is $2x$ and the height is y , area is:

$$A(x) = 2x \cdot y = 2x\sqrt{R^2 - x^2}$$

Step 2: Maximize Area

$$A(x) = 2x\sqrt{R^2 - x^2}$$

Differentiate using the product rule:

$$A'(x) = 2\sqrt{R^2 - x^2} + 2x \cdot \frac{-x}{\sqrt{R^2 - x^2}} = \frac{2(R^2 - x^2) - 2x^2}{\sqrt{R^2 - x^2}} = \frac{2R^2 - 4x^2}{\sqrt{R^2 - x^2}}$$

Set $A'(x) = 0$:

$$2R^2 - 4x^2 = 0 \Rightarrow x^2 = \frac{R^2}{2} \Rightarrow x = \frac{R}{\sqrt{2}}, \quad y = \sqrt{R^2 - x^2} = \frac{R}{\sqrt{2}}$$

Step 3: Max Area and Dimensions

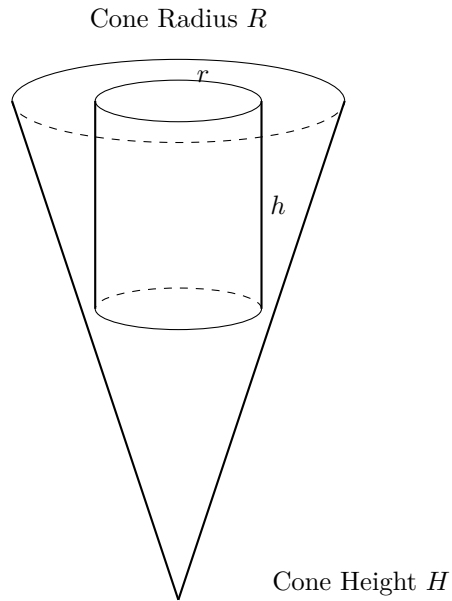
$$\text{Width} = 2x = \sqrt{2}R, \quad \text{Height} = y = \frac{R}{\sqrt{2}}, \quad \text{Area} = 2x \cdot y = R^2$$

Maximum area: $A = R^2$
Width: $2x = \sqrt{2}R$
Height: $y = \frac{R}{\sqrt{2}}$

Question 10

A right circular cylinder is inscribed in a right circular cone of height H and base radius R , such that the cylinder shares the same central axis as the cone.

Find the dimensions of the cylinder that maximize its volume.



Question 10 - Solution

Let the height of the cylinder be h , and its base radius be r . We want to maximize:

$$V = \pi r^2 h$$

Step 1: Relationship between r and h

From similar triangles (cone vs cylinder section), we have:

$$\frac{r}{H-h} = \frac{R}{H} \Rightarrow r = \frac{R(H-h)}{H}$$

Substitute into volume:

$$V(h) = \pi \left(\frac{R(H-h)}{H} \right)^2 h = \pi \cdot \frac{R^2(H-h)^2 h}{H^2}$$

Step 2: Maximize $V(h)$

Let's simplify:

$$V(h) = \frac{\pi R^2}{H^2} (H-h)^2 h$$

Let $k = \frac{\pi R^2}{H^2}$, then:

$$V(h) = k(H-h)^2 h$$

Use the product rule:

$$V'(h) = k [2(H-h)(-1) \cdot h + (H-h)^2] = k [-2h(H-h) + (H-h)^2]$$

$$= k(H-h)[(H-h) - 2h] = k(H-h)(H-3h)$$

Set $V'(h) = 0$:

$$(H-h)(H-3h) = 0 \Rightarrow h = H \text{ or } h = \frac{H}{3}$$

But $h = H$ gives 0 volume. So maximum occurs at $\boxed{h = \frac{H}{3}}$

Substitute into r :

$$r = \frac{R(H-H/3)}{H} = \frac{2R}{3}$$

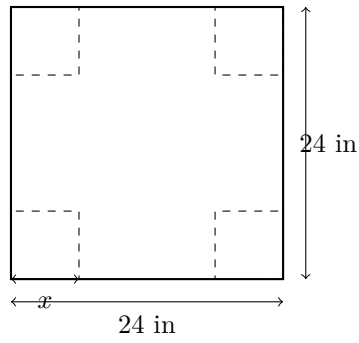
Step 3: Final Answer

Height of cylinder: $h = \frac{H}{3}$
Radius of cylinder: $r = \frac{2R}{3}$
Maximum Volume: $V = \pi r^2 h = \pi \cdot \left(\frac{2R}{3} \right)^2 \cdot \frac{H}{3} = \frac{4\pi R^2 H}{27}$

Question 1

A square piece of cardboard 24 inches on each side is to be made into an open-top box by cutting equal squares from each corner and folding up the sides. What size square should be cut from each corner to maximize the volume of the box?

Diagram:



Question 1 - Solution

Let x be the length of the square cut from each corner.

After folding, the dimensions of the box will be:

Height = x

Length and width = $24 - 2x$

Volume:

$$V(x) = x(24 - 2x)^2$$

Step 1: Expand volume function

$$V(x) = x(576 - 96x + 4x^2) = 576x - 96x^2 + 4x^3$$

Step 2: Find critical points

$$V'(x) = 576 - 192x + 12x^2 \Rightarrow 12x^2 - 192x + 576 = 0 \Rightarrow x^2 - 16x + 48 = 0 \Rightarrow x = 8 \pm \sqrt{16} \Rightarrow x = 8 \pm 4$$

$$x = 4 \quad \text{or} \quad x = 12$$

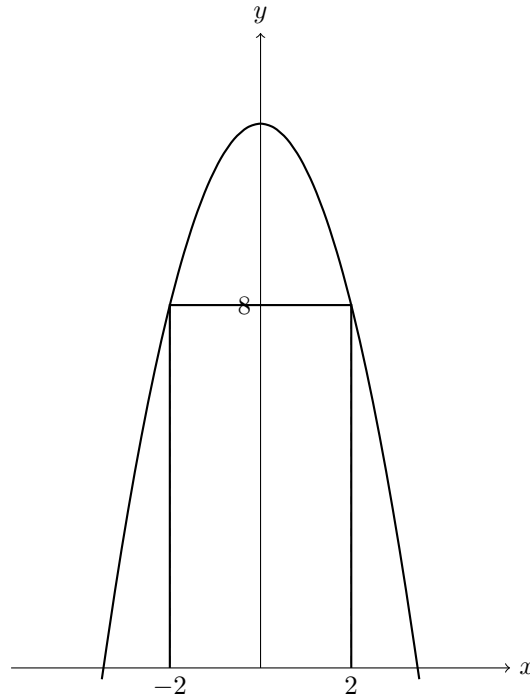
But $x = 12$ makes $24 - 2x = 0 \rightarrow$ not valid

Valid critical point: $x = 4$

Answer: Cut out squares of size 4 in to maximize volume

Question 2

A rectangle is inscribed under the parabola $y = 12 - x^2$ such that its base lies on the x -axis and its upper corners lie on the parabola.



Find the dimensions of the rectangle of maximum area.

Question 2 - Solution

Let the top corners of the rectangle be (x, y) and $(-x, y)$, where the rectangle is symmetric about the y -axis.

From the parabola:

$$y = 12 - x^2$$

The width of the rectangle is $2x$, and the height is $y = 12 - x^2$

Area Function:

$$A(x) = \text{width} \times \text{height} = 2x(12 - x^2) = 24x - 2x^3$$

Differentiate:

$$A'(x) = 24 - 6x^2$$

Set derivative to zero:

$$24 - 6x^2 = 0 \Rightarrow x^2 = 4 \Rightarrow x = 2$$

Check Maximum:

Second derivative:

$$A''(x) = -12x \Rightarrow A''(2) = -24 < 0$$

So, $x = 2$ gives a maximum.

Final Dimensions:

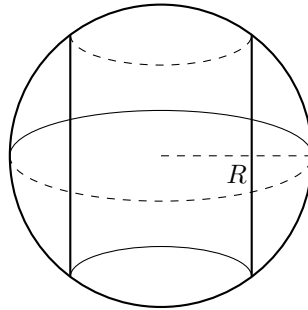
$$x = 2 \Rightarrow \text{width} = 2x = 4$$

$$y = 12 - x^2 = 12 - 4 = 8$$

Width = 4
Height = 8
Maximum Area = 32

Question 3

A right circular cylinder is inscribed inside a sphere of radius R .



Find the dimensions (radius and height) of the cylinder that ****maximize its volume****, and express the maximum volume in terms of R .

Question 3 - Solution

Let the radius of the cylinder be r and height h . Since the cylinder fits in the sphere of radius R , we use the Pythagorean relation from the triangle formed by r , $h/2$, and R :

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2 \Rightarrow h = 2\sqrt{R^2 - r^2}$$

Volume of the Cylinder:

$$V = \pi r^2 h = \pi r^2 \cdot 2\sqrt{R^2 - r^2} = 2\pi r^2 \sqrt{R^2 - r^2}$$

Maximize Volume:

Let:

$$V(r) = 2\pi r^2 \sqrt{R^2 - r^2}$$

Take the derivative (using product and chain rule):

$$\begin{aligned} V'(r) &= 2\pi \left[2r\sqrt{R^2 - r^2} + r^2 \cdot \frac{1}{2}(R^2 - r^2)^{-1/2}(-2r) \right] \\ &= 2\pi \left[2r\sqrt{R^2 - r^2} - \frac{r^3}{\sqrt{R^2 - r^2}} \right] \end{aligned}$$

Set $V'(r) = 0$:

$$2r\sqrt{R^2 - r^2} = \frac{r^3}{\sqrt{R^2 - r^2}} \Rightarrow 2(R^2 - r^2) = r^2 \Rightarrow 2R^2 - 2r^2 = r^2 \Rightarrow 3r^2 = 2R^2 \Rightarrow r^2 = \frac{2}{3}R^2$$

$$r = R\sqrt{\frac{2}{3}}, \quad h = 2\sqrt{R^2 - r^2} = 2\sqrt{R^2 - \frac{2}{3}R^2} = 2\sqrt{\frac{1}{3}R^2} = 2R\sqrt{\frac{1}{3}}$$

Maximum Volume:

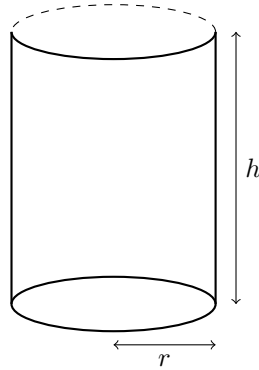
$$V = \pi r^2 h = \pi \cdot \frac{2}{3}R^2 \cdot 2R\sqrt{\frac{1}{3}} = \frac{4\pi R^3}{3} \cdot \frac{1}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}$$

$\begin{aligned} r &= R\sqrt{\frac{2}{3}} \\ h &= 2R\sqrt{\frac{1}{3}} \\ V_{\max} &= \frac{4\pi R^3}{3\sqrt{3}} \end{aligned}$

Question 4

A closed right circular cylinder has a fixed volume of $V = 1000 \text{ cm}^3$.

Find the radius and height of the cylinder that minimizes its surface area. Also determine the minimum surface area.



Question 4 - Solution

Let the radius be r , and the height be h . We are given:

$$V = \pi r^2 h = 1000 \Rightarrow h = \frac{1000}{\pi r^2}$$

Surface Area of the Cylinder:

$$A = 2\pi r^2 + 2\pi r h \Rightarrow A(r) = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}$$

Minimize $A(r)$:

Take derivative:

$$A'(r) = 4\pi r - \frac{2000}{r^2}$$

Set $A'(r) = 0$:

$$4\pi r = \frac{2000}{r^2} \Rightarrow 4\pi r^3 = 2000 \Rightarrow r^3 = \frac{500}{\pi} \Rightarrow r = \sqrt[3]{\frac{500}{\pi}}$$

Now compute h :

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi \left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = \frac{1000}{\pi} \cdot \left(\frac{\pi}{500}\right)^{2/3} = 1000 \cdot \pi^{-1/3} \cdot 500^{-2/3}$$

Minimum Surface Area:

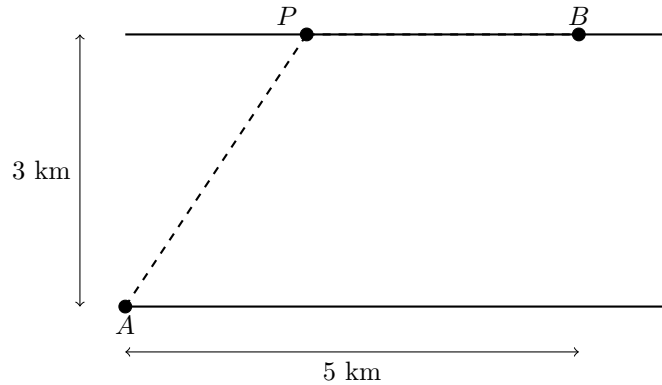
$$A = 2\pi r^2 + \frac{2000}{r} = 2\pi \left(\sqrt[3]{\frac{500}{\pi}}\right)^2 + \frac{2000}{\sqrt[3]{\frac{500}{\pi}}}$$

$r = \sqrt[3]{\frac{500}{\pi}}$
$h = \frac{1000}{\pi r^2}$
$A_{\min} = 2\pi r^2 + \frac{2000}{r}$

Question 5

A person is standing at point A on the shore of a straight river that is 3 km wide. Point B lies on the opposite shore, 5 km downstream from the point directly opposite A . The person can run at 10 km/h along the shore and swim at 5 km/h directly across the river or at an angle.

The person first swims from A to P , then runs along the opposite shore to B . At what point P between the point directly opposite A and B should the person land to minimize total time to reach B ?



Question 5 - Solution

Let the swimmer land at point P , which is x km from the point directly across from A .

Step 1: Write distances:

Swim distance: $\sqrt{x^2 + 3^2} = \sqrt{x^2 + 9}$, Run distance: $5 - x$

Step 2: Write time expression:

Swimming speed: 5 km/h , Running speed: 10 km/h

$$T(x) = \frac{\sqrt{x^2 + 9}}{5} + \frac{5 - x}{10}$$

Step 3: Minimize time — take derivative:

$$T'(x) = \frac{x}{5\sqrt{x^2 + 9}} - \frac{1}{10}$$

Set $T'(x) = 0$:

$$\frac{x}{5\sqrt{x^2 + 9}} = \frac{1}{10} \Rightarrow \frac{2x}{\sqrt{x^2 + 9}} = 1 \Rightarrow 4x^2 = x^2 + 9 \Rightarrow 3x^2 = 9 \Rightarrow x^2 = 3 \Rightarrow x = \sqrt{3}$$

Step 4: Answer

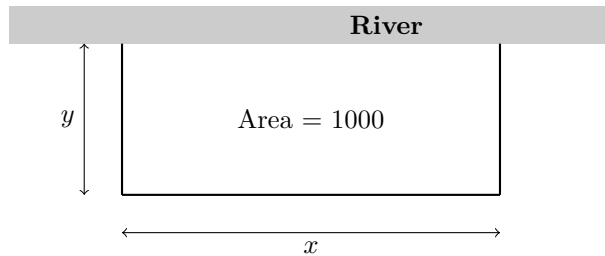
$$x = \sqrt{3} \approx 1.732 \text{ km}$$

$T_{\min}$ = when the person lands $\sqrt{3}$ km from the point directly across from A .

Question 6

A rectangular enclosure is to be built adjacent to a river (no fence is needed along the river side). The other three sides are to be fenced. The area of the enclosure must be 1000 square meters.

If the fencing along the two sides perpendicular to the river costs \$10 per meter and the fencing along the side parallel to the river costs \$5 per meter, what dimensions will minimize the total cost?



Question 6 - Solution

Let: - x = length along the river (no fence needed) - y = depth away from river (fence on both sides)

Area constraint:

$$xy = 1000 \Rightarrow y = \frac{1000}{x}$$

Cost function: - Two sides of length y at \$10/m: cost = $2(10)(y) = 20y$ - One side of length x at \$5/m: cost = $5x$

$$C(x) = 5x + 20 \cdot \frac{1000}{x} = 5x + \frac{20000}{x}$$

Minimize cost — take derivative:

$$C'(x) = 5 - \frac{20000}{x^2}$$

Set derivative to zero:

$$5 = \frac{20000}{x^2} \Rightarrow x^2 = 4000 \Rightarrow x = \sqrt{4000} = 20\sqrt{10}$$

Find y :

$$y = \frac{1000}{x} = \frac{1000}{20\sqrt{10}} = \frac{50}{\sqrt{10}} = 5\sqrt{10}$$

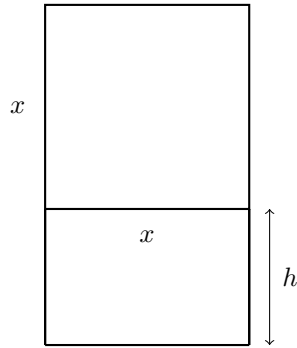
Answer:

$x = 20\sqrt{10} \approx 63.25 \text{ m}$ $y = 5\sqrt{10} \approx 15.81 \text{ m}$

These dimensions minimize the fencing cost.

Question 7

A rectangular box with a square base and no top is to have a volume of 500 cm^3 . What dimensions will minimize the surface area?



Open-top square box

Question 7 - Solution

Let: - x = side length of the square base (in cm) - h = height of the box (in cm)

Volume constraint:

$$V = x^2h = 500 \Rightarrow h = \frac{500}{x^2}$$

Surface area to minimize: - One bottom square: x^2 - Four sides: $4(xh)$

$$A(x) = x^2 + 4xh = x^2 + 4x \cdot \frac{500}{x^2} = x^2 + \frac{2000}{x}$$

Take derivative:

$$A'(x) = 2x - \frac{2000}{x^2}$$

Set derivative to zero:

$$2x = \frac{2000}{x^2} \Rightarrow 2x^3 = 2000 \Rightarrow x^3 = 1000 \Rightarrow x = 10$$

Find h :

$$h = \frac{500}{x^2} = \frac{500}{100} = 5$$

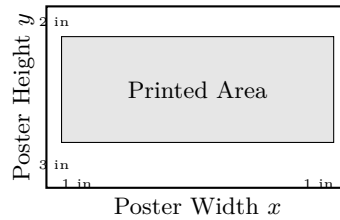
Answer:

$x = 10 \text{ cm}, \quad h = 5 \text{ cm}$

These dimensions minimize the surface area.

Question 8

A poster is to have an area of 192 in^2 . It must have 1-inch margins on the sides, a 2-inch margin at the top, and a 3-inch margin at the bottom. What dimensions of the poster will allow the largest printed area?



Question 8 - Solution

Let x be poster width and y its height. Then $xy = 192$, and positive printed dimensions require $2 < x < 192/5$.

$$A(x) = (x - 2)(192/x - 5) = 202 - 5x - 384/x.$$

$$A'(x) = -5 + 384/x^2 = 0 \implies x = \sqrt{384/5}.$$

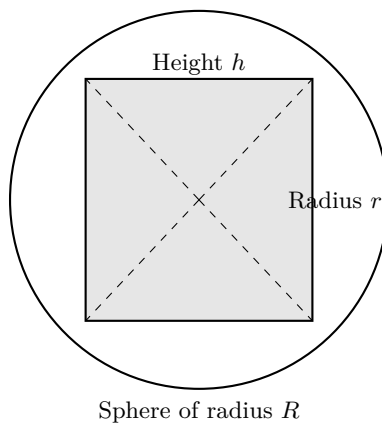
Since $A''(x) = -768/x^3 < 0$, this is the unique maximum.

$x = \sqrt{384/5} \approx 8.76 \text{ in}, \quad y = \sqrt{480} \approx 21.91 \text{ in.}$
--

The exact width, rather than its rounded value, is used to compute the height.

Question 9

A cylinder is inscribed in a sphere of radius R . Find the dimensions (radius and height) of the cylinder that has the maximum volume.



Question 9 - Solution

Let the cylinder have: - Radius: r - Height: h

It is inscribed in a sphere of radius R , so by the Pythagorean Theorem applied to the right triangle formed from the center of the sphere to the top edge of the cylinder:

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2$$

Volume of the cylinder:

$$V = \pi r^2 h$$

From the constraint:

$$r^2 = R^2 - \left(\frac{h}{2}\right)^2 = R^2 - \frac{h^2}{4}$$

Substitute into the volume:

$$V(h) = \pi \left(R^2 - \frac{h^2}{4}\right) h = \pi \left(R^2 h - \frac{h^3}{4}\right)$$

Differentiate and maximize:

$$V'(h) = \pi \left(R^2 - \frac{3h^2}{4}\right)$$

Set $V'(h) = 0$:

$$R^2 - \frac{3h^2}{4} = 0 \Rightarrow h^2 = \frac{4R^2}{3} \Rightarrow h = \frac{2R}{\sqrt{3}}$$

Then:

$$r^2 = R^2 - \left(\frac{h}{2}\right)^2 = R^2 - \frac{1}{4} \cdot \frac{4R^2}{3} = R^2 - \frac{R^2}{3} = \frac{2R^2}{3} \Rightarrow r = \sqrt{\frac{2}{3}}R$$

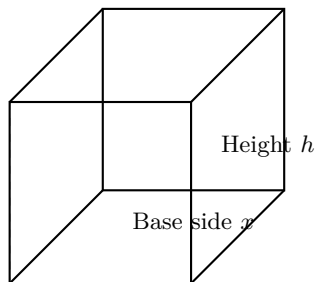
Answer:

$$\boxed{h = \frac{2R}{\sqrt{3}}, \quad r = \sqrt{\frac{2}{3}}R}$$

These are the dimensions of the inscribed cylinder with maximum volume.

Question 10

A box with a square base and an open top must have a volume of V cubic units. Find the dimensions of the box that minimize its surface area.



Question 10 - Solution

Let: - x = length of the base side (square base) - h = height of the box - Volume: $V = x^2h \Rightarrow h = \frac{V}{x^2}$

Surface Area (SA): The box has no top, so:

$$A(x) = \text{Base} + 4 \text{ sides} = x^2 + 4xh$$

$$A(x) = x^2 + 4x \left(\frac{V}{x^2} \right) = x^2 + \frac{4V}{x}$$

Minimize $A(x)$:

Differentiate:

$$A'(x) = 2x - \frac{4V}{x^2}$$

Set $A'(x) = 0$:

$$2x - \frac{4V}{x^2} = 0 \Rightarrow 2x^3 = 4V \Rightarrow x^3 = 2V \Rightarrow x = \sqrt[3]{2V}$$

Then:

$$h = \frac{V}{x^2} = \frac{V}{\left(\sqrt[3]{2V}\right)^2} = \frac{V}{\sqrt[3]{4V^2}} = \sqrt[3]{\frac{V}{4}}$$

Answer:

$$x = \sqrt[3]{2V}, \quad h = \sqrt[3]{\frac{V}{4}}$$

These dimensions give the minimum surface area for the box.

Question 1

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x}$$

- (a) Verify that the limit results in an indeterminate form.
- (b) Use L'Hospital's Rule to evaluate the limit.

Question 1 - Solution

(a) As $x \rightarrow 0$:

- $\sin(3x) \rightarrow 0$, and $x \rightarrow 0$ - So, the limit is of the indeterminate form $\frac{0}{0}$

(b) Apply L'Hospital's Rule:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}[\sin(3x)]}{\frac{d}{dx}[x]} \\ &= \lim_{x \rightarrow 0} \frac{3 \cos(3x)}{1} = 3 \cos(0) = 3(1) = \boxed{3}\end{aligned}$$

Question 2

Evaluate the limit:

$$\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$$

Question 2 - Solution

We observe that both $\frac{1}{x} \rightarrow \infty$ and $\frac{1}{\sin x} \rightarrow \infty$ as $x \rightarrow 0^+$, so the expression is of the indeterminate form:

$$\infty - \infty$$

We first combine into a single fraction:

$$\frac{1}{x} - \frac{1}{\sin x} = \frac{\sin x - x}{x \sin x}$$

Now the limit becomes:

$$\lim_{x \rightarrow 0^+} \frac{\sin x - x}{x \sin x}$$

As $x \rightarrow 0^+$, both numerator and denominator $\rightarrow 0$. This is now a $\frac{0}{0}$ indeterminate form, so we apply L'Hospital's Rule:

$$\lim_{x \rightarrow 0^+} \frac{\sin x - x}{x \sin x} = \lim_{x \rightarrow 0^+} \frac{\cos x - 1}{\sin x + x \cos x}$$

Now evaluate the new limit:

- Numerator: $\cos x - 1 \rightarrow 0$ - Denominator: $\sin x + x \cos x \rightarrow 0$

Still indeterminate! Apply L'Hospital's Rule again:

$$= \lim_{x \rightarrow 0^+} \frac{-\sin x}{\cos x + \cos x - x \sin x} = \lim_{x \rightarrow 0^+} \frac{-\sin x}{2 \cos x - x \sin x}$$

Now evaluate: - Numerator: $-\sin x \rightarrow 0$ - Denominator: $2 \cos x - x \sin x \rightarrow 2$

So the final result is:

$$\boxed{0}$$

Question 3

Evaluate the limit:

$$\lim_{x \rightarrow \infty} \left(x \ln \left(1 + \frac{1}{x} \right) \right)$$

Question 3 - Solution

We observe that as $x \rightarrow \infty$:

$$- \ln\left(1 + \frac{1}{x}\right) \rightarrow 0 \quad - x \rightarrow \infty$$

So the expression becomes:

$$\infty \cdot 0 \quad (\text{indeterminate form})$$

We rewrite it as a quotient:

$$x \ln\left(1 + \frac{1}{x}\right) = \frac{\ln\left(1 + \frac{1}{x}\right)}{1/x}$$

Now as $x \rightarrow \infty$, we get a $\frac{0}{0}$ form, so we can apply L'Hospital's Rule.

Let's compute the limit using L'Hospital:

$$\lim_{x \rightarrow \infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{1/x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln\left(1 + \frac{1}{x}\right)}{\frac{d}{dx} \left(\frac{1}{x}\right)}$$

Differentiate numerator and denominator:

- Numerator derivative:

$$\frac{d}{dx} \ln\left(1 + \frac{1}{x}\right) = \frac{-1}{x^2\left(1 + \frac{1}{x}\right)} = \frac{-1}{x(x+1)}$$

- Denominator derivative:

$$\frac{d}{dx} \left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

Now plug in:

$$\begin{aligned} &= \lim_{x \rightarrow \infty} \frac{-1/(x(x+1))}{-1/x^2} = \lim_{x \rightarrow \infty} \frac{x^2}{x(x+1)} = \lim_{x \rightarrow \infty} \frac{x}{x+1} \\ &\quad \lim_{x \rightarrow \infty} \frac{x}{x+1} = 1 \end{aligned}$$

Final Answer:

$$\boxed{1}$$

Question 4

Evaluate the limit:

$$\lim_{x \rightarrow 0^+} (1 + 2x)^{\frac{1}{x}}$$

Question 4 - Solution

We identify the limit as the indeterminate form:

$$\lim_{x \rightarrow 0^+} (1 + 2x)^{\frac{1}{x}} = 1^\infty$$

Let:

$$y = (1 + 2x)^{\frac{1}{x}} \Rightarrow \ln y = \frac{1}{x} \ln(1 + 2x)$$

Now compute the limit:

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{\ln(1 + 2x)}{x}$$

This is a $\frac{0}{0}$ form. Apply L'Hospital's Rule:

$$= \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} \ln(1 + 2x)}{\frac{d}{dx} x} = \lim_{x \rightarrow 0^+} \frac{\frac{2}{1+2x}}{1} = \frac{2}{1} = 2$$

So:

$$\ln y \rightarrow 2 \quad \Rightarrow \quad y \rightarrow e^2$$

Final Answer:

$$\boxed{e^2}$$

Question 5

Evaluate the limit:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}}$$

Question 5 - Solution

As $x \rightarrow \infty$, both numerator and denominator approach infinity:

$$\frac{\ln x}{\sqrt{x}} \rightarrow \frac{\infty}{\infty} \quad (\text{indeterminate form})$$

Apply L'Hospital's Rule:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} \sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{2\sqrt{x}}{x} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0$$

Final Answer:

$$\boxed{0}$$

Question 6

Evaluate the limit:

$$\lim_{x \rightarrow \infty} (x - \ln x)$$

Question 6 - Solution

As $x \rightarrow \infty$, we observe:

$$x \rightarrow \infty, \quad \ln x \rightarrow \infty$$

So the limit is of the form:

$$\infty - \infty \quad (\text{indeterminate form})$$

To apply L'Hospital's Rule, we rewrite the expression to form a quotient. Factor:

$$x - \ln x = x \left(1 - \frac{\ln x}{x} \right)$$

Now evaluate:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0 \quad (\text{since exponential grows faster})$$

So:

$$\lim_{x \rightarrow \infty} (x - \ln x) = \infty \cdot (1 - 0) = \infty$$

Final Answer: ∞

Question 7

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$$

Question 7 - Solution

As $x \rightarrow 0$, we know:

$$\tan x \rightarrow 0, \quad x \rightarrow 0 \Rightarrow \tan x - x \rightarrow 0 \quad \text{and} \quad x^3 \rightarrow 0 \Rightarrow \text{Indeterminate form } \frac{0}{0}$$

Apply L'Hospital's Rule:

First derivative:

$$\lim_{x \rightarrow 0} \frac{\frac{d}{dx}[\tan x - x]}{\frac{d}{dx}[x^3]} = \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2}$$

As $x \rightarrow 0$, $\sec^2 x \rightarrow 1$, so numerator $\rightarrow 0$, denominator $\rightarrow 0$, still $\frac{0}{0}$

Second derivative:

$$\lim_{x \rightarrow 0} \frac{\frac{d}{dx}[\sec^2 x - 1]}{\frac{d}{dx}[3x^2]} = \lim_{x \rightarrow 0} \frac{2 \sec^2 x \cdot \sec x \tan x}{6x}$$

As $x \rightarrow 0$, $\tan x \rightarrow 0$, so numerator $\rightarrow 0$, denominator $\rightarrow 0$

Third derivative:

Differentiate numerator and denominator again. Let's simplify instead:

Use the series:

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \Rightarrow \tan x - x = \frac{x^3}{3} + \cdots$$

Then:

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\frac{x^3}{3} + \cdots}{x^3} = \frac{1}{3}$$

Final Answer:

$$\boxed{\frac{1}{3}}$$

Question 8

Evaluate the limit:

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x)$$

Question 8 - Solution

As $x \rightarrow \infty$, we have:

$$\sqrt{x^2 + x} \rightarrow \infty, \quad x \rightarrow \infty \Rightarrow \text{Indeterminate form: } \infty - \infty$$

To resolve this, multiply by the conjugate:

$$\begin{aligned} \lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x) &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + x} - x)(\sqrt{x^2 + x} + x)}{\sqrt{x^2 + x} + x} \\ &= \lim_{x \rightarrow \infty} \frac{(x^2 + x) - x^2}{\sqrt{x^2 + x} + x} = \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + x} + x} \end{aligned}$$

Now simplify numerator and denominator:

Factor x from the square root:

$$= \lim_{x \rightarrow \infty} \frac{x}{x \left(\sqrt{1 + \frac{1}{x}} + 1 \right)} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{x}} + 1}$$

As $x \rightarrow \infty$, $\frac{1}{x} \rightarrow 0$, so:

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{x}} + 1} = \frac{1}{1 + 1} = \frac{1}{2}$$

Final Answer:

$$\boxed{\frac{1}{2}}$$

Question 9

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$$

Question 9 - Solution

As $x \rightarrow 0$, we have:

$$\tan x \rightarrow 0, \quad x \rightarrow 0 \quad \Rightarrow \quad \frac{\tan x - x}{x^3} \rightarrow \frac{0}{0} \quad (\text{indeterminate form})$$

Apply L'Hospital's Rule:

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2}$$

As $x \rightarrow 0$, $\sec^2 x \rightarrow 1$, so the numerator goes to 0, denominator to 0:

$$\Rightarrow \frac{0}{0} \quad \text{Apply L'Hospital's Rule again}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{6x} = \lim_{x \rightarrow 0} \frac{2 \cdot 1 \cdot 0}{6 \cdot 0} = \frac{0}{0} \quad (\text{Apply once more})$$

Take third derivative:

Numerator derivative:

$$\frac{d}{dx} [2 \sec^2 x \tan x] = 2 (2 \sec^2 x \tan^2 x + \sec^4 x)$$

Denominator derivative:

$$\frac{d}{dx} (6x) = 6$$

Now take the limit:

$$\lim_{x \rightarrow 0} \frac{2(2 \sec^2 x \tan^2 x + \sec^4 x)}{6} = \frac{2(2 \cdot 1 \cdot 0 + 1)}{6} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

Question 10

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(5x)}{\tan(3x)}$$

Question 10 - Solution

As $x \rightarrow 0$:

$$\sin(5x) \rightarrow 0, \quad \tan(3x) \rightarrow 0 \Rightarrow \text{Indeterminate form } \frac{0}{0}$$

Apply L'Hospital's Rule:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin(5x)}{\tan(3x)} &= \lim_{x \rightarrow 0} \frac{d}{dx} \left[\frac{\sin(5x)}{\tan(3x)} \right] = \lim_{x \rightarrow 0} \frac{5 \cos(5x)}{3 \sec^2(3x)} \\ &= \lim_{x \rightarrow 0} \frac{5 \cos(5x)}{3 \cdot \frac{1}{\cos^2(3x)}} = \lim_{x \rightarrow 0} \frac{5 \cos(5x) \cos^2(3x)}{3} \end{aligned}$$

As $x \rightarrow 0$:

$$\cos(5x) \rightarrow 1, \quad \cos(3x) \rightarrow 1 \Rightarrow \cos^2(3x) \rightarrow 1$$

So,

$$\frac{5 \cdot 1 \cdot 1}{3} = \frac{5}{3}$$

Final Answer: $\boxed{\frac{5}{3}}$

Question 1

Use a linear approximation to estimate $\sqrt{4.1}$.

- (a) Define the function and the point where you linearize.
- (b) Find the linearization $L(x)$ at the chosen point.
- (c) Use $L(x)$ to estimate $\sqrt{4.1}$.

Question 1 - Solution

(a) Let $f(x) = \sqrt{x}$

Choose point $a = 4$, since $\sqrt{4} = 2$ is easy to compute.

(b) Linearization formula:

$$L(x) = f(a) + f'(a)(x - a)$$

Compute derivative:

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(4) = \frac{1}{2 \cdot 2} = \frac{1}{4}$$

$$L(x) = 2 + \frac{1}{4}(x - 4)$$

(c) Estimate $\sqrt{4.1} \approx L(4.1)$:

$$L(4.1) = 2 + \frac{1}{4}(0.1) = 2 + 0.025 = \boxed{2.025}$$

Question 2

Use a linear approximation to estimate the value of $\sqrt[3]{8.1}$.

- (a) Identify a function $f(x)$, a point a , and write the linear approximation $L(x)$.
- (b) Use $L(x)$ to estimate $\sqrt[3]{8.1}$.
- (c) Compare the estimate to the actual value using a calculator.

Question 2 - Solution

Let $f(x) = \sqrt[3]{x} = x^{1/3}$. We choose $a = 8$, since $\sqrt[3]{8} = 2$ is easy to compute.

(a) Linear approximation formula:

$$L(x) = f(a) + f'(a)(x - a)$$

Compute:

$$f(x) = x^{1/3}, \quad f'(x) = \frac{1}{3}x^{-2/3} \Rightarrow f'(8) = \frac{1}{3(8)^{2/3}} = \frac{1}{3 \cdot 4} = \frac{1}{12}$$

So:

$$L(x) = f(8) + f'(8)(x - 8) = 2 + \frac{1}{12}(x - 8)$$

(b) Estimate:

$$L(8.1) = 2 + \frac{1}{12}(8.1 - 8) = 2 + \frac{1}{12}(0.1) = 2 + \frac{1}{120} \approx 2.0083$$

(c) Actual value:

$$\sqrt[3]{8.1} \approx 2.0083 \quad (\text{using calculator})$$

Conclusion: The linear approximation gives a very close estimate:

$\sqrt[3]{8.1} \approx 2.0083$

Question 3

Use a linear approximation to estimate the value of $\ln(1.05)$.

- (a) Identify a function $f(x)$, choose a point a , and write the linear approximation $L(x)$.
- (b) Use $L(x)$ to estimate $\ln(1.05)$.
- (c) Compare the estimate to the actual value.

Question 3 - Solution

Let $f(x) = \ln(x)$, and choose $a = 1$ because $\ln(1) = 0$ is easy to compute.

(a) Linear approximation formula:

$$L(x) = f(a) + f'(a)(x - a)$$

Compute:

$$f(x) = \ln(x), \quad f'(x) = \frac{1}{x}, \quad f'(1) = 1$$

So:

$$L(x) = \ln(1) + 1 \cdot (x - 1) = 0 + (x - 1) = x - 1$$

(b) Estimate:

$$L(1.05) = 1.05 - 1 = 0.05$$

(c) Actual value:

$$\ln(1.05) \approx 0.04879 \quad (\text{using calculator})$$

Conclusion: The linear approximation gives:

$\ln(1.05) \approx 0.05$

which is a very good estimate.

Question 4

Use a linear approximation to estimate $\sqrt[3]{8.1}$.

- (a) Identify a function $f(x)$, choose a suitable point a , and write the linear approximation $L(x)$.
- (b) Use $L(x)$ to estimate $\sqrt[3]{8.1}$.
- (c) Compare the estimate to the actual value.

Question 4 - Solution

Let $f(x) = \sqrt[3]{x} = x^{1/3}$, and choose $a = 8$, since $\sqrt[3]{8} = 2$.

(a) **Linear approximation formula:**

$$L(x) = f(a) + f'(a)(x - a)$$

Compute:

$$f'(x) = \frac{1}{3}x^{-2/3}, \quad f'(8) = \frac{1}{3} \cdot 8^{-2/3} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

So:

$$L(x) = 2 + \frac{1}{12}(x - 8)$$

(b) **Estimate:**

$$L(8.1) = 2 + \frac{1}{12}(0.1) = 2 + \frac{1}{120} = \frac{241}{120} \approx 2.0083$$

(c) **Actual value:**

$$\sqrt[3]{8.1} \approx 2.00828 \quad (\text{using calculator})$$

Conclusion: The linear approximation gives:

$\sqrt[3]{8.1} \approx 2.0083$

which is very close to the actual value.

Question 5

Use linear approximation to estimate $\ln(1.05)$.

- (a) Define an appropriate function $f(x)$, and choose a convenient point a near 1.05.
- (b) Find the linear approximation $L(x)$ of $f(x)$ at a .
- (c) Use $L(x)$ to approximate $\ln(1.05)$, and compare it with the actual value.

Question 5 - Solution

Let $f(x) = \ln(x)$. Choose $a = 1$, since $\ln(1) = 0$ is simple and near 1.05.

(a) Derivative of $f(x)$:

$$f'(x) = \frac{1}{x}, \quad f'(1) = 1$$

(b) Linear approximation:

$$L(x) = f(a) + f'(a)(x - a) = \ln(1) + 1(x - 1) = x - 1$$

(c) Estimate:

$$L(1.05) = 1.05 - 1 = 0.05$$

Actual value:

$$\ln(1.05) \approx 0.04879 \quad (\text{using calculator})$$

Conclusion:

$$\boxed{\ln(1.05) \approx 0.05} \quad (\text{Linear Approximation})$$

Actual: $\ln(1.05) \approx 0.04879 \Rightarrow$ Very close!

Question 6

Use linear approximation to estimate $\sqrt[3]{8.2}$.

- (a) Define an appropriate function $f(x)$, and choose a point a near 8.2.
- (b) Find the linear approximation $L(x)$ of $f(x)$ at a .
- (c) Use $L(x)$ to estimate $\sqrt[3]{8.2}$, and compare it to the actual value.

Question 6 - Solution

Let $f(x) = \sqrt[3]{x} = x^{1/3}$, and choose $a = 8$ since $\sqrt[3]{8} = 2$ is exact and close to 8.2.

(a) Compute derivative:

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}, \quad \text{so} \quad f'(8) = \frac{1}{3\sqrt[3]{64}} = \frac{1}{3 \cdot 4} = \frac{1}{12}$$

(b) Linear approximation:

$$L(x) = f(a) + f'(a)(x - a) = 2 + \frac{1}{12}(x - 8)$$

(c) Estimate:

$$L(8.2) = 2 + \frac{1}{12}(0.2) = 2 + \frac{0.2}{12} = 2 + \frac{1}{60} \approx 2.0167$$

Actual value:

$$\sqrt[3]{8.2} \approx 2.0164 \quad (\text{from calculator})$$

Conclusion:

$$\boxed{\sqrt[3]{8.2} \approx 2.0167} \quad (\text{Linear Approximation})$$

$$\text{Actual: } \sqrt[3]{8.2} \approx 2.0164 \Rightarrow \text{Very close!}$$

Question 7

Use linear approximation to estimate $\ln(1.05)$.

- (a) Define a function $f(x)$ and choose a suitable point a near 1.05.
- (b) Find the linear approximation $L(x)$ of $f(x)$ at $x = a$.
- (c) Use $L(x)$ to estimate $\ln(1.05)$ and compare to the actual value.

Question 7 - Solution

Let $f(x) = \ln(x)$. We choose $a = 1$ since $\ln(1) = 0$ is easy to compute.

(a) Compute the derivative:

$$f'(x) = \frac{1}{x}, \quad \text{so } f'(1) = 1$$

(b) Linear approximation:

$$L(x) = f(a) + f'(a)(x - a) = 0 + 1(x - 1) = x - 1$$

(c) Estimate:

$$L(1.05) = 1.05 - 1 = 0.05$$

Compare to actual:

$$\ln(1.05) \approx 0.04879 \quad (\text{from calculator})$$

Conclusion:

$$\boxed{\ln(1.05) \approx 0.05} \quad (\text{Linear Approximation})$$

Actual: $\ln(1.05) \approx 0.04879 \Rightarrow$ Very close!

Question 8

Use a linear approximation to estimate $\sqrt{10.2}$.

- (a) Define an appropriate function $f(x)$ and choose a point a close to 10.2 where $f(a)$ is easy to compute.
- (b) Find the linear approximation $L(x)$ to $f(x)$ at $x = a$.
- (c) Use the approximation to estimate $\sqrt{10.2}$, and compare it with the actual value.

Question 8 - Solution

For $f(x) = \sqrt{x}$, use the nearby perfect square $a = 9$.

$$f(9) = 3, \quad f'(9) = \frac{1}{6}, \quad L(x) = 3 + \frac{x-9}{6}.$$

Therefore

$$\boxed{\sqrt{10.2} \approx L(10.2) = 3.2.}$$

Since $f''(x) = -1/(4x^{3/2}) < 0$ for $x > 0$, the tangent line lies above the curve, so this is an overestimate. For comparison, $\sqrt{10.2} \approx 3.193743885$; the error is approximately 0.006256115.

Question 9

Use a linear approximation to estimate $\ln(1.05)$.

- (a) Define a function $f(x)$ such that $f(x) = \ln(x)$, and choose a value a near 1.05 where $f(a)$ is easy to compute.
- (b) Find the linear approximation $L(x)$ at $x = a$.
- (c) Use the approximation to estimate $\ln(1.05)$ and compare with the actual value.

Question 9 - Solution

(a) Let $f(x) = \ln(x)$, and choose $a = 1$ since $\ln(1) = 0$ is easy to compute.

$$f(a) = \ln(1) = 0, \quad f'(x) = \frac{1}{x}, \quad f'(1) = 1$$

(b) Linear approximation:

$$L(x) = f(a) + f'(a)(x - a) = 0 + 1(x - 1) = x - 1$$

(c) Estimate $\ln(1.05)$ using:

$$L(1.05) = 1.05 - 1 = 0.05$$

Actual: $\ln(1.05) \approx 0.04879$

Conclusion:

$$\boxed{\ln(1.05) \approx 0.05} \quad (\text{Linear Approximation})$$

Question 10

Use a linear approximation to estimate $\sqrt{10.2}$.

- (a) Let $f(x) = \sqrt{x}$. Choose a value a near 10.2 where the square root is easy to compute.
- (b) Find the linear approximation $L(x)$ at $x = a$.
- (c) Use the approximation to estimate $\sqrt{10.2}$ and compare with the actual value.

Question 10 - Solution

For $f(x) = \sqrt{x}$, use the nearby perfect square $a = 9$.

$$f(9) = 3, \quad f'(9) = \frac{1}{6}, \quad L(x) = 3 + \frac{x-9}{6}.$$

Therefore

$$\boxed{\sqrt{10.2} \approx L(10.2) = 3.2.}$$

Since $f''(x) = -1/(4x^{3/2}) < 0$ for $x > 0$, the tangent line lies above the curve, so this is an overestimate. For comparison, $\sqrt{10.2} \approx 3.193743885$; the error is approximately 0.006256115.

Question 1

Let $f(x) = x^3$.

- (a) Use differentials to approximate the change in f as x changes from 2 to 2.05.
- (b) Compare your result to the actual change $f(2.05) - f(2)$.

Question 1 - Solution

We are given:

$$f(x) = x^3, \quad f'(x) = 3x^2$$

(a) Use:

$$df = f'(x) dx$$

Let $x = 2$, $dx = 0.05$:

$$f'(2) = 3(2)^2 = 12$$

$$df = 12 \cdot 0.05 = \boxed{0.6}$$

So, the approximate change in f is $\boxed{0.6}$

(b) Actual change:

$$f(2.05) = (2.05)^3 = 8.615125, \quad f(2) = 8$$

$$\text{Actual change} = 8.615125 - 8 = \boxed{0.615125}$$

Conclusion: The differential estimate 0.6 is close to the actual change 0.615125

Question 2

Suppose the radius r of a sphere is measured to be 5 cm with a possible error of 0.1 cm.

- (a) Use differentials to approximate the possible error in the calculated volume of the sphere.
- (b) Estimate the relative error and percentage error in the volume.

Question 2 - Solution

We use the volume formula of a sphere:

$$V = \frac{4}{3}\pi r^3$$

(a) **Approximate error using differentials:**

$$dV = \frac{dV}{dr} dr = 4\pi r^2 dr$$

$$dV = 4\pi(5)^2(0.1) = 4\pi(25)(0.1) = 10\pi \approx 31.42 \text{ cm}^3$$

(b) **Relative error:**

$$\frac{dV}{V} = \frac{10\pi}{\frac{4}{3}\pi(5)^3} = \frac{10\pi}{\frac{500}{3}\pi} = \frac{10 \cdot 3}{500} = \frac{30}{500} = 0.06$$

Percentage error:

$$0.06 \times 100 = \boxed{6\%}$$

Final Answer:

- Approximate error in volume: $\boxed{31.42 \text{ cm}^3}$
- Relative error: $\boxed{0.06}$
- Percentage error: $\boxed{6\%}$

Question 3

The length of the side of a cube is measured to be 10 cm with a possible error of 0.2 cm.

- (a) Use differentials to approximate the maximum error in the calculated surface area of the cube.
- (b) Estimate the relative error and the percentage error in the surface area.

Question 3 - Solution

Let the surface area of the cube be:

$$S = 6x^2$$

where x is the length of a side.

(a) **Approximate error using differentials:**

$$dS = \frac{dS}{dx} dx = 12x dx$$

$$dS = 12(10)(0.2) = 24 \text{ cm}^2$$

(b) **Relative error:**

$$\frac{dS}{S} = \frac{24}{6(10)^2} = \frac{24}{600} = 0.04$$

Percentage error:

$$0.04 \times 100 = \boxed{4\%}$$

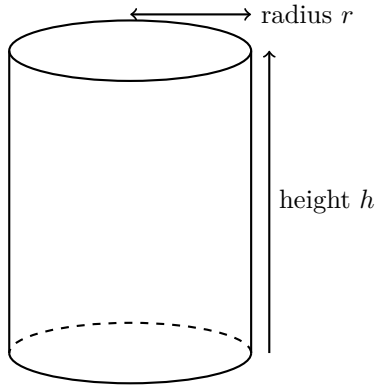
Final Answer:

- Maximum error in surface area: $\boxed{24 \text{ cm}^2}$
- Relative error: $\boxed{0.04}$
- Percentage error: $\boxed{4\%}$

Question 4

A right circular cylinder has a height of 12 cm, and its radius is measured to be 5 cm with a possible error of 0.1 cm.

- (a) Use differentials to approximate the maximum error in the calculated volume of the cylinder.
- (b) Estimate the relative error and percentage error in the volume.



Question 4 - Solution

The volume of a right circular cylinder is given by:

$$V = \pi r^2 h$$

Given: $r = 5$ cm, $dr = 0.1$ cm, $h = 12$ cm

(a) Use differentials to approximate the error:

$$dV = \frac{dV}{dr} \cdot dr = 2\pi r h \cdot dr$$

$$dV = 2\pi(5)(12)(0.1) = 12\pi \text{ cm}^3 \approx \boxed{37.70 \text{ cm}^3}$$

(b) Estimate the relative error and percentage error:

$$V = \pi r^2 h = \pi(5)^2(12) = 300\pi \approx 942.48 \text{ cm}^3$$

$$\text{Relative error} = \frac{dV}{V} = \frac{12\pi}{300\pi} = \frac{12}{300} = 0.04$$

$$\text{Percentage error} = 0.04 \times 100 = \boxed{4\%}$$

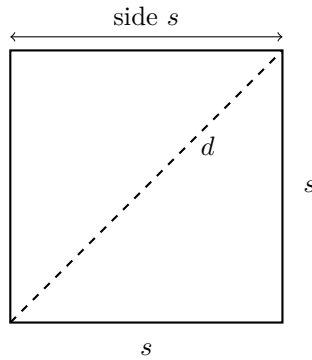
Final Answer:

- Maximum error in volume: $\boxed{37.70 \text{ cm}^3}$
- Relative error: $\boxed{0.04}$
- Percentage error: $\boxed{4\%}$

Question 5

The length of the diagonal d of a square is measured to be 10 cm with a possible error of 0.2 cm. Use differentials to approximate:

- (a) The maximum error in computing the area of the square.
- (b) The relative and percentage errors.



Question 5 - Solution

We are given that the diagonal of a square is $d = 10$ cm with an error of $dd = 0.2$ cm. The area of a square in terms of its diagonal is:

$$s = \frac{d}{\sqrt{2}}, \quad A = s^2 = \left(\frac{d}{\sqrt{2}} \right)^2 = \frac{d^2}{2}$$

Differentiate:

$$dA = \frac{d}{dx} \left(\frac{d^2}{2} \right) \cdot dd = d \cdot dd$$

(a) **Maximum error in the area:**

$$dA = d \cdot dd = 10 \cdot 0.2 = \boxed{2 \text{ cm}^2}$$

(b) **Relative and percentage error:**

Actual area:

$$A = \frac{10^2}{2} = 50 \text{ cm}^2$$

Relative error:

$$\frac{dA}{A} = \frac{2}{50} = 0.04$$

Percentage error:

$$0.04 \times 100 = \boxed{4\%}$$

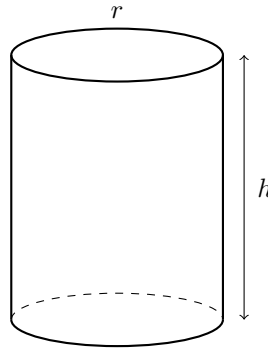
Final Answer:

- Maximum error in area: $\boxed{2 \text{ cm}^2}$
- Relative error: $\boxed{0.04}$
- Percentage error: $\boxed{4\%}$

Question 6

The radius r of a right circular cylinder is measured to be 5 cm with a possible error of 0.1 cm. The height h is 20 cm with a possible error of 0.2 cm.

- (a) Use differentials to approximate the maximum error in calculating the volume of the cylinder.
- (b) Find the relative and percentage errors in the volume.



Question 6 - Solution

The volume of a cylinder is:

$$V = \pi r^2 h$$

We compute the differential dV :

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh = 2\pi r h dr + \pi r^2 dh$$

Given:

$$r = 5 \text{ cm}, \quad dr = 0.1 \text{ cm}, \quad h = 20 \text{ cm}, \quad dh = 0.2 \text{ cm}$$

Substitute into the differential:

$$dV = 2\pi(5)(20)(0.1) + \pi(5)^2(0.2) = 20\pi + 5\pi = \boxed{25\pi \approx 78.54 \text{ cm}^3}$$

(a) Maximum error:

$$\boxed{dV \approx 78.54 \text{ cm}^3}$$

(b) Relative and percentage error:

Actual volume:

$$V = \pi(5)^2(20) = 500\pi \approx 1570.8 \text{ cm}^3$$

Relative error:

$$\frac{dV}{V} = \frac{25\pi}{500\pi} = \frac{1}{20} = 0.05$$

Percentage error:

$$0.05 \times 100 = \boxed{5\%}$$

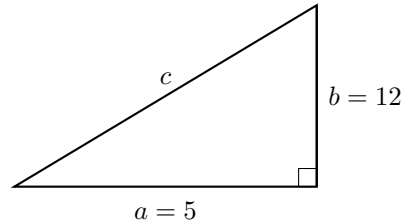
Final Answers:

- Maximum error in volume: $\boxed{25\pi \approx 78.54 \text{ cm}^3}$
- Relative error: $\boxed{0.05}$
- Percentage error: $\boxed{5\%}$

Question 7

A right triangle has one leg of fixed length 5 cm. The other leg is measured to be 12 cm with a possible measurement error of 0.1 cm.

- (a) Use differentials to approximate the maximum error in computing the hypotenuse.
- (b) Find the relative and percentage error in the hypotenuse.



Question 7 - Solution

We are given that $a = 5$ is fixed and $b = 12$ with a possible error $db = 0.1$. The hypotenuse is:

$$c = \sqrt{a^2 + b^2}$$

Differentiate both sides:

$$dc = \frac{1}{2}(a^2 + b^2)^{-1/2} \cdot 2b db = \frac{b}{\sqrt{a^2 + b^2}} \cdot db$$

Substitute known values:

$$a = 5, \quad b = 12, \quad db = 0.1$$
$$c = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

$$dc = \frac{12}{13} \cdot 0.1 \approx \boxed{0.0923}$$

(a) **Maximum error:**

$$\boxed{dc \approx 0.0923 \text{ cm}}$$

(b) **Relative and percentage error:**

$$\text{Relative error} = \frac{dc}{c} = \frac{0.0923}{13} \approx 0.0071$$

$$\text{Percentage error} = 0.0071 \times 100 \approx \boxed{0.71\%}$$

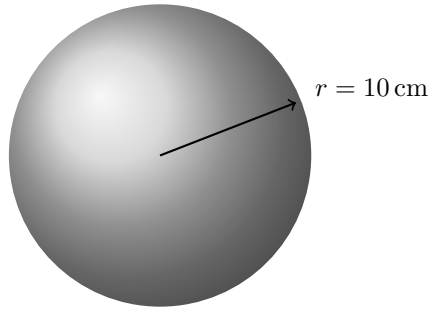
Final Answers:

- Maximum error in hypotenuse: $\boxed{0.0923 \text{ cm}}$
- Relative error: $\boxed{0.0071}$
- Percentage error: $\boxed{0.71\%}$

Question 8

The radius of a spherical balloon is measured to be 10 cm with a possible error of 0.05 cm.

- (a) Use differentials to estimate the maximum error in calculating the volume of the balloon.
- (b) Estimate the relative and percentage error in the volume.



Question 8 - Solution

We are given:

$$r = 10 \text{ cm}, \quad dr = 0.05 \text{ cm}$$

The volume of a sphere is:

$$V = \frac{4}{3}\pi r^3$$

Differentiate using differentials:

$$dV = \frac{d}{dr} \left(\frac{4}{3}\pi r^3 \right) \cdot dr = 4\pi r^2 \cdot dr$$

Substitute known values:

$$dV = 4\pi(10)^2 \cdot 0.05 = 4\pi \cdot 100 \cdot 0.05 = 20\pi \approx \boxed{62.83 \text{ cm}^3}$$

(a) Maximum error in volume:

$$\boxed{dV \approx 62.83 \text{ cm}^3}$$

(b) Relative and percentage error:

Actual volume:

$$V = \frac{4}{3}\pi(10)^3 = \frac{4000}{3}\pi \approx 4188.79 \text{ cm}^3$$

$$\text{Relative error} = \frac{dV}{V} \approx \frac{62.83}{4188.79} \approx 0.015$$

$$\text{Percentage error} = 0.015 \times 100 = \boxed{1.5\%}$$

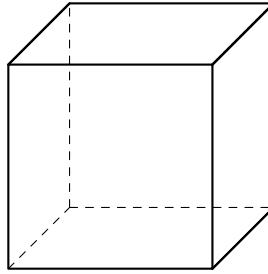
Final Answers:

- Maximum error in volume: $\boxed{62.83 \text{ cm}^3}$
- Relative error: $\boxed{0.015}$
- Percentage error: $\boxed{1.5\%}$

Question 9

The side of a cube is measured to be 5 cm with a possible error of 0.1 cm.

- (a) Use differentials to estimate the maximum error in computing the volume of the cube.
- (b) Estimate the relative and percentage error in the volume.



$$s = 5 \text{ cm}$$

Question 9 - Solution

We are given:

$$s = 5 \text{ cm}, \quad ds = 0.1 \text{ cm}$$

The volume of a cube is:

$$V = s^3$$

Differentiate using differentials:

$$dV = \frac{d}{ds}(s^3) ds = 3s^2 ds$$

Substitute values:

$$dV = 3(5)^2(0.1) = 3 \cdot 25 \cdot 0.1 = \boxed{7.5 \text{ cm}^3}$$

(a) **Maximum error in volume:**

$$\boxed{7.5 \text{ cm}^3}$$

(b) **Relative and percentage error:**

$$V = 5^3 = 125 \text{ cm}^3$$

$$\text{Relative error} = \frac{dV}{V} = \frac{7.5}{125} = 0.06$$

$$\text{Percentage error} = 0.06 \times 100 = \boxed{6\%}$$

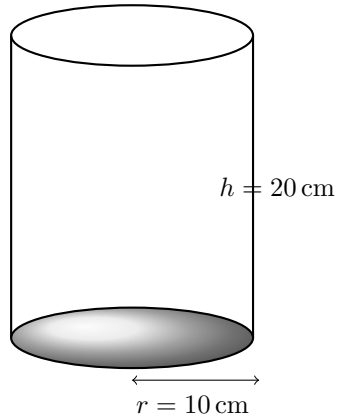
Final Answers:

- Maximum error in volume: $\boxed{7.5 \text{ cm}^3}$
- Relative error: $\boxed{0.06}$
- Percentage error: $\boxed{6\%}$

Question 10

The radius of a right circular cylinder is measured as $r = 10$ cm with a possible error of 0.2 cm, and the height is measured as $h = 20$ cm with a possible error of 0.3 cm.

- (a) Use differentials to estimate the maximum error in the volume of the cylinder.
- (b) Estimate the percentage error in the volume.



Question 10 - Solution

We are given:

$$r = 10 \text{ cm}, \quad dr = 0.2 \text{ cm}, \quad h = 20 \text{ cm}, \quad dh = 0.3 \text{ cm}$$

The volume of a cylinder is:

$$V = \pi r^2 h$$

Use differentials:

$$dV = \frac{\partial}{\partial r} dr + \frac{\partial}{\partial h} dh = (2\pi r h) \cdot dr + (\pi r^2) \cdot dh$$

Substitute values:

$$\begin{aligned} dV &= 2\pi(10)(20)(0.2) + \pi(10)^2(0.3) \\ &= 2\pi(200)(0.2) + \pi(100)(0.3) = \pi(80) + \pi(30) = \pi(110) \end{aligned}$$

$$\boxed{dV \approx 345.58 \text{ cm}^3}$$

(a) **Estimated maximum error:** $\boxed{345.58 \text{ cm}^3}$

(b) **Percentage error:**

$$V = \pi r^2 h = \pi(100)(20) = 2000\pi \approx 6283.19 \text{ cm}^3$$

$$\text{Percentage error} = \frac{345.58}{6283.19} \times 100 \approx \boxed{5.5\%}$$

Final Answers:

- Maximum error in volume: $\boxed{345.58 \text{ cm}^3}$
- Percentage error: $\boxed{5.5\%}$

Question 1

Use Newton's Method to approximate a root of the function

$$f(x) = x^3 - x - 1$$

starting with $x_0 = 1.5$, and compute the first two iterations: x_1 and x_2 .

Question 1 - Solution

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f(x) = x^3 - x - 1, \quad f'(x) = 3x^2 - 1.$$

Keeping full precision internally gives

$$x_1 = 1.5000000000 - \frac{0.8750000000}{5.7500000000} \approx \boxed{1.3478}$$

$$x_2 = 1.3478260870 - \frac{0.1006821731}{4.4499054820} \approx \boxed{1.3252}$$

Question 2

Use Newton's Method to approximate the root of the equation

$$x^3 - 2x - 5 = 0$$

starting with the initial guess $x_1 = 2$. Perform three iterations to compute x_2 , x_3 , and x_4 . Round your answers to four decimal places.

Question 2 - Solution

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f(x) = x^3 - 2x - 5, \quad f'(x) = 3x^2 - 2.$$

Keeping full precision internally gives

$$x_2 = 2.0000000000 - \frac{-1.0000000000}{10.0000000000} \approx \boxed{2.1000}$$

$$x_3 = 2.1000000000 - \frac{0.0610000000}{11.2300000000} \approx \boxed{2.0946}$$

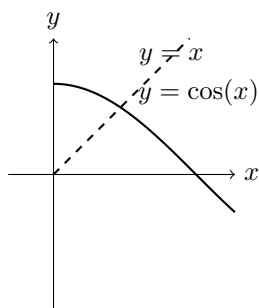
$$x_4 = 2.0945681211 - \frac{0.0001857232}{11.1616468418} \approx \boxed{2.0946}$$

Question 3

Use Newton's Method to approximate a solution to the equation:

$$\cos(x) = x$$

Start with an initial guess of $x_1 = 0.5$, and compute the next three iterations: x_2 , x_3 , and x_4 , rounding answers to four decimal places.



Question 3 - Solution

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f(x) = \cos x - x, \quad f'(x) = -\sin x - 1.$$

Keeping full precision internally gives

$$x_2 = 0.5000000000 - \frac{0.3775825619}{-1.4794255386} \approx \boxed{0.7552}$$

$$x_3 = 0.7552224171 - \frac{-0.0271033119}{-1.6854506318} \approx \boxed{0.7391}$$

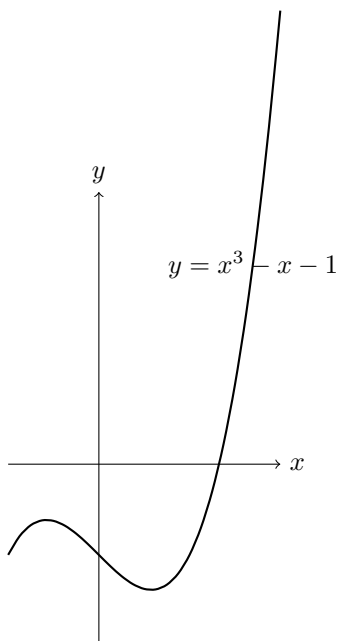
$$x_4 = 0.7391416661 - \frac{-0.0000946154}{-1.6736538108} \approx \boxed{0.7391}$$

Question 4

Use Newton's Method to approximate a root of the equation:

$$x^3 - x - 1 = 0$$

Start with an initial guess of $x_1 = 1$, and compute the next three iterations: x_2 , x_3 , and x_4 , rounding answers to four decimal places.



Question 4 - Solution

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f(x) = x^3 - x - 1, \quad f'(x) = 3x^2 - 1.$$

Keeping full precision internally gives

$$x_2 = 1.0000000000 - \frac{-1.0000000000}{2.0000000000} \approx \boxed{1.5000}$$

$$x_3 = 1.5000000000 - \frac{0.8750000000}{5.7500000000} \approx \boxed{1.3478}$$

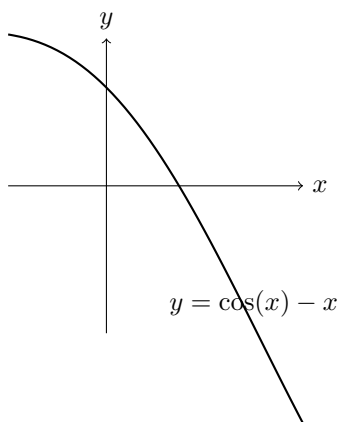
$$x_4 = 1.3478260870 - \frac{0.1006821731}{4.4499054820} \approx \boxed{1.3252}$$

Question 5

Use Newton's Method to approximate a root of the equation:

$$\cos(x) - x = 0$$

Start with an initial guess of $x_1 = 0.5$, and compute the next three iterations: x_2 , x_3 , and x_4 , rounding to four decimal places.



Question 5 - Solution

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f(x) = \cos x - x, \quad f'(x) = -\sin x - 1.$$

Keeping full precision internally gives

$$x_2 = 0.5000000000 - \frac{0.3775825619}{-1.4794255386} \approx \boxed{0.7552}$$

$$x_3 = 0.7552224171 - \frac{-0.0271033119}{-1.6854506318} \approx \boxed{0.7391}$$

$$x_4 = 0.7391416661 - \frac{-0.0000946154}{-1.6736538108} \approx \boxed{0.7391}$$

Question 6

Use Newton's Method to approximate a root of the equation:

$$x^3 - 2x - 5 = 0$$

Start with an initial guess of $x_1 = 2$, and compute the next three approximations: x_2 , x_3 , and x_4 . Round answers to four decimal places.

Question 6 - Solution

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f(x) = x^3 - 2x - 5, \quad f'(x) = 3x^2 - 2.$$

Keeping full precision internally gives

$$x_2 = 2.0000000000 - \frac{-1.0000000000}{10.0000000000} \approx \boxed{2.1000}$$

$$x_3 = 2.1000000000 - \frac{0.0610000000}{11.2300000000} \approx \boxed{2.0946}$$

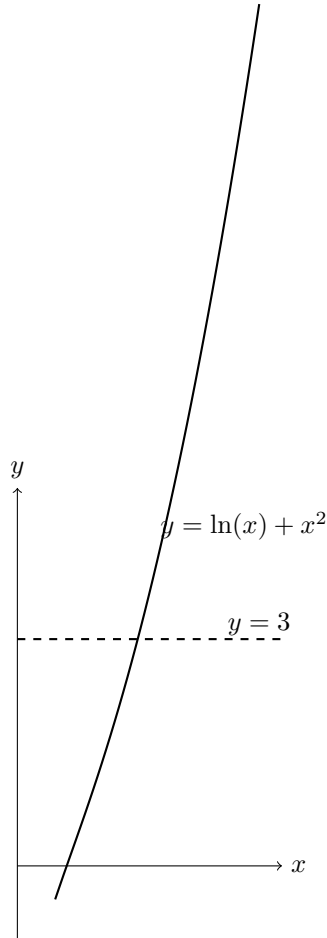
$$x_4 = 2.0945681211 - \frac{0.0001857232}{11.1616468418} \approx \boxed{2.0946}$$

Question 7

Use Newton's Method to approximate a solution to the equation:

$$\ln(x) + x^2 = 3$$

Start with an initial guess of $x_1 = 1.5$, and compute x_2 , x_3 , and x_4 . Round each to four decimal places.



Question 7 - Solution

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f(x) = \ln x + x^2 - 3, \quad f'(x) = 1/x + 2x.$$

Keeping full precision internally gives

$$x_2 = 1.5000000000 - \frac{-0.3445348919}{3.6666666667} \approx \boxed{1.5940}$$

$$x_3 = 1.5939640614 - \frac{0.0069454631}{3.8152948397} \approx \boxed{1.5921}$$

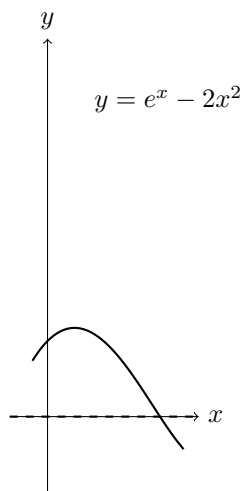
$$x_4 = 1.5921436351 - \frac{0.0000026613}{3.8123713061} \approx \boxed{1.5921}$$

Question 8

Approximate a root of the equation:

$$e^x - 2x^2 = 0$$

using Newton's Method. Start with $x_1 = 0.5$, and compute x_2 , x_3 , and x_4 to four decimal places.



Question 8 - Solution

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f(x) = e^x - 2x^2, \quad f'(x) = e^x - 4x.$$

Keeping full precision internally gives

$$x_2 = 0.5000000000 - \frac{1.1487212707}{-0.3512787293} \approx \boxed{3.7701}$$

$$x_3 = 3.7701133740 - \frac{14.9574735817}{28.3045297918} \approx \boxed{3.2417}$$

$$x_4 = 3.2416653238 - \frac{4.5594909448}{12.6096177926} \approx \boxed{2.8801}$$

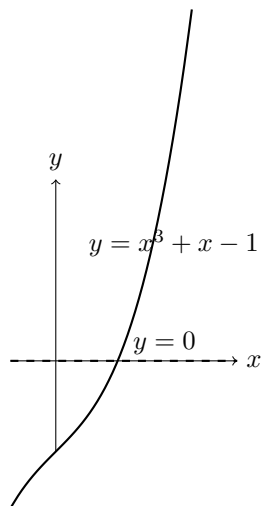
These are the requested iterates; three steps do not yet give an accurate root. Continue iterating if a converged root is required.

Question 9

Approximate a solution to the equation:

$$x^3 + x - 1 = 0$$

using Newton's Method. Start with $x_1 = 0.5$, and compute x_2 , x_3 , and x_4 , rounding each to four decimal places.



Question 9 - Solution

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f(x) = x^3 + x - 1, \quad f'(x) = 3x^2 + 1.$$

Keeping full precision internally gives

$$x_2 = 0.5000000000 - \frac{-0.3750000000}{1.7500000000} \approx \boxed{0.7143}$$

$$x_3 = 0.7142857143 - \frac{0.0787172012}{2.5306122449} \approx \boxed{0.6832}$$

$$x_4 = 0.6831797235 - \frac{0.0020432938}{2.4002036038} \approx \boxed{0.6823}$$

Question 10

Consider the equation

$$x^3 - 3x + 1 = 0.$$

- (a) Explain why Newton's method fails with the initial guess $x_1 = 1$.
- (b) Restart with $x_1 = 0$, and compute x_2 , x_3 , and x_4 . Round the answers to four decimal places.

Question 10 - Solution

At $x_1 = 1$, $f(1) = -1$ and $f'(1) = 0$, so the Newton step is undefined. Restart at $x_1 = 0$. Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f(x) = x^3 - 3x + 1, \quad f'(x) = 3x^2 - 3.$$

Keeping full precision internally gives

$$x_2 = 0.0000000000 - \frac{1.0000000000}{-3.0000000000} \approx \boxed{0.3333}$$

$$x_3 = 0.3333333333 - \frac{0.0370370370}{-2.6666666667} \approx \boxed{0.3472}$$

$$x_4 = 0.3472222222 - \frac{0.0001955804}{-2.6383101852} \approx \boxed{0.3473}$$

Question 1

A company's revenue (in dollars) from selling x items is given by:

$$R(x) = 50x - 0.2x^2$$

and the cost (in dollars) to produce x items is:

$$C(x) = 10x + 100$$

- (a) Find the profit function $P(x)$.
- (b) Find the number of items that maximizes profit.
- (c) What is the maximum profit?

Question 1 - Solution

(a) Profit is revenue minus cost:

$$P(x) = R(x) - C(x) = (50x - 0.2x^2) - (10x + 100) = 40x - 0.2x^2 - 100$$

(b) Maximize $P(x) = -0.2x^2 + 40x - 100$.

This is a downward-opening parabola. The maximum occurs at:

$$x = \frac{-b}{2a} = \frac{-40}{2(-0.2)} = \frac{-40}{-0.4} = \boxed{100}$$

(c) Maximum profit:

$$P(100) = 40(100) - 0.2(100)^2 - 100 = 4000 - 2000 - 100 = \boxed{1900}$$

Answer:

Max profit occurs at $x = 100$ items and the maximum profit is \$1900.

Question 2

A company manufactures and sells x units of a product per week. The revenue (in thousands of dollars) and cost (in thousands of dollars) functions are given by:

$$R(x) = 50x - 0.5x^2, \quad C(x) = 20x + 100$$

- (a) Find the profit function $P(x)$.
- (b) Determine the number of units x that maximizes the profit.
- (c) Compute the maximum profit.

Question 2 - Solution**(a) Profit Function:**

$$P(x) = R(x) - C(x) = (50x - 0.5x^2) - (20x + 100) = -0.5x^2 + 30x - 100$$

(b) Maximize Profit:

$$P'(x) = -x + 30$$

Set $P'(x) = 0$:

$$-x + 30 = 0 \quad \Rightarrow \quad x = 30$$

Second derivative:

$$P''(x) = -1 < 0$$

Hence, profit is maximized at $x = 30$.**(c) Maximum Profit:**

$$P(30) = -0.5(30)^2 + 30(30) - 100 = -450 + 900 - 100 = 350$$

Maximum profit occurs at $x = 30$ units, $P_{\max} = 350$ thousand dollars

Question 3

A company's profit $P(x)$ (in thousands of dollars) depends on the number of units x produced per week:

$$P(x) = -2x^2 + 40x - 120$$

- (a) Determine the production level x that maximizes profit.
- (b) Compute the maximum profit.
- (c) Find the production level x at which the profit is zero.

Question 3 - Solution**(a) Maximize Profit:**

$$P'(x) = \frac{d}{dx}(-2x^2 + 40x - 120) = -4x + 40$$

Set $P'(x) = 0$:

$$-4x + 40 = 0 \quad \Rightarrow \quad x = 10$$

Second derivative:

$$P''(x) = -4 < 0$$

Hence, profit is maximized at $x = 10$.**(b) Maximum Profit:**

$$P(10) = -2(10)^2 + 40(10) - 120 = -200 + 400 - 120 = 80$$

(c) Break-even Points (Profit = 0):

$$-2x^2 + 40x - 120 = 0$$

Divide through by 2:

$$x^2 - 20x + 60 = 0$$

Solve using the quadratic formula:

$$x = \frac{20 \pm \sqrt{400 - 240}}{2} = \frac{20 \pm \sqrt{160}}{2} = 10 \pm 2\sqrt{10}$$

Maximum profit occurs at $x = 10$ units, $P_{\max} = 80$ thousand dollars,
Profit is zero at $x = 10 - 2\sqrt{10}$, $10 + 2\sqrt{10}$.

Question 4

The demand for a product is modeled by

$$p(x) = 100 - 2x,$$

where $p(x)$ is the price per unit in dollars and x is the number of units sold in hundreds. The total cost of producing $100x$ units is

$$C(x) = 20x + 200 \quad \text{dollars.}$$

Require $x \geq 0$ and $p(x) \geq 0$.

- (a) Find the revenue and profit functions.
- (b) Find the production level and maximum profit in the continuous model.
- (c) Find the best whole-number quantity of units and its profit.

Question 4 - Solution**(a) Revenue and profit**

The number sold is $100x$, so revenue is $100x p(x)$:

$$R(x) = 10000x - 200x^2, \quad P(x) = -200x^2 + 9980x - 200.$$

$$P'(x) = -400x + 9980, \quad P''(x) = -400 < 0.$$

(b) Continuous maximum

The continuous maximum occurs at

$$x = \frac{9980}{400} \approx 24.950000, \quad P_{\max} \approx \$124,300.50.$$

(c) Whole units

This represents 2495.000000 units. Comparing the adjacent whole-unit counts gives 2495 units, with profit \$124,300.50. The price is nonnegative at these levels.

Question 5

A company produces and sells x units of a product per week. The weekly revenue and cost functions (in thousands of dollars) are:

$$R(x) = 200x - 2x^2, \quad C(x) = 50x + 300$$

- (a) Find the profit function $P(x)$.
- (b) Determine the number of units that maximizes the profit.
- (c) Compute the maximum profit.

Question 5 - Solution

$$P(x) = R(x) - C(x) = -2x^2 + 150x - 300.$$

In the continuous model, $P'(x) = -4x + 150 = 0$ gives $x = 37.5$ and $P''(x) = -4 < 0$.

The continuous maximum is 2512.5 thousand dollars.

For whole units, compare $x = 37$ and $x = 38$, the integers nearest the vertex:

$$P(37) = P(38) = 2512.$$

Thus the feasible maximum is

37 or 38 units,	2512 thousand dollars.
-----------------	------------------------

Question 6

The demand for a product is modeled by

$$p(x) = 150 - 3x,$$

where $p(x)$ is the price per unit in dollars and x is the number of units sold in hundreds. The total cost of producing $100x$ units is

$$C(x) = 50x + 300 \quad \text{dollars.}$$

Require $x \geq 0$ and $p(x) \geq 0$.

- (a) Find the revenue and profit functions.
- (b) Find the production level and maximum profit in the continuous model.
- (c) Find the best whole-number quantity of units and its profit.

Question 6 - Solution**(a) Revenue and profit**

The number sold is $100x$, so revenue is $100x p(x)$:

$$R(x) = 15000x - 300x^2, \quad P(x) = -300x^2 + 14950x - 300.$$

$$P'(x) = -600x + 14950, \quad P''(x) = -600 < 0.$$

(b) Continuous maximum

The continuous maximum occurs at

$$x = \frac{14950}{600} \approx 24.916667, \quad P_{\max} \approx \$185,952.08.$$

(c) Whole units

This represents 2491.666667 units. Comparing the adjacent whole-unit counts gives 2492 units, with profit \$185,952.08. The price is nonnegative at these levels.

Question 7

A company's profit $P(x)$ (in dollars) depends on the number of items x produced per week:

$$P(x) = -5x^2 + 300x - 2000$$

- (a) Determine the production level x that maximizes profit.
- (b) Compute the maximum profit.
- (c) Find the production levels x at which the profit is zero.

Question 7 - Solution**(a) Maximize Profit:**

$$P'(x) = \frac{d}{dx}(-5x^2 + 300x - 2000) = -10x + 300$$

Set $P'(x) = 0$:

$$-10x + 300 = 0 \quad \Rightarrow \quad x = 30$$

Second derivative:

$$P''(x) = -10 < 0$$

Hence, profit is maximized at $x = 30$.**(b) Maximum Profit:**

$$P(30) = -5(30)^2 + 300(30) - 2000 = -4500 + 9000 - 2000 = 2500$$

(c) Break-even Points (Profit = 0):

$$-5x^2 + 300x - 2000 = 0$$

Divide through by 5:

$$x^2 - 60x + 400 = 0$$

Solve using the quadratic formula:

$$x = \frac{60 \pm \sqrt{3600 - 1600}}{2} = \frac{60 \pm \sqrt{2000}}{2} = 30 \pm 10\sqrt{5}$$

Maximum profit occurs at $x = 30$ units, $P_{\max} = 2500$,
Profit is zero at $x = 30 - 10\sqrt{5}$, $30 + 10\sqrt{5}$.

Question 8

The weekly revenue (in thousands of dollars) for producing x units is:

$$R(x) = 120x - x^2$$

and the weekly cost is:

$$C(x) = 40x + 200$$

- (a) Determine the profit function $P(x)$.
- (b) Find the number of units x that maximizes profit.
- (c) Compute the maximum profit.

Question 8 - Solution**(a) Profit Function:**

$$P(x) = R(x) - C(x) = (120x - x^2) - (40x + 200) = -x^2 + 80x - 200$$

(b) Maximize Profit:

$$P'(x) = -2x + 80$$

Set $P'(x) = 0$:

$$-2x + 80 = 0 \quad \Rightarrow \quad x = 40$$

Second derivative:

$$P''(x) = -2 < 0$$

Hence, profit is maximized when $x = 40$.**(c) Maximum Profit:**

$$P(40) = -(40)^2 + 80(40) - 200 = -1600 + 3200 - 200 = 1400$$

Maximum profit occurs at $x = 40$ units, $P_{\max} = 1400$ thousand dollars

Question 9

The demand for a product is modeled by

$$p(x) = 80 - x,$$

where $p(x)$ is the price per unit in dollars and x is the number of units sold in hundreds. The total cost of producing $100x$ units is

$$C(x) = 20x + 100 \quad \text{dollars.}$$

Require $x \geq 0$ and $p(x) \geq 0$.

- (a) Find the revenue and profit functions.
- (b) Find the production level and maximum profit in the continuous model.
- (c) Find the best whole-number quantity of units and its profit.

Question 9 - Solution**(a) Revenue and profit**

The number sold is $100x$, so revenue is $100x p(x)$:

$$R(x) = 8000x - 100x^2, \quad P(x) = -100x^2 + 7980x - 100.$$

$$P'(x) = -200x + 7980, \quad P''(x) = -200 < 0.$$

(b) Continuous maximum

The continuous maximum occurs at

$$x = \frac{7980}{200} \approx 39.900000, \quad P_{\max} \approx \$159,101.00.$$

(c) Whole units

This represents 3990.000000 units. Comparing the adjacent whole-unit counts gives 3990 units, with profit \$159,101.00. The price is nonnegative at these levels.

Question 10

A company manufactures and sells a product. The weekly revenue and cost functions (in thousands of dollars) are:

$$R(x) = 500x - 5x^2, \quad C(x) = 200x + 500$$

- (a) Determine the profit function $P(x)$.
- (b) Find the production level x that maximizes profit.
- (c) Compute the maximum profit.

Question 10 - Solution**(a) Profit Function:**

$$P(x) = R(x) - C(x) = (500x - 5x^2) - (200x + 500) = -5x^2 + 300x - 500$$

(b) Maximize Profit:

$$P'(x) = -10x + 300$$

Set $P'(x) = 0$:

$$-10x + 300 = 0 \quad \Rightarrow \quad x = 30$$

Check the second derivative:

$$P''(x) = -10 < 0$$

Thus, profit is maximized when $x = 30$.**(c) Maximum Profit:**

$$P(30) = -5(30)^2 + 300(30) - 500 = -4500 + 9000 - 500 = 4000$$

Maximum profit occurs at $x = 30$ units, $P_{\max} = 4000$ thousand dollars