

Question 1

Let

$$f(x) = \frac{1}{x+2}$$

Use the definition of the derivative to compute $f'(1)$. Show all steps.

Question 1 - Solution

We are given:

$$f(x) = \frac{1}{x+2}$$

Using the definition of the derivative:

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

Compute:

$$f(1+h) = \frac{1}{(1+h)+2} = \frac{1}{h+3}, \quad f(1) = \frac{1}{1+2} = \frac{1}{3}$$

So:

$$f'(1) = \lim_{h \rightarrow 0} \frac{\frac{1}{h+3} - \frac{1}{3}}{h}$$

Find a common denominator for the numerator:

$$\frac{1}{h+3} - \frac{1}{3} = \frac{3 - (h+3)}{3(h+3)} = \frac{-h}{3(h+3)}$$

Now plug back into the limit:

$$f'(1) = \lim_{h \rightarrow 0} \frac{-h}{3(h+3)} \cdot \frac{1}{h} = \lim_{h \rightarrow 0} \frac{-1}{3(h+3)}$$

Now evaluate the limit:

$$f'(1) = \frac{-1}{3(0+3)} = \boxed{-\frac{1}{9}}$$

Question 2

Let

$$f(x) = \sqrt{4x + 1}$$

Use the definition of the derivative to compute $f'(2)$. Show all steps clearly without using shortcut rules.

Question 2 - Solution

We are given:

$$f(x) = \sqrt{4x + 1}$$

Using the limit definition of the derivative:

$$f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$$

Compute:

$$f(2+h) = \sqrt{4(2+h) + 1} = \sqrt{8+4h+1} = \sqrt{4h+9}$$

$$f(2) = \sqrt{4(2) + 1} = \sqrt{9} = 3$$

So,

$$f'(2) = \lim_{h \rightarrow 0} \frac{\sqrt{4h+9} - 3}{h}$$

To evaluate this limit, multiply numerator and denominator by the conjugate of the numerator:

$$f'(2) = \lim_{h \rightarrow 0} \frac{\sqrt{4h+9} - 3}{h} \cdot \frac{\sqrt{4h+9} + 3}{\sqrt{4h+9} + 3}$$

Apply difference of squares in the numerator:

$$= \lim_{h \rightarrow 0} \frac{(4h+9) - 9}{h(\sqrt{4h+9} + 3)} = \lim_{h \rightarrow 0} \frac{4h}{h(\sqrt{4h+9} + 3)}$$

Cancel h in numerator and denominator:

$$= \lim_{h \rightarrow 0} \frac{4}{\sqrt{4h+9} + 3}$$

Now evaluate the limit:

$$f'(2) = \frac{4}{\sqrt{9+3}} = \frac{4}{3+3} = \boxed{\frac{2}{3}}$$

Question 3

Let

$$f(x) = \frac{1}{x+3}$$

Use the definition of the derivative to compute $f'(1)$. Show all steps clearly without using differentiation shortcuts.

Question 3 - Solution

We are given:

$$f(x) = \frac{1}{x+3}$$

Using the definition of the derivative:

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

Compute:

$$f(1+h) = \frac{1}{1+h+3} = \frac{1}{h+4}, \quad f(1) = \frac{1}{4}$$

So,

$$f'(1) = \lim_{h \rightarrow 0} \frac{\frac{1}{h+4} - \frac{1}{4}}{h}$$

Combine the terms in the numerator:

$$= \lim_{h \rightarrow 0} \frac{\frac{4-(h+4)}{4(h+4)}}{h} = \lim_{h \rightarrow 0} \frac{\frac{-h}{4(h+4)}}{h}$$

Simplify:

$$= \lim_{h \rightarrow 0} \frac{-1}{4(h+4)}$$

Now take the limit:

$$f'(1) = \frac{-1}{4(4)} = \boxed{-\frac{1}{16}}$$

Question 4

Let

$$f(x) = x^2$$

Use the definition of the derivative to compute $f'(3)$. Do not use differentiation rules.

Question 4 - Solution

We are given:

$$f(x) = x^2$$

Using the definition of the derivative at $x = 3$:

$$f'(3) = \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h}$$

Compute each term:

$$f(3+h) = (3+h)^2 = 9 + 6h + h^2$$

$$f(3) = 3^2 = 9$$

Substitute into the limit:

$$f'(3) = \lim_{h \rightarrow 0} \frac{(9 + 6h + h^2) - 9}{h} = \lim_{h \rightarrow 0} \frac{6h + h^2}{h}$$

Factor h from the numerator:

$$= \lim_{h \rightarrow 0} \frac{h(6 + h)}{h}$$

Cancel h :

$$= \lim_{h \rightarrow 0} (6 + h)$$

Now take the limit:

$$f'(3) = \boxed{6}$$

Question 5

Let

$$f(x) = \frac{1}{x}$$

Use the definition of the derivative to find an expression for $f'(x)$. Then evaluate $f'(2)$ using your expression.

Question 5 - Solution

We are given:

$$f(x) = \frac{1}{x}$$

Use the definition of the derivative:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

Combine the terms in the numerator:

$$= \lim_{h \rightarrow 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h} = \lim_{h \rightarrow 0} \frac{\frac{-h}{x(x+h)}}{h}$$

Simplify:

$$= \lim_{h \rightarrow 0} \frac{-1}{x(x+h)}$$

Take the limit:

$$f'(x) = \frac{-1}{x^2}$$

Now evaluate at $x = 2$:

$$f'(2) = \frac{-1}{2^2} = \frac{-1}{4}$$

Final Answer:

$$f'(x) = \frac{-1}{x^2}, \quad f'(2) = \boxed{-\frac{1}{4}}$$

Question 6

Let

$$f(x) = \sqrt{x+1}$$

Use the definition of the derivative to find $f'(x)$. Show all steps clearly without using derivative shortcuts.

Question 6 - Solution

We are given:

$$f(x) = \sqrt{x+1}$$

Use the definition of the derivative:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$$

To simplify the numerator, multiply by the conjugate:

$$\lim_{h \rightarrow 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}}$$

This becomes:

$$= \lim_{h \rightarrow 0} \frac{(x+h+1) - (x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h+1} + \sqrt{x+1})}$$

Cancel the h 's:

$$= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}}$$

Now take the limit as $h \rightarrow 0$:

$$f'(x) = \frac{1}{2\sqrt{x+1}}$$

Final Answer:

$$f'(x) = \boxed{\frac{1}{2\sqrt{x+1}}}$$

Question 7

Let

$$f(x) = \frac{1}{x}$$

Use the definition of the derivative to compute $f'(x)$. Show all steps clearly without using derivative rules.

Question 7 - Solution

We are given:

$$f(x) = \frac{1}{x}$$

Use the definition:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

Combine the terms in the numerator:

$$= \lim_{h \rightarrow 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h} = \lim_{h \rightarrow 0} \frac{\frac{-h}{x(x+h)}}{h}$$

Simplify:

$$= \lim_{h \rightarrow 0} \frac{-1}{x(x+h)}$$

Now take the limit as $h \rightarrow 0$:

$$f'(x) = \frac{-1}{x^2}$$

Final Answer:

$$f'(x) = \boxed{-\frac{1}{x^2}}$$

Question 8

Let

$$f(x) = \sqrt{x}$$

Use the definition of the derivative to compute $f'(x)$. Do not use derivative shortcuts. Show all algebraic simplification and steps clearly.

Question 8 - Solution

We are given:

$$f(x) = \sqrt{x}$$

Use the definition of the derivative:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Multiply numerator and denominator by the conjugate:

$$= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h \rightarrow 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

Simplify numerator:

$$= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

Now take the limit as $h \rightarrow 0$:

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Final Answer:

$$f'(x) = \boxed{\frac{1}{2\sqrt{x}}}$$

Question 9

Let

$$f(x) = \sqrt{x^2 + 4x + 5}$$

Task: Use the *definition of the derivative* to find $f'(x)$.

- Use first principles only.
- Do not apply derivative shortcuts.
- Show all algebraic steps.

Question 9 - Solution

We are given:

$$f(x) = \sqrt{x^2 + 4x + 5}$$

Apply the definition of the derivative:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)^2 + 4(x+h) + 5} - \sqrt{x^2 + 4x + 5}}{h}$$

Expand the expression inside the first square root:

$$(x+h)^2 + 4(x+h) + 5 = x^2 + 2xh + h^2 + 4x + 4h + 5$$

So we have:

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} - \sqrt{x^2 + 4x + 5}}{h}$$

Multiply numerator and denominator by the conjugate:

$$= \lim_{h \rightarrow 0} \frac{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} - \sqrt{x^2 + 4x + 5}}{h} \cdot \frac{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5}}{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5}}$$

Use the identity $(a-b)(a+b) = a^2 - b^2$:

$$= \lim_{h \rightarrow 0} \frac{(x^2 + 2xh + h^2 + 4x + 4h + 5) - (x^2 + 4x + 5)}{h (\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5})}$$

Simplify the numerator:

$$= \lim_{h \rightarrow 0} \frac{2xh + h^2 + 4h}{h (\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5})}$$

Factor h from the numerator:

$$= \lim_{h \rightarrow 0} \frac{h(2x + h + 4)}{h (\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5})}$$

Cancel h :

$$= \lim_{h \rightarrow 0} \frac{2x + h + 4}{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5}}$$

Now take the limit as $h \rightarrow 0$:

$$f'(x) = \frac{2x + 4}{2\sqrt{x^2 + 4x + 5}}$$

Final Answer:

$$f'(x) = \boxed{\frac{2x + 4}{2\sqrt{x^2 + 4x + 5}}}$$

Question 10

Let

$$f(x) = \sqrt{x+3}$$

Use the definition of the derivative to find $f'(x)$. Do not apply any derivative shortcuts — use first principles only.

Question 10 - Solution

We are given:

$$f(x) = \sqrt{x+3}$$

Apply the definition of the derivative:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h}$$

To evaluate this limit, multiply numerator and denominator by the conjugate of the numerator:

$$= \lim_{h \rightarrow 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h} \cdot \frac{\sqrt{x+h+3} + \sqrt{x+3}}{\sqrt{x+h+3} + \sqrt{x+3}}$$

Use the identity $(a-b)(a+b) = a^2 - b^2$:

$$= \lim_{h \rightarrow 0} \frac{(x+h+3) - (x+3)}{h(\sqrt{x+h+3} + \sqrt{x+3})} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h+3} + \sqrt{x+3})}$$

Cancel h in numerator and denominator:

$$= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h+3} + \sqrt{x+3}}$$

Now take the limit as $h \rightarrow 0$:

$$f'(x) = \frac{1}{2\sqrt{x+3}}$$

Final Answer:

$$f'(x) = \boxed{\frac{1}{2\sqrt{x+3}}}$$

Question 1

The position of a particle moving along a straight line is given by

$$s(t) = t^3 - 6t^2 + 9t \quad \text{for } t \geq 0$$

- (a) Find the velocity and acceleration functions.
- (b) At what time(s) is the particle at rest?
- (c) Determine when the particle is speeding up and when it is slowing down.

Question 1 - Solution

We are given the position function:

$$s(t) = t^3 - 6t^2 + 9t$$

(a) Velocity and Acceleration:

Velocity is the first derivative of position:

$$v(t) = s'(t) = 3t^2 - 12t + 9$$

Acceleration is the derivative of velocity:

$$a(t) = v'(t) = 6t - 12$$

(b) Particle at Rest:

The particle is at rest when $v(t) = 0$:

$$3t^2 - 12t + 9 = 0 \Rightarrow t^2 - 4t + 3 = 0 \Rightarrow (t - 1)(t - 3) = 0 \Rightarrow t = 1, 3$$

So, the particle is at rest at $t = \boxed{1}$ and $t = \boxed{3}$.

(c) Speeding Up / Slowing Down:

Speed increases when velocity and acceleration have the same sign; slows down when signs differ.

Critical points from previous steps:

$$v(t) = 3t^2 - 12t + 9, \quad a(t) = 6t - 12$$

Velocity is 0 at $t = 1, 3$, acceleration is 0 at $t = 2$

Test signs in intervals:

$t < 1$: $v(t) > 0$, $a(t) < 0 \rightarrow$ slowing down

$1 < t < 2$: $v(t) < 0$, $a(t) < 0 \rightarrow$ speeding up

$2 < t < 3$: $v(t) < 0$, $a(t) > 0 \rightarrow$ slowing down

$t > 3$: $v(t) > 0$, $a(t) > 0 \rightarrow$ speeding up

Conclusion:

- Speeding up: $t \in (1, 2) \cup (3, \infty)$
- Slowing down: $t \in (0, 1) \cup (2, 3)$

Question 2

A rock is thrown vertically upward from the top of a cliff. Its height in meters after t seconds is given by:

$$h(t) = -5t^2 + 20t + 50$$

- (a) What is the velocity of the rock after 2 seconds?
- (b) When does the rock reach its maximum height?
- (c) When does the rock hit the ground?
- (d) What is the velocity of the rock at the moment it hits the ground?

Question 2 - Solution

We are given:

$$h(t) = -5t^2 + 20t + 50$$

(a) Velocity after 2 seconds:

Velocity is the derivative of height:

$$v(t) = h'(t) = -10t + 20$$

So,

$$v(2) = -10(2) + 20 = -20 + 20 = \boxed{0 \text{ m/s}}$$

(b) Maximum Height:

Maximum height occurs when velocity is zero:

$$v(t) = -10t + 20 = 0 \Rightarrow t = 2 \text{ seconds}$$

(c) When does the rock hit the ground?

The rock hits the ground when $h(t) = 0$:

$$-5t^2 + 20t + 50 = 0 \Rightarrow t^2 - 4t - 10 = 0 \Rightarrow t = \frac{4 \pm \sqrt{16 + 40}}{2} = \frac{4 \pm \sqrt{56}}{2} = \frac{4 \pm 2\sqrt{14}}{2} = 2 \pm \sqrt{14}$$

Since time must be positive, we choose the positive root:

$$t = 2 + \sqrt{14} \approx \boxed{5.74 \text{ seconds}}$$

(d) Velocity at impact:

Plug into the velocity function:

$$v(2 + \sqrt{14}) = -10(2 + \sqrt{14}) + 20 = -20 - 10\sqrt{14} + 20 = -10\sqrt{14}$$

So the velocity at impact is:

$$\boxed{-10\sqrt{14} \text{ m/s}} \quad (\text{downward})$$

Question 3

The position of a particle moving along a straight line is given by:

$$s(t) = 3t^3 - 15t^2 + 18t \quad (\text{in meters}), \text{ for } t \geq 0.$$

- (a) Find the velocity and acceleration functions of the particle.
- (b) At what times is the particle at rest?
- (c) Determine the intervals when the particle is moving forward.
- (d) When is the acceleration zero, and what does it signify?

Question 3 - Solution

Given:

$$s(t) = 3t^3 - 15t^2 + 18t$$

(a) Velocity and Acceleration:

Velocity is the derivative of position:

$$v(t) = s'(t) = 9t^2 - 30t + 18$$

Acceleration is the derivative of velocity:

$$a(t) = v'(t) = 18t - 30$$

(b) Particle at Rest:

Set velocity to zero:

$$9t^2 - 30t + 18 = 0 \Rightarrow t = \frac{30 \pm \sqrt{(-30)^2 - 4(9)(18)}}{2(9)} = \frac{30 \pm \sqrt{900 - 648}}{18} = \frac{30 \pm \sqrt{252}}{18}$$

$$\sqrt{252} = 6\sqrt{7}, \quad t = \frac{30 \pm 6\sqrt{7}}{18} = \frac{5 \pm \sqrt{7}}{3}$$

So the particle is at rest at:

$$t = \frac{5 \pm \sqrt{7}}{3}$$

(c) Particle Moving Forward:

Forward motion means $v(t) > 0$. Use sign chart for:

$$v(t) = 9t^2 - 30t + 18 = 9(t - t_1)(t - t_2), \quad \text{where } t_1 = \frac{5 - \sqrt{7}}{3}, t_2 = \frac{5 + \sqrt{7}}{3}$$

This quadratic opens upward (coefficient of t^2 is positive), so:

$$v(t) > 0 \quad \text{on } (0, t_1) \cup (t_2, \infty)$$

So the particle moves forward on:

$$\left(0, \frac{5 - \sqrt{7}}{3}\right) \cup \left(\frac{5 + \sqrt{7}}{3}, \infty\right)$$

(d) Acceleration Zero:

$$a(t) = 18t - 30 = 0 \Rightarrow t = \frac{30}{18} = \frac{5}{3}$$

This is when the particle changes concavity or when velocity changes most rapidly.

Conclusion:

$$a(t) = 0 \text{ when } t = \frac{5}{3} \text{ seconds}$$

Question 4

The cost (in dollars) to produce x items is given by the function:

$$C(x) = 50x + 300\sqrt{x}$$

- (a) Find the marginal cost function.
- (b) Interpret the meaning of $C'(100)$.
- (c) Approximate the cost of producing the 101st item using the derivative.
- (d) Compute the actual cost of producing the 101st item and compare with your approximation.

Question 4 - Solution

Marginal cost. For $x > 0$,

$$C'(x) = 50 + 150/\sqrt{x}.$$

Thus $C'(100) = 65$ dollars per item. This is the instantaneous cost rate and predicts a cost of approximately $\boxed{\$65}$ for one additional item.

Actual additional cost.

$$C(101) - C(100) = 50 + 300(\sqrt{101} - 10) \approx \boxed{\$64.96}.$$

The derivative estimate exceeds the actual cost by about \$0.04. This agrees with $C''(x) = -75x^{-3/2} < 0$: the marginal cost decreases over the next item.

Question 5

The population of a certain city (in thousands) $P(t)$, as a function of time t (in years since 2000), is modeled by the function:

$$P(t) = 500 + 40t - 0.5t^2$$

- (a) Find the rate of change of the population with respect to time.
- (b) What is the population growth rate in the year 2010?
- (c) At what year is the population growth rate zero?
- (d) Interpret your result from (c) in the context of the population.

Question 5 - Solution

We are given:

$$P(t) = 500 + 40t - 0.5t^2$$

(a) Rate of Change of Population:

The rate of change of the population is given by the derivative:

$$P'(t) = \frac{d}{dt}(500 + 40t - 0.5t^2) = 40 - t$$

$$P'(t) = 40 - t$$

(b) Growth Rate in 2010:

Since t is years since 2000, 2010 corresponds to $t = 10$.

$$P'(10) = 40 - 10 = 30$$

Interpretation: In 2010, the population was increasing at a rate of 30 thousand people per year.

$$P'(10) = 30 \quad (\text{thousand people/year})$$

(c) When is the Population Growth Rate Zero?

Set $P'(t) = 0$:

$$40 - t = 0 \Rightarrow t = 40$$

$$t = 40$$

(d) Interpretation:

Since $t = 40$ corresponds to the year $2000 + 40 = 2040$, the population stops growing in 2040.

Conclusion: In 2040, the population reaches its maximum and stops increasing. After that, it begins to decline.

$$\text{The population stops growing in 2040.}$$

Question 6

The temperature T (in °C) in a chemical reaction chamber is modeled by the function

$$T(t) = 100 - 20e^{-0.5t}$$

where t is the time in minutes since the reaction started.

- (a) Find the rate at which the temperature is changing with respect to time.
- (b) What is the rate of temperature change at $t = 0$?
- (c) What is the long-term behavior of the rate of temperature change? Explain.

Question 6 - Solution

We are given:

$$T(t) = 100 - 20e^{-0.5t}$$

(a) Rate of Change of Temperature:

Differentiate $T(t)$ with respect to t :

$$T'(t) = \frac{d}{dt}(100 - 20e^{-0.5t}) = -20 \cdot \frac{d}{dt}(e^{-0.5t}) = -20 \cdot (-0.5e^{-0.5t}) = 10e^{-0.5t}$$

$$T'(t) = 10e^{-0.5t}$$

This represents the rate of temperature change at time t .

(b) Rate of Change at $t = 0$:

$$T'(0) = 10e^{-0.5 \cdot 0} = 10e^0 = 10$$

$$T'(0) = 10 \quad (\text{°C per minute})$$

Interpretation: At the beginning of the reaction, the temperature is increasing at 10°C per minute.

(c) Long-Term Behavior:

As $t \rightarrow \infty$, $e^{-0.5t} \rightarrow 0$, so:

$$T'(t) = 10e^{-0.5t} \rightarrow 0$$

Conclusion: The rate of temperature change decreases over time and approaches zero. This means the temperature stabilizes and stops changing as time progresses.

$$\lim_{t \rightarrow \infty} T'(t) = 0$$

Question 7

The position of a particle moving along a straight line is given by the function

$$s(t) = t^3 - 6t^2 + 9t$$

where $s(t)$ is in meters and $t \geq 0$ is in seconds.

- (a) Find the velocity function $v(t)$.
- (b) At what time(s) is the particle at rest?
- (c) Determine the direction of motion (moving forward or backward) on the intervals determined by part (b).
- (d) Is the particle speeding up or slowing down at $t = 1$? Justify using acceleration.

Question 7 - Solution

Velocity and rest times.

$$v(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3), \quad a(t) = 6t - 12.$$

Hence the particle is at rest at $t = 1, 3$.

For $t \geq 0$, it moves forward on $[0, 1) \cup (3, \infty)$ and backward on $(1, 3)$.

Behavior at $t = 1$. Here $v(1) = 0$ and $a(1) = -6$. Just before 1, velocity and acceleration have opposite signs, so speed decreases. Just after 1, both are negative, so speed increases.

The speed $|v(t)|$ has left derivative -6 and right derivative 6 at 1 ; it is not differentiable there.

At $t = 1$ it is at rest and reverses direction.

It slows down immediately before that instant and speeds up immediately afterward.

Question 8

The height of a ball thrown vertically upward is modeled by the function:

$$h(t) = -16t^2 + 64t + 80$$

where $h(t)$ is the height in feet after t seconds.

- (a) Determine the velocity function of the ball.
- (b) When does the ball reach its maximum height?
- (c) What is the maximum height the ball reaches?
- (d) What is the acceleration of the ball at any time t ?

Question 8 - Solution

We are given:

$$h(t) = -16t^2 + 64t + 80$$

(a) Velocity Function:

The velocity is the first derivative of height:

$$v(t) = h'(t) = -32t + 64$$

$$v(t) = -32t + 64$$

(b) Time of Maximum Height:

The ball reaches its maximum height when the velocity is zero:

$$-32t + 64 = 0 \Rightarrow t = 2$$

$$t = 2 \text{ seconds}$$

(c) Maximum Height:

Plug $t = 2$ into the original height function:

$$h(2) = -16(2)^2 + 64(2) + 80 = -64 + 128 + 80 = 144$$

$$144 \text{ feet}$$

(d) Acceleration Function:

Acceleration is the derivative of velocity:

$$a(t) = v'(t) = -32$$

$$a(t) = -32 \text{ ft/s}^2 \text{ (constant acceleration due to gravity)}$$

Question 9

The position $s(t)$ of a particle moving along a straight line is given by:

$$s(t) = \frac{3t^2 + 2}{t + 1}, \quad t > 0$$

- (a) Find the instantaneous velocity of the particle at $t = 1$.
- (b) Interpret the sign of the velocity at $t = 1$.
- (c) Find all critical points of $s(t)$ and determine whether the particle changes direction.

Question 9 - Solution

We are given:

$$s(t) = \frac{3t^2 + 2}{t + 1}, \quad t > 0$$

(a) Instantaneous Velocity at $t = 1$:

Velocity is the derivative $s'(t)$. Using the quotient rule:

$$\begin{aligned} s'(t) &= \frac{(t+1)(6t) - (3t^2+2)(1)}{(t+1)^2} \\ &= \frac{6t^2 + 6t - 3t^2 - 2}{(t+1)^2} = \frac{3t^2 + 6t - 2}{(t+1)^2} \end{aligned}$$

Evaluate at $t = 1$:

$$s'(1) = \frac{3 + 6 - 2}{4} = \frac{7}{4}$$

$$\boxed{s'(1) = \frac{7}{4}}$$

(b) Interpretation of the Sign of Velocity:

Since $s'(1) = \frac{7}{4} > 0$, the particle is moving in the positive direction at $t = 1$.

The particle is moving to the right (increasing position) at $t = 1$

(c) Critical Points and Direction Change:

Critical points occur where $s'(t) = 0$ or where $s'(t)$ is undefined.

The derivative is undefined at $t = -1$, which is outside the domain $t > 0$ and therefore ignored.

Set the numerator equal to zero:

$$3t^2 + 6t - 2 = 0$$

$$t = \frac{-6 \pm \sqrt{36 + 24}}{6} = \frac{-6 \pm \sqrt{60}}{6} = \frac{-3 \pm \sqrt{15}}{3}$$

This gives two solutions:

$$t = \frac{-3 - \sqrt{15}}{3} \quad \text{and} \quad t = \frac{-3 + \sqrt{15}}{3}$$

Since time must satisfy $t > 0$, the negative solution is rejected. The only valid critical time is:

$$\boxed{t = \frac{-3 + \sqrt{15}}{3}}$$

To determine whether the particle changes direction, test the sign of $s'(t)$ on either side of this value (within $t > 0$). The derivative changes from negative to positive, indicating a direction change.

The particle changes direction at $t = \frac{-3 + \sqrt{15}}{3}$

Question 10

Suppose that the function $C(p)$ models the total cost (in dollars) of producing p items, and is given by:

$$C(p) = 200 + 8p + 0.05p^2$$

- (a) Find and interpret $C'(p)$.
- (b) Find $C'(50)$ and interpret its meaning in context.
- (c) Determine whether the marginal cost is increasing or decreasing as p increases.

Question 10 - Solution

We are given:

$$C(p) = 200 + 8p + 0.05p^2$$

(a) Find and interpret $C'(p)$:

Differentiate:

$$C'(p) = \frac{d}{dp}(200 + 8p + 0.05p^2) = 0 + 8 + 0.1p = 8 + 0.1p$$

Interpretation: $C'(p)$ is the marginal cost — the approximate cost of producing one additional item when p items have already been produced.

(b) Compute and interpret $C'(50)$:

$$C'(50) = 8 + 0.1(50) = 8 + 5 = \boxed{13}$$

Interpretation: When 50 items have been produced, the cost of producing one more item is approximately \$13.

(c) Is marginal cost increasing or decreasing?

Since:

$$C'(p) = 8 + 0.1p$$

and the derivative of this is:

$$C''(p) = 0.1 > 0$$

The marginal cost is increasing as p increases.

Marginal cost increases with production level.

Question 1

Let

$$f(x) = \frac{\sin(x)}{x^2 + 1} + x \cos(x).$$

- (a) Find the derivative $f'(x)$ using standard differentiation rules.
- (b) Determine the slope of the tangent line to the graph of $f(x)$ at $x = 0$.

Question 1 - Solution

We are given:

$$f(x) = \frac{\sin(x)}{x^2 + 1} + x \cos(x)$$

(a) Find $f'(x)$:

Differentiate term by term.

First term (quotient rule):

$$\frac{d}{dx} \left(\frac{\sin(x)}{x^2 + 1} \right) = \frac{(x^2 + 1) \cos(x) - \sin(x)(2x)}{(x^2 + 1)^2}$$

Second term (product rule):

$$\frac{d}{dx} (x \cos(x)) = \cos(x) - x \sin(x)$$

Putting both together:

$$f'(x) = \frac{(x^2 + 1) \cos(x) - 2x \sin(x)}{(x^2 + 1)^2} + \cos(x) - x \sin(x)$$

(b) Find the slope at $x = 0$:

Evaluate $f'(0)$.

$$\begin{aligned} f'(0) &= \frac{(0^2 + 1) \cos(0) - 2(0) \sin(0)}{(0^2 + 1)^2} + \cos(0) - 0 \cdot \sin(0) \\ &= \frac{1 \cdot 1 - 0}{1^2} + 1 - 0 \\ &= 1 + 1 = \boxed{2} \end{aligned}$$

So, the slope of the tangent line to the curve at $x = 0$ is $\boxed{2}$.

Question 2

Let

$$f(x) = \sqrt{x^2 + 3x} + \frac{1}{\sqrt{x^2 + 3x}}.$$

- (a) Compute $f'(x)$ using differentiation rules.
- (b) Determine all points where the derivative does not exist.

Question 2 - Solution

Domain. Set $u = x^2 + 3x = x(x + 3)$. The reciprocal square root requires $u > 0$, so

$$D = (-\infty, -3) \cup (0, \infty).$$

Differentiate.

$$f = u^{1/2} + u^{-1/2}, \quad \boxed{f'(x) = \frac{(2x+3)(x^2+3x-1)}{2(x^2+3x)^{3/2}}}, \quad x \in D.$$

The derivative exists at every point of the function's domain. On $[-3, 0]$, the function itself is undefined; these are not nondifferentiable points of its graph.

Question 3

Let

$$f(x) = \frac{e^x}{\cos x} + \ln(x^2 + 1).$$

- (a) Find $f'(x)$.
- (b) Determine the domain of $f'(x)$.

Question 3 - Solution

We are given:

$$f(x) = \frac{e^x}{\cos x} + \ln(x^2 + 1)$$

(a) Find $f'(x)$:

Differentiate each term separately.

For the first term $\frac{e^x}{\cos x}$, use the quotient rule:

$$\frac{d}{dx} \left(\frac{e^x}{\cos x} \right) = \frac{(\cos x)(e^x) - e^x(-\sin x)}{(\cos x)^2} = \frac{e^x(\cos x + \sin x)}{\cos^2 x}$$

For the second term $\ln(x^2 + 1)$, use the chain rule:

$$\frac{d}{dx} \ln(x^2 + 1) = \frac{2x}{x^2 + 1}$$

Combine both:

$$f'(x) = \frac{e^x(\cos x + \sin x)}{\cos^2 x} + \frac{2x}{x^2 + 1}$$

(b) Domain of $f'(x)$:

We must exclude values that cause division by zero.

$\cos x = 0$ when $x = \frac{\pi}{2} + n\pi$, where $n \in \mathbb{Z}$, $x^2 + 1 > 0$ for all real x , so no restrictions there

Domain: All real x such that $\cos x \neq 0$

$$f'(x) = \frac{e^x(\cos x + \sin x)}{\cos^2 x} + \frac{2x}{x^2 + 1}, \quad x \in \mathbb{R} \setminus \left\{ \frac{\pi}{2} + n\pi \mid n \in \mathbb{Z} \right\}$$

Question 4

Let

$$f(x) = \frac{\arctan x}{1 + x^2} + x^5 e^x.$$

- (a) Compute $f'(x)$.
- (b) Write the equation for critical numbers and approximate one in $(-5, -4)$ to six decimal places.

Question 4 - Solution**(a) Differentiate**

The quotient and product rules give

$$f'(x) = \frac{1 - 2x \arctan x}{(1 + x^2)^2} + e^x x^4 (x + 5).$$

(b) Critical numbers

The derivative exists for every real x , since $1 + x^2 > 0$.

Thus critical numbers solve $f'(x) = 0$. Bisection on $(-5, -4)$ gives

$$x \approx -4.995520.$$

The derivative has opposite signs at the ends of this interval, so continuity guarantees a root there.

Question 5

Let

$$f(x) = \frac{\ln(x)}{x^3}$$

- (a) Find the derivative $f'(x)$ using the quotient rule.
- (b) Determine any local extrema of $f(x)$ for $x > 0$.

Question 5 - Solution

We are given:

$$f(x) = \frac{\ln(x)}{x^3}$$

(a) Find $f'(x)$:

Use the quotient rule:

$$f'(x) = \frac{x^3 \cdot \frac{1}{x} - \ln(x) \cdot 3x^2}{x^6} = \frac{x^2 - 3x^2 \ln(x)}{x^6}$$

Factor the numerator:

$$f'(x) = \frac{x^2(1 - 3\ln(x))}{x^6} = \frac{1 - 3\ln(x)}{x^4}$$

So,

$$f'(x) = \boxed{\frac{1 - 3\ln(x)}{x^4}}$$

(b) Local Extrema:

Critical points occur where $f'(x) = 0$:

$$\frac{1 - 3\ln(x)}{x^4} = 0 \Rightarrow 1 - 3\ln(x) = 0 \Rightarrow \ln(x) = \frac{1}{3} \Rightarrow x = e^{1/3}$$

Now determine if this is a maximum or minimum by checking the sign of $f'(x)$ around $x = e^{1/3}$:

For $x < e^{1/3}$, say $x = 1$: $f'(1) = \frac{1 - 3\ln(1)}{1^4} = 1 > 0$

For $x > e^{1/3}$, say $x = 2$: $f'(2) = \frac{1 - 3\ln(2)}{2^4} \approx \frac{1 - 3(0.693)}{16} \approx \frac{-1.079}{16} < 0$

So $f(x)$ has a local maximum at $x = e^{1/3}$.

Conclusion:

$f'(x) = \frac{1 - 3\ln(x)}{x^4}$, Local maximum at $x = \boxed{e^{1/3}}$

Question 6

Let

$$f(x) = \sqrt{x^2 + 4x + 5}$$

- (a) Differentiate $f(x)$ using the chain rule.
- (b) Find all values of x where the tangent line to $f(x)$ is horizontal.

Question 6 - Solution

We are given:

$$f(x) = \sqrt{x^2 + 4x + 5} = (x^2 + 4x + 5)^{1/2}$$

(a) Differentiate $f(x)$:

Using the chain rule:

$$f'(x) = \frac{1}{2}(x^2 + 4x + 5)^{-1/2} \cdot (2x + 4)$$

Simplify:

$$f'(x) = \frac{2x + 4}{2\sqrt{x^2 + 4x + 5}}$$

So,

$$f'(x) = \boxed{\frac{2x + 4}{2\sqrt{x^2 + 4x + 5}}}$$

(b) Horizontal Tangents:

The tangent is horizontal when $f'(x) = 0$. That occurs when the numerator is zero:

$$2x + 4 = 0 \Rightarrow x = -2$$

Check that the denominator is defined at $x = -2$:

$$\sqrt{(-2)^2 + 4(-2) + 5} = \sqrt{4 - 8 + 5} = \sqrt{1} = 1$$

So it's defined, and the tangent is horizontal at $x = -2$.

Conclusion:

- $f'(x) = \frac{2x + 4}{2\sqrt{x^2 + 4x + 5}}$
- Horizontal tangent at $x = \boxed{-2}$

Question 7

Let

$$f(x) = \frac{e^x \sin x}{x^2 + 1}$$

- (a) Differentiate $f(x)$ using the quotient rule.
- (b) Evaluate $f'(0)$.

Question 7 - Solution

We are given:

$$f(x) = \frac{e^x \sin x}{x^2 + 1}$$

(a) Use the Quotient Rule:

The quotient rule is:

$$f(x) = \frac{u(x)}{v(x)} \Rightarrow f'(x) = \frac{u'v - uv'}{v^2}$$

Let:

$$u(x) = e^x \sin x, \quad v(x) = x^2 + 1$$

Differentiate $u(x)$ using the product rule:

$$u'(x) = \frac{d}{dx}[e^x \sin x] = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$$

Differentiate $v(x)$:

$$v'(x) = \frac{d}{dx}[x^2 + 1] = 2x$$

Now apply the quotient rule:

$$f'(x) = \frac{e^x(\sin x + \cos x)(x^2 + 1) - e^x \sin x \cdot 2x}{(x^2 + 1)^2}$$

Factor out e^x :

$$f'(x) = \frac{e^x [(\sin x + \cos x)(x^2 + 1) - 2x \sin x]}{(x^2 + 1)^2}$$

(b) Evaluate $f'(0)$:

Substitute $x = 0$:

Numerator:

$$e^0 [(\sin 0 + \cos 0)(0^2 + 1) - 2(0) \sin 0] = 1 \cdot (0 + 1)(1) = 1$$

Denominator:

$$(0^2 + 1)^2 = 1^2 = 1$$

So:

$$f'(0) = \boxed{1}$$

Question 8

Let

$$f(x) = \ln \left(\sqrt{x^2 + 3x + 2} \right)$$

- (a) Simplify the function before differentiating.
- (b) Find $f'(x)$.

Question 8 - Solution**(a) Simplify the Function:**

We start with:

$$f(x) = \ln\left(\sqrt{x^2 + 3x + 2}\right)$$

Since

$$\sqrt{x^2 + 3x + 2} = (x^2 + 3x + 2)^{1/2},$$

we can write:

$$f(x) = \ln\left((x^2 + 3x + 2)^{1/2}\right)$$

Using the logarithm rule

$$\ln(A^r) = r \ln(A),$$

we get:

$$f(x) = \frac{1}{2} \ln(x^2 + 3x + 2)$$

So the simplified function is:

$$f(x) = \frac{1}{2} \ln(x^2 + 3x + 2)$$

(b) Find $f'(x)$:

Now differentiate:

$$f(x) = \frac{1}{2} \ln(x^2 + 3x + 2)$$

Using

$$\frac{d}{dx} \ln(u) = \frac{u'}{u},$$

where

$$u = x^2 + 3x + 2$$

and

$$u' = 2x + 3,$$

we get:

$$f'(x) = \frac{1}{2} \cdot \frac{2x + 3}{x^2 + 3x + 2}$$

Therefore,

$$f'(x) = \frac{2x + 3}{2(x^2 + 3x + 2)}$$

Question 9

Let

$$f(x) = \tan^2(x) + \sec^2(x)$$

- (a) Differentiate $f(x)$ using the standard differentiation formulas for trigonometric functions.
- (b) Simplify your answer.

Question 9 - Solution**(a) Differentiate $f(x)$:**

We are given:

$$f(x) = \tan^2(x) + \sec^2(x)$$

We apply the chain rule and known derivatives:

$$\frac{d}{dx}[\tan^2(x)] = 2 \tan(x) \sec^2(x) \text{ (using chain rule)}$$

$$\frac{d}{dx}[\sec^2(x)] = 2 \sec^2(x) \tan(x) \text{ (using chain rule)}$$

So:

$$f'(x) = 2 \tan(x) \sec^2(x) + 2 \sec^2(x) \tan(x)$$

(b) Simplify:

Both terms are the same:

$$f'(x) = 4 \tan(x) \sec^2(x)$$

Final Answer:

$f'(x) = 4 \tan(x) \sec^2(x)$

Question 10

Let

$$f(x) = \frac{\sin(x)}{1 + \cos(x)}$$

- (a) Differentiate $f(x)$ using the quotient rule.
- (b) Simplify the result as much as possible.

Question 10 - Solution

We are given:

$$f(x) = \frac{\sin(x)}{1 + \cos(x)}$$

Let:

$$u = \sin(x), \quad v = 1 + \cos(x)$$

Then:

$$f'(x) = \frac{v \cdot u' - u \cdot v'}{v^2}$$

Compute the derivatives:

$$u' = \cos(x), \quad v' = -\sin(x)$$

Apply the quotient rule:

$$f'(x) = \frac{(1 + \cos(x)) \cos(x) - \sin(x)(-\sin(x))}{(1 + \cos(x))^2}$$

Simplify numerator:

$$(1 + \cos(x)) \cos(x) + \sin^2(x) = \cos(x) + \cos^2(x) + \sin^2(x)$$

Use the identity $\sin^2(x) + \cos^2(x) = 1$:

$$= \cos(x) + 1$$

So:

$$f'(x) = \frac{\cos(x) + 1}{(1 + \cos(x))^2}$$

Cancel common factor:

$$f'(x) = \frac{1}{1 + \cos(x)}$$

Final Answer:

$f'(x) = \frac{1}{1 + \cos(x)}$

Question 1

Let

$$f(x) = x^2 \cdot \ln(x)$$

- (a) Differentiate $f(x)$ using the product rule.
- (b) Find the equation of the tangent line to the graph of $f(x)$ at $x = 1$.

Question 1 - Solution

We are given:

$$f(x) = x^2 \cdot \ln(x)$$

Let:

$$u = x^2, \quad v = \ln(x) \Rightarrow u' = 2x, \quad v' = \frac{1}{x}$$

Apply the product rule:

$$f'(x) = u'v + uv' = 2x \ln(x) + x^2 \cdot \frac{1}{x} = 2x \ln(x) + x$$

Final derivative:

$$f'(x) = 2x \ln(x) + x$$

Now evaluate the tangent line at $x = 1$:

$$f(1) = 1^2 \cdot \ln(1) = 0$$

$$f'(1) = 2(1) \ln(1) + 1 = 0 + 1 = 1$$

Point on the curve: $(1, 0)$ Slope of tangent: 1

Use point-slope form:

$$y - 0 = 1(x - 1) \Rightarrow y = x - 1$$

Equation of the tangent line:

$$y = x - 1$$

Question 2

Let

$$f(x) = \frac{x^2 \sin(x)}{x + 1}$$

- (a) Differentiate $f(x)$ using the quotient rule.
- (b) Simplify your answer as much as possible.

Question 2 - Solution

We are given:

$$f(x) = \frac{x^2 \sin(x)}{x+1}$$

Let

$$u(x) = x^2 \sin(x), \quad v(x) = x+1$$

Then using the **quotient rule**:

$$f'(x) = \frac{v \cdot u' - u \cdot v'}{v^2}$$

We compute:

$$u'(x) = \frac{d}{dx}[x^2 \sin(x)] = 2x \sin(x) + x^2 \cos(x) \quad (\text{product rule})$$

$$v'(x) = \frac{d}{dx}[x+1] = 1$$

Now plug into the quotient rule:

$$f'(x) = \frac{(x+1)(2x \sin(x) + x^2 \cos(x)) - x^2 \sin(x) \cdot 1}{(x+1)^2}$$

Now expand and simplify the numerator:

$$= \frac{(x+1)(2x \sin(x) + x^2 \cos(x)) - x^2 \sin(x)}{(x+1)^2}$$

Distribute:

$$= \frac{2x(x+1) \sin(x) + x^2(x+1) \cos(x) - x^2 \sin(x)}{(x+1)^2}$$

Final simplified expression:

$f'(x) = \frac{2x(x+1) \sin(x) - x^2 \sin(x) + x^2(x+1) \cos(x)}{(x+1)^2}$
--

Question 3

Let

$$f(x) = \frac{\ln(x)}{x \cos(x)}$$

- (a) Differentiate $f(x)$ using the quotient and product rules.
- (b) Simplify your answer as much as possible.

Question 3 - Solution

We are given:

$$f(x) = \frac{\ln(x)}{x \cos(x)}$$

Let

$$u(x) = \ln(x), \quad v(x) = x \cos(x)$$

Then by the **quotient rule**:

$$f'(x) = \frac{v \cdot u' - u \cdot v'}{v^2}$$

Compute each derivative:

$$u'(x) = \frac{1}{x}$$

For $v(x) = x \cos(x)$, use the **product rule**:

$$v'(x) = \frac{d}{dx}[x \cos(x)] = \cos(x) - x \sin(x)$$

Now substitute into the quotient rule:

$$f'(x) = \frac{x \cos(x) \cdot \frac{1}{x} - \ln(x)(\cos(x) - x \sin(x))}{(x \cos(x))^2}$$

Simplify numerator:

$$f'(x) = \frac{\cos(x) - \ln(x) \cos(x) + \ln(x)x \sin(x)}{x^2 \cos^2(x)}$$

Final Answer:

$$f'(x) = \frac{\cos(x)(1 - \ln(x)) + \ln(x)x \sin(x)}{x^2 \cos^2(x)}$$

Question 4

Let

$$f(x) = (x^2 + 3x) \cdot e^x$$

- (a) Use the product rule to differentiate $f(x)$.
- (b) Factor and simplify your final expression.

Question 4 - Solution

We are given:

$$f(x) = (x^2 + 3x) \cdot e^x$$

Let:

$$u(x) = x^2 + 3x, \quad v(x) = e^x$$

By the product rule:

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$

Differentiate each part:

$$u'(x) = 2x + 3, \quad v'(x) = e^x$$

Now plug into the formula:

$$f'(x) = (2x + 3)e^x + (x^2 + 3x)e^x$$

Factor out e^x :

$$f'(x) = e^x [(2x + 3) + (x^2 + 3x)]$$

Simplify inside the brackets:

$$(2x + 3) + (x^2 + 3x) = x^2 + 5x + 3$$

Final Answer:

$$\boxed{f'(x) = e^x(x^2 + 5x + 3)}$$

Question 5

Let

$$f(x) = x^2 e^x \cos x$$

- (a) Compute $f'(x)$ using appropriate differentiation rules.
- (b) Factor the result where possible.

Question 5 - Solution

We are given:

$$f(x) = x^2 e^x \cos x$$

This is a product of three functions: - $u(x) = x^2$ - $v(x) = e^x$ - $w(x) = \cos x$

Use the **product rule** for three functions:

$$f'(x) = u'(x)v(x)w(x) + u(x)v'(x)w(x) + u(x)v(x)w'(x)$$

Compute derivatives: - $u'(x) = 2x$ - $v'(x) = e^x$ - $w'(x) = -\sin x$

Now substitute:

$$f'(x) = 2xe^x \cos x + x^2 e^x \cos x - x^2 e^x \sin x$$

Factor where possible:

$$f'(x) = e^x (\cos x (2x + x^2) - x^2 \sin x)$$

Final Answer:

$$\boxed{f'(x) = e^x [\cos x (x^2 + 2x) - x^2 \sin x]}$$

Question 6

Let

$$f(x) = \frac{x^2 e^{2x} \sin x}{1 + x^3}$$

- (a) Compute $f'(x)$.
- (b) Simplify and factor your answer as much as possible.

Question 6 - Solution

We are given:

$$f(x) = \frac{x^2 e^{2x} \sin x}{1 + x^3}$$

Let

$$u(x) = x^2 e^{2x} \sin x, \quad v(x) = 1 + x^3$$

Using the quotient rule:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Step 1: Differentiate $u(x)$

The function $u(x)$ is a product of three functions:

$$u(x) = x^2 \cdot e^{2x} \cdot \sin x$$

Apply the product rule:

$$u'(x) = (x^2)' e^{2x} \sin x + x^2 (e^{2x})' \sin x + x^2 e^{2x} (\sin x)'$$

Compute derivatives:

$$(x^2)' = 2x, \quad (e^{2x})' = 2e^{2x}, \quad (\sin x)' = \cos x$$

So:

$$u'(x) = 2x e^{2x} \sin x + 2x^2 e^{2x} \sin x + x^2 e^{2x} \cos x$$

Factor:

$$u'(x) = e^{2x} [\sin x (2x + 2x^2) + x^2 \cos x]$$

Step 2: Differentiate $v(x)$

$$v'(x) = 3x^2$$

Step 3: Apply the Quotient Rule

$$f'(x) = \frac{e^{2x} [\sin x (2x + 2x^2) + x^2 \cos x] (1 + x^3) - x^2 e^{2x} \sin x (3x^2)}{(1 + x^3)^2}$$

Factor common terms in the numerator:

$$f'(x) = \frac{e^{2x} x [(1 + x^3)(2 \sin x + 2x \sin x + x \cos x) - 3x^3 \sin x]}{(1 + x^3)^2}$$

Final Answer:

$$\boxed{f'(x) = \frac{e^{2x} x [(1 + x^3)(2 \sin x + 2x \sin x + x \cos x) - 3x^3 \sin x]}{(1 + x^3)^2}}$$

Question 7

Let

$$f(x) = \frac{x \sin x}{x^2 + 1}$$

- (a) Compute $f'(x)$ using the quotient rule.
- (b) Simplify the derivative fully.

Question 7 - Solution

We are given:

$$f(x) = \frac{x \sin x}{x^2 + 1}$$

Let:

$$u(x) = x \sin x, \quad v(x) = x^2 + 1$$

We apply the **quotient rule**:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

First, compute $u'(x)$ using the product rule:

$$u'(x) = \sin x + x \cos x$$

Next, compute $v'(x)$:

$$v'(x) = 2x$$

Now apply the quotient rule:

$$f'(x) = \frac{(\sin x + x \cos x)(x^2 + 1) - x \sin x(2x)}{(x^2 + 1)^2}$$

Expand the numerator:

$$(\sin x)(x^2 + 1) + x \cos x(x^2 + 1) - 2x^2 \sin x$$

Distribute:

$$x^2 \sin x + \sin x + x^3 \cos x + x \cos x - 2x^2 \sin x$$

Combine like terms:

$$-x^2 \sin x + \sin x + x^3 \cos x + x \cos x$$

Factor where possible:

$$\sin x(1 - x^2) + x \cos x(x^2 + 1)$$

Final Answer:

$$f'(x) = \frac{\sin x(1 - x^2) + x \cos x(x^2 + 1)}{(x^2 + 1)^2}$$

Question 8

Let

$$f(x) = \frac{e^x \cos x}{\ln(x)}$$

- (a) Compute $f'(x)$ using the quotient rule.
- (b) State the domain of $f'(x)$.

Question 8 - Solution

We are given:

$$f(x) = \frac{e^x \cos x}{\ln(x)}$$

Let: $u(x) = e^x \cos x$ - $v(x) = \ln(x)$

We'll use the **quotient rule**:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Step 1: Differentiate $u(x) = e^x \cos x$ using the **product rule**:

$$u'(x) = e^x \cos x + e^x(-\sin x) = e^x(\cos x - \sin x)$$

Step 2: Differentiate $v(x) = \ln(x)$:

$$v'(x) = \frac{1}{x}$$

Now plug into the quotient rule:

$$f'(x) = \frac{e^x(\cos x - \sin x) \cdot \ln x - e^x \cos x \cdot \frac{1}{x}}{(\ln x)^2}$$

Factor e^x from the numerator:

$$f'(x) = \frac{e^x \left[(\cos x - \sin x) \ln x - \frac{\cos x}{x} \right]}{(\ln x)^2}$$

Final Answer:

$$f'(x) = \frac{e^x \left[(\cos x - \sin x) \ln x - \frac{\cos x}{x} \right]}{(\ln x)^2}$$

Domain of $f'(x)$:

$\ln x$ is defined only for $x > 0$

Also, $\ln x \neq 0 \Rightarrow x \neq 1$

And denominator has $\ln(x)^2$, so $x = 1$ is excluded

The term $\frac{\cos x}{x}$ is defined for $x > 0$

Domain:

$$(0, 1) \cup (1, \infty)$$

Question 9

Let

$$f(x) = (x^2 + 1) \tan(x)$$

- (a) Differentiate $f(x)$ using the product rule.
- (b) Determine all x for which $f'(x)$ is undefined in the interval $(-\pi, \pi)$.

Question 9 - Solution

We are given:

$$f(x) = (x^2 + 1) \tan(x)$$

Let: $u(x) = x^2 + 1$ $v(x) = \tan(x)$

We'll use the product rule:

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$

Step 1: Differentiate $u(x)$:

$$u'(x) = 2x$$

Step 2: Differentiate $v(x) = \tan(x)$:

$$v'(x) = \sec^2(x)$$

Now apply the product rule:

$$f'(x) = 2x \tan(x) + (x^2 + 1) \sec^2(x)$$

Final Answer:

$$f'(x) = 2x \tan(x) + (x^2 + 1) \sec^2(x)$$

(b) Where is $f'(x)$ undefined on $(-\pi, \pi)$?

$\tan(x)$ and $\sec^2(x)$ are undefined where $\cos(x) = 0$

$\cos(x) = 0$ at $x = -\frac{\pi}{2}, \frac{\pi}{2}$ in the given interval

So, $f'(x)$ is undefined at:

$$x = -\frac{\pi}{2}, \frac{\pi}{2}$$

Question 10

Let

$$f(x) = \frac{x^2 e^x \sin(x)}{x^2 + 1}$$

- (a) Differentiate $f(x)$ using appropriate differentiation rules.
- (b) Simplify and factor your answer as much as possible.

Question 10 - Solution

We are given:

$$f(x) = \frac{x^2 e^x \sin(x)}{x^2 + 1}$$

This is a quotient of two functions:

$$u(x) = x^2 e^x \sin(x), \quad v(x) = x^2 + 1$$

We apply the **quotient rule**:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Step 1: Differentiate the numerator

The numerator is a product of three functions:

$$u(x) = x^2 \cdot e^x \cdot \sin(x)$$

Using the product rule for three functions:

$$u'(x) = (x^2)' e^x \sin(x) + x^2 (e^x)' \sin(x) + x^2 e^x (\sin(x))'$$

Compute derivatives:

$$(x^2)' = 2x, \quad (e^x)' = e^x, \quad (\sin x)' = \cos x$$

Substitute:

$$u'(x) = 2x e^x \sin(x) + x^2 e^x \sin(x) + x^2 e^x \cos(x)$$

Factor:

$$u'(x) = e^x [\sin(x)(2x + x^2) + x^2 \cos(x)]$$

Step 2: Differentiate the denominator

$$v'(x) = 2x$$

Step 3: Apply the quotient rule

$$f'(x) = \frac{e^x [\sin(x)(2x + x^2) + x^2 \cos(x)] (x^2 + 1) - x^2 e^x \sin(x)(2x)}{(x^2 + 1)^2}$$

Step 4: Simplify

Factor out e^x :

$$f'(x) = \frac{e^x \{ (x^2 + 1) [\sin(x)(2x + x^2) + x^2 \cos(x)] - 2x^3 \sin(x) \}}{(x^2 + 1)^2}$$

This is a fully factored and simplified form.

Final Answer:

$$f'(x) = \frac{e^x [(x^2 + 1) (\sin(x)(2x + x^2) + x^2 \cos(x)) - 2x^3 \sin(x)]}{(x^2 + 1)^2}$$

Question 1

Let

$$f(x) = \sin(x) \cos(x) + \tan(x)$$

- (a) Find $f'(x)$, the derivative of $f(x)$, using standard derivative rules.
- (b) Simplify the result as much as possible.

Question 1 - Solution

We are given:

$$f(x) = \sin(x) \cos(x) + \tan(x)$$

(a) Differentiate each term:

We apply the product rule to the first term $\sin(x) \cos(x)$, and recall the derivative of $\tan(x)$ is $\sec^2(x)$:

$$\frac{d}{dx} [\sin(x) \cos(x)] = \cos(x) \cos(x) + \sin(x)(-\sin(x)) = \cos^2(x) - \sin^2(x)$$

So the derivative of the entire function is:

$$f'(x) = \cos^2(x) - \sin^2(x) + \sec^2(x)$$

(b) Simplify the result:

Note that:

$$\cos^2(x) - \sin^2(x) = \cos(2x)$$

Thus:

$$\boxed{f'(x) = \cos(2x) + \sec^2(x)}$$

Question 2

Let

$$f(x) = x^2 \sec(x) + x \cot(x)$$

- (a) Compute the derivative $f'(x)$.
- (b) Simplify your result as much as possible.

Question 2 - Solution

We are given:

$$f(x) = x^2 \sec(x) + x \cot(x)$$

(a) Differentiate each term:

We apply the product rule to both terms:

First Term: $x^2 \sec(x)$

Let $u = x^2$, $v = \sec(x)$, then:

$$\frac{d}{dx}[x^2 \sec(x)] = \frac{d}{dx}[u \cdot v] = u'v + uv' = 2x \sec(x) + x^2 \sec(x) \tan(x)$$

Second Term: $x \cot(x)$

Let $u = x$, $v = \cot(x)$, then:

$$\frac{d}{dx}[x \cot(x)] = u'v + uv' = 1 \cdot \cot(x) + x(-\csc^2(x)) = \cot(x) - x \csc^2(x)$$

(b) Combine all terms:

$$f'(x) = 2x \sec(x) + x^2 \sec(x) \tan(x) + \cot(x) - x \csc^2(x)$$

Final Answer:

$f'(x) = 2x \sec(x) + x^2 \sec(x) \tan(x) + \cot(x) - x \csc^2(x)$
--

Question 3

Let

$$f(x) = \sin(x) \cos(x) + \tan(x) \sec(x)$$

- (a) Find $f'(x)$.
- (b) Simplify the expression.

Question 3 - Solution

We are given:

$$f(x) = \sin(x) \cos(x) + \tan(x) \sec(x)$$

(a) Differentiate each term:

Term 1: $\sin(x) \cos(x)$

Using the product rule:

$$\frac{d}{dx}[\sin(x) \cos(x)] = \cos(x) \cos(x) + \sin(x)(-\sin(x)) = \cos^2(x) - \sin^2(x)$$

Term 2: $\tan(x) \sec(x)$

Using the product rule:

$$\frac{d}{dx}[\tan(x) \sec(x)] = \sec^2(x) \sec(x) + \tan(x) \sec(x) \tan(x) = \sec^3(x) + \sec(x) \tan^2(x)$$

(b) Combine and simplify:

$$f'(x) = \cos^2(x) - \sin^2(x) + \sec^3(x) + \sec(x) \tan^2(x)$$

Use trigonometric identities:

$$\cos^2(x) - \sin^2(x) = \cos(2x)$$

$$\tan^2(x) = \sec^2(x) - 1$$

$$\sec(x) \tan^2(x) = \sec^3(x) - \sec(x)$$

Substitute and combine like terms:

$$f'(x) = \cos(2x) + 2 \sec^3(x) - \sec(x)$$

Final Answer:

$f'(x) = \cos(2x) + 2 \sec^3(x) - \sec(x)$
--

Question 4

Let

$$f(x) = \cot(x) \csc(x) + \cos^2(x)$$

- (a) Find the derivative $f'(x)$.
- (b) Simplify the expression as much as possible.

Question 4 - Solution

We are given:

$$f(x) = \cot(x) \csc(x) + \cos^2(x)$$

(a) Differentiate each term separately.

Term 1: $\cot(x) \csc(x)$

Using the product rule:

$$\begin{aligned}\frac{d}{dx}[\cot(x) \csc(x)] &= \cot(x)(-\csc(x)) + \csc(x)(-\csc^2(x)) \\ &= -\cot^2(x) \csc(x) - \csc^3(x)\end{aligned}$$

Term 2: $\cos^2(x)$

Using the chain rule:

$$\frac{d}{dx}[\cos^2(x)] = 2 \cos(x)(-\sin(x)) = -2 \sin(x) \cos(x)$$

(b) Combine and simplify:

$$f'(x) = -\cot^2(x) \csc(x) - \csc^3(x) - 2 \sin(x) \cos(x)$$

Use the identities:

$$\cot^2(x) = \csc^2(x) - 1 \quad 2 \sin(x) \cos(x) = \sin(2x)$$

$$-\cot^2(x) \csc(x) = -(\csc^2(x) - 1) \csc(x) = -\csc^3(x) + \csc(x)$$

Substitute and combine like terms:

$$f'(x) = \csc(x) - 2 \csc^3(x) - \sin(2x)$$

Final Answer:

$$\boxed{f'(x) = \csc(x) - 2 \csc^3(x) - \sin(2x)}$$

Question 5

Let

$$f(x) = x \tan(x) - \sec(x)$$

- (a) Find the derivative $f'(x)$.
- (b) Identify any values of x where $f'(x)$ is undefined.

Question 5 - Solution

We are given:

$$f(x) = x \tan(x) - \sec(x)$$

(a) Differentiate each term.

Term 1: $x \tan(x)$ Use the product rule:

$$\frac{d}{dx}[x \tan(x)] = x \cdot \sec^2(x) + \tan(x)$$

Term 2: $-\sec(x)$

$$\frac{d}{dx}[-\sec(x)] = -\sec(x) \tan(x)$$

Combine the terms:

$$f'(x) = x \sec^2(x) + \tan(x) - \sec(x) \tan(x)$$

Final Answer:

$$\boxed{f'(x) = x \sec^2(x) + \tan(x) - \sec(x) \tan(x)}$$

(b) Values where $f'(x)$ is undefined:

Note that $\tan(x)$ and $\sec(x)$ are undefined at:

$$x = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}$$

So, the derivative $f'(x)$ is undefined at:

$$\boxed{x = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}}$$

Question 6

Let

$$f(x) = \frac{\sin(x)}{1 + \cos(x)}$$

- (a) Find the derivative $f'(x)$.
- (b) Simplify the derivative as much as possible.

Question 6 - Solution

We are given:

$$f(x) = \frac{\sin(x)}{1 + \cos(x)}$$

(a) Use the Quotient Rule:

The quotient rule is:

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$$

Let:

$$g(x) = \sin(x), \quad h(x) = 1 + \cos(x)$$

$$g'(x) = \cos(x), \quad h'(x) = -\sin(x)$$

Then:

$$\begin{aligned} f'(x) &= \frac{\cos(x)(1 + \cos(x)) - \sin(x)(-\sin(x))}{(1 + \cos(x))^2} \\ &= \frac{\cos(x)(1 + \cos(x)) + \sin^2(x)}{(1 + \cos(x))^2} \end{aligned}$$

(b) Simplify the numerator:

First expand:

$$\cos(x)(1 + \cos(x)) = \cos(x) + \cos^2(x)$$

So the full numerator is:

$$\cos(x) + \cos^2(x) + \sin^2(x)$$

Use the identity:

$$\cos^2(x) + \sin^2(x) = 1$$

So the numerator becomes:

$$\cos(x) + 1$$

Therefore,

$$f'(x) = \frac{\cos(x) + 1}{(1 + \cos(x))^2}$$

Simplify:

$$f'(x) = \frac{1}{1 + \cos(x)}$$

Final Answer:

$$\boxed{f'(x) = \frac{1}{1 + \cos(x)}}$$

Question 7

Let

$$f(x) = \tan(x) \cdot \sec(x)$$

- (a) Find the derivative $f'(x)$.
- (b) Simplify the result using trigonometric identities.

Question 7 - Solution

We are given:

$$f(x) = \tan(x) \cdot \sec(x)$$

(a) Apply the Product Rule:

The product rule states:

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$

Let:

$$u(x) = \tan(x), \quad v(x) = \sec(x)$$

$$u'(x) = \sec^2(x), \quad v'(x) = \sec(x) \tan(x)$$

Then:

$$\begin{aligned} f'(x) &= \sec^2(x) \cdot \sec(x) + \tan(x) \cdot \sec(x) \cdot \tan(x) \\ &= \sec^3(x) + \sec(x) \tan^2(x) \end{aligned}$$

(b) Factor the result:

Both terms have a factor of $\sec(x)$, so factor it out:

$$f'(x) = \sec(x) (\sec^2(x) + \tan^2(x))$$

Final Answer:

$$\boxed{f'(x) = \sec(x) (\sec^2(x) + \tan^2(x))}$$

Question 8

Let

$$f(x) = \frac{\sin(x) \cos(x)}{1 + \sin^2(x)}$$

- (a) Differentiate $f(x)$.
- (b) Simplify the result as much as possible.

Question 8 - Solution

We are given:

$$f(x) = \frac{\sin(x) \cos(x)}{1 + \sin^2(x)}$$

Step 1: Rewrite using a double-angle identity.

Recall:

$$\sin(x) \cos(x) = \frac{1}{2} \sin(2x)$$

So we rewrite the function:

$$f(x) = \frac{\frac{1}{2} \sin(2x)}{1 + \sin^2(x)}$$

Step 2: Apply the quotient rule.

Let:

$$u(x) = \frac{1}{2} \sin(2x), \quad v(x) = 1 + \sin^2(x)$$

Differentiate each function:

$$u'(x) = \cos(2x)$$

$$v'(x) = 2 \sin(x) \cos(x) = \sin(2x)$$

The quotient rule states:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Substitute:

$$f'(x) = \frac{\cos(2x)(1 + \sin^2(x)) - \frac{1}{2} \sin(2x) \sin(2x)}{(1 + \sin^2(x))^2}$$

Step 3: Simplify carefully.

Rewrite the squared sine:

$$\sin^2(2x) = 1 - \cos^2(2x)$$

So the numerator becomes:

$$\cos(2x)(1 + \sin^2(x)) - \frac{1}{2}(1 - \cos^2(2x))$$

Expand:

$$= \cos(2x) + \cos(2x) \sin^2(x) - \frac{1}{2} + \frac{1}{2} \cos^2(2x)$$

Now use:

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

Substitute:

$$\cos(2x) \sin^2(x) = \frac{1}{2} \cos(2x) - \frac{1}{2} \cos^2(2x)$$

Combine terms:

$$= \frac{3}{2} \cos(2x) - \frac{1}{2}$$

Final Simplified Result:

$$f'(x) = \frac{3 \cos(2x) - 1}{2(1 + \sin^2(x))^2}$$

Final Answer:

$$f'(x) = \frac{3 \cos(2x) - 1}{2(1 + \sin^2(x))^2}$$

Question 9

Let

$$f(x) = x \tan(x)$$

- (a) Use the product rule to find the derivative of $f(x)$.
- (b) Find the equation of the tangent line to $f(x)$ at $x = \frac{\pi}{4}$.

Question 9 - Solution

We are given:

$$f(x) = x \tan(x)$$

(a) Differentiate Using the Product Rule:

Let:

$$u(x) = x, \quad v(x) = \tan(x)$$

Then:

$$u'(x) = 1, \quad v'(x) = \sec^2(x)$$

Product rule:

$$f'(x) = u'(x)v(x) + u(x)v'(x) = 1 \cdot \tan(x) + x \cdot \sec^2(x) = \tan(x) + x \sec^2(x)$$

$$f'(x) = \tan(x) + x \sec^2(x)$$

(b) Tangent Line at $x = \frac{\pi}{4}$:

Evaluate:

$$f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} \cdot \tan\left(\frac{\pi}{4}\right) = \frac{\pi}{4} \cdot 1 = \frac{\pi}{4}$$

$$f'\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) + \frac{\pi}{4} \cdot \sec^2\left(\frac{\pi}{4}\right) = 1 + \frac{\pi}{4} \cdot 2 = 1 + \frac{\pi}{2}$$

Use point-slope form:

$$y - f\left(\frac{\pi}{4}\right) = f'\left(\frac{\pi}{4}\right) \left(x - \frac{\pi}{4}\right) \Rightarrow y - \frac{\pi}{4} = \left(1 + \frac{\pi}{2}\right) \left(x - \frac{\pi}{4}\right)$$

Final Answer: The tangent line is:

$$y = \left(1 + \frac{\pi}{2}\right) \left(x - \frac{\pi}{4}\right) + \frac{\pi}{4}$$

Question 10

Consider the function:

$$f(x) = \sin(x) \cos(x)$$

- (a) Differentiate $f(x)$ using the product rule.
- (b) Simplify the derivative using trigonometric identities.
- (c) Evaluate $f'(x)$ at $x = \frac{\pi}{3}$.

Question 10 - Solution

We are given:

$$f(x) = \sin(x) \cos(x)$$

(a) Differentiate Using the Product Rule:

Let:

$$u(x) = \sin(x), \quad v(x) = \cos(x)$$

Then:

$$u'(x) = \cos(x), \quad v'(x) = -\sin(x)$$

Using the product rule:

$$f'(x) = u'(x)v(x) + u(x)v'(x) = \cos(x)\cos(x) + \sin(x)(-\sin(x)) = \cos^2(x) - \sin^2(x)$$

$$\boxed{f'(x) = \cos^2(x) - \sin^2(x)}$$

(b) Simplify Using Trig Identity:

We use the identity:

$$\cos^2(x) - \sin^2(x) = \cos(2x)$$

So:

$$\boxed{f'(x) = \cos(2x)}$$

(c) Evaluate at $x = \frac{\pi}{3}$:

$$f'\left(\frac{\pi}{3}\right) = \cos\left(2 \cdot \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$$\boxed{f'\left(\frac{\pi}{3}\right) = -\frac{1}{2}}$$

Question 1

Let

$$f(x) = \ln(x^2 + 1) \cdot e^{3x}$$

- (a) Differentiate $f(x)$ using the product rule and the chain rule.
- (b) Simplify your answer.

Question 1 - Solution

We are given:

$$f(x) = \ln(x^2 + 1) \cdot e^{3x}$$

(a) Differentiate Using Product Rule:

Let:

$$u(x) = \ln(x^2 + 1), \quad v(x) = e^{3x}$$

Then:

$$u'(x) = \frac{1}{x^2 + 1} \cdot 2x = \frac{2x}{x^2 + 1}, \quad v'(x) = 3e^{3x}$$

Now apply the product rule:

$$f'(x) = u'(x)v(x) + u(x)v'(x) = \frac{2x}{x^2 + 1} \cdot e^{3x} + \ln(x^2 + 1) \cdot 3e^{3x}$$

(b) Final Simplified Answer:

Factor out e^{3x} :

$$f'(x) = e^{3x} \left(\frac{2x}{x^2 + 1} + 3 \ln(x^2 + 1) \right)$$

$$f'(x) = e^{3x} \left(\frac{2x}{x^2 + 1} + 3 \ln(x^2 + 1) \right)$$

Question 2

Consider the function:

$$f(x) = \frac{e^x}{\ln(x)}$$

- (a) Find the derivative $f'(x)$.
- (b) State the domain of $f(x)$ and $f'(x)$.

Question 2 - Solution

We are given:

$$f(x) = \frac{e^x}{\ln(x)}$$

(a) Differentiate $f(x)$:

This is a quotient, so we apply the quotient rule:

$$f'(x) = \frac{(\ln(x) \cdot \frac{d}{dx}[e^x]) - (e^x \cdot \frac{d}{dx}[\ln(x)])}{(\ln(x))^2}$$

Differentiate:

$$\frac{d}{dx}[e^x] = e^x, \quad \frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

Now substitute:

$$f'(x) = \frac{\ln(x) \cdot e^x - e^x \cdot \frac{1}{x}}{(\ln(x))^2} = \frac{e^x (\ln(x) - \frac{1}{x})}{(\ln(x))^2}$$

$$f'(x) = \frac{e^x (\ln(x) - \frac{1}{x})}{(\ln(x))^2}$$

(b) Domain of $f(x)$:

$\ln(x)$ is only defined for $x > 0$

But we also need $\ln(x) \neq 0 \Rightarrow x \neq 1$

So domain of $f(x)$ is:

$$(0, 1) \cup (1, \infty)$$

The derivative involves the same terms (no new restrictions), so:

$$\text{Domain of } f'(x) = \boxed{(0, 1) \cup (1, \infty)}$$

Question 3

Let $f(x) = x^x$ for $x > 0$.

- (a) Use logarithmic differentiation to find $f'(x)$.
- (b) Evaluate $f'(x)$ at $x = 2$.

Question 3 - Solution

We are given:

$$f(x) = x^x, \quad x > 0$$

(a) Logarithmic Differentiation

Take natural log on both sides:

$$\ln(f(x)) = \ln(x^x) = x \ln(x)$$

Differentiate both sides implicitly:

$$\frac{1}{f(x)} \cdot f'(x) = \frac{d}{dx}[x \ln(x)]$$

Differentiate the right-hand side using product rule:

$$\frac{d}{dx}[x \ln(x)] = \ln(x) + 1$$

So:

$$\frac{1}{f(x)} \cdot f'(x) = \ln(x) + 1 \Rightarrow f'(x) = f(x) \cdot (\ln(x) + 1)$$

Recall $f(x) = x^x$, so:

$$\boxed{f'(x) = x^x(\ln(x) + 1)}$$

(b) Evaluate at $x = 2$:

$$f'(2) = 2^2(\ln(2) + 1) = 4(\ln(2) + 1)$$

Exact Answer:

$$\boxed{f'(2) = 4(\ln(2) + 1)}$$

Question 4

Let the function $f(x) = e^{x \ln(\sin x)}$, for $0 < x < \pi$.

- (a) Simplify the expression $f(x)$.
- (b) Find the derivative $f'(x)$.

Question 4 - Solution

We are given:

$$f(x) = e^{x \ln(\sin x)}, \quad 0 < x < \pi$$

(a) Simplify the expression:

Recall the identity:

$$e^{x \ln(\sin x)} = (e^{\ln(\sin x)})^x = (\sin x)^x$$

So:

$$f(x) = (\sin x)^x$$

(b) Differentiate $f(x) = (\sin x)^x$ using logarithmic differentiation:

Take natural log on both sides:

$$\ln(f(x)) = x \ln(\sin x)$$

Differentiate both sides:

$$\frac{1}{f(x)} \cdot f'(x) = \frac{d}{dx}[x \ln(\sin x)]$$

Apply product rule on the right-hand side:

$$\frac{d}{dx}[x \ln(\sin x)] = \ln(\sin x) + x \cdot \frac{\cos x}{\sin x} = \ln(\sin x) + x \cot x$$

Multiply both sides by $f(x) = (\sin x)^x$:

$$f'(x) = (\sin x)^x \cdot (\ln(\sin x) + x \cot x)$$

Final Answer:

$$\boxed{f'(x) = (\sin x)^x (\ln(\sin x) + x \cot x)}$$

Question 5

Let the function $f(x) = \ln\left(\frac{x^2\sqrt{e^x+1}}{\ln x}\right)$, for $x > 1$.

- (a) Simplify the expression using logarithmic properties.
- (b) Find the derivative $f'(x)$.

Question 5 - Solution

We are given:

$$f(x) = \ln \left(\frac{x^2 \sqrt{e^x + 1}}{\ln x} \right), \quad x > 1$$

(a) Simplify the expression:

Use logarithmic identities:

$$\ln \left(\frac{A \cdot B}{C} \right) = \ln(A) + \ln(B) - \ln(C)$$

Apply to this expression:

$$f(x) = \ln(x^2) + \ln \left((e^x + 1)^{1/2} \right) - \ln(\ln x)$$

Now simplify powers:

$$f(x) = 2 \ln x + \frac{1}{2} \ln(e^x + 1) - \ln(\ln x)$$

(b) Differentiate $f(x)$:

Differentiate term-by-term:

$$\begin{aligned} \frac{d}{dx} [2 \ln x] &= \frac{2}{x} \\ \frac{d}{dx} \left[\frac{1}{2} \ln(e^x + 1) \right] &= \frac{1}{2} \cdot \frac{e^x}{e^x + 1} \\ \frac{d}{dx} [\ln(\ln x)] &= \frac{1}{x \ln x} \end{aligned}$$

Putting it all together:

$$f'(x) = \frac{2}{x} + \frac{1}{2} \cdot \frac{e^x}{e^x + 1} - \frac{1}{x \ln x}$$

Final Answer:

$f'(x) = \frac{2}{x} + \frac{e^x}{2(e^x + 1)} - \frac{1}{x \ln x}$
--

Question 6

Let $f(x) = x^x$, defined for $x > 0$.

- (a) Find the derivative $f'(x)$.
- (b) Evaluate $f'(1)$.

Question 6 - Solution

We are given:

$$f(x) = x^x, \quad x > 0$$

This is a function where the base and exponent both depend on x . To differentiate it, we take logarithms on both sides.

Step 1: Take natural log of both sides:

Let $y = x^x$. Then,

$$\ln y = \ln(x^x) = x \ln x$$

Step 2: Differentiate both sides implicitly:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(x \ln x) = \ln x + 1$$

Step 3: Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y(\ln x + 1) = x^x(\ln x + 1)$$

Final answer:

$$\boxed{f'(x) = x^x(\ln x + 1)}$$

(b) Evaluate $f'(1)$:

$$f'(1) = 1^1(\ln 1 + 1) = 1(0 + 1) = \boxed{1}$$

Question 7

Let $f(x) = \ln\left(\frac{e^x+1}{e^x-1}\right)$, for $x > 0$.

- (a) Find $f'(x)$.
- (b) Simplify your answer as much as possible.

Question 7 - Solution

We are given:

$$f(x) = \ln \left(\frac{e^x + 1}{e^x - 1} \right)$$

Step 1: Use the derivative of the natural log of a quotient:

$$\frac{d}{dx} \ln \left(\frac{u(x)}{v(x)} \right) = \frac{d}{dx} [\ln(u(x)) - \ln(v(x))] = \frac{u'(x)}{u(x)} - \frac{v'(x)}{v(x)}$$

Here, $u(x) = e^x + 1$, and $v(x) = e^x - 1$

So,

$$f'(x) = \frac{d}{dx} [\ln(e^x + 1) - \ln(e^x - 1)] = \frac{e^x}{e^x + 1} - \frac{e^x}{e^x - 1}$$

Step 2: Combine the two terms:

$$f'(x) = \frac{e^x}{e^x + 1} - \frac{e^x}{e^x - 1}$$

Get a common denominator:

$$f'(x) = e^x \left(\frac{1}{e^x + 1} - \frac{1}{e^x - 1} \right) = e^x \left(\frac{(e^x - 1) - (e^x + 1)}{(e^x + 1)(e^x - 1)} \right) = e^x \left(\frac{e^x - 1 - e^x - 1}{e^{2x} - 1} \right)$$

$$f'(x) = e^x \left(\frac{-2}{e^{2x} - 1} \right) = \boxed{\frac{-2e^x}{e^{2x} - 1}}$$

Question 8

Let $f(x) = x^x$, where $x > 0$.

- (a) Find the derivative $f'(x)$.
- (b) Simplify your answer as much as possible.

Question 8 - Solution

We are given:

$$f(x) = x^x \quad \text{for } x > 0$$

This is a function where the variable is both the base and the exponent. To differentiate it, we take the natural logarithm of both sides:

Let

$$y = x^x$$

Take the natural log of both sides:

$$\ln y = \ln(x^x) = x \ln x$$

Differentiate both sides implicitly:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(x \ln x)$$

Now differentiate the right-hand side using the product rule:

$$\frac{d}{dx}(x \ln x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

Now solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y(\ln x + 1)$$

Recall that $y = x^x$, so:

$$f'(x) = \boxed{x^x(\ln x + 1)}$$

Question 9

Let $a > 0$ be a real constant and define the function

$$f(x) = \ln\left(\frac{e^{ax} + 1}{e^{ax} - 1}\right).$$

- (a) Find the derivative $f'(x)$ in terms of a .
- (b) Simplify your answer as much as possible.
- (c) Determine the domain of $f(x)$.
- (d) Show that $f'(x)$ can be written in the form

$$f'(x) = -\frac{2a}{e^{ax} - e^{-ax}}.$$

- (e) Hence, or otherwise, determine whether $f(x)$ is increasing or decreasing on its domain.

Question 9 - Solution

We are given

$$f(x) = \ln\left(\frac{e^{ax} + 1}{e^{ax} - 1}\right), \quad a > 0.$$

(a) Differentiation

Rewrite the function using logarithmic identities:

$$f(x) = \ln(e^{ax} + 1) - \ln(e^{ax} - 1).$$

Differentiate term by term using the chain rule:

$$f'(x) = \frac{ae^{ax}}{e^{ax} + 1} - \frac{ae^{ax}}{e^{ax} - 1}.$$

(b) Simplification

Factor out ae^{ax} :

$$f'(x) = ae^{ax} \left(\frac{1}{e^{ax} + 1} - \frac{1}{e^{ax} - 1} \right).$$

Combine the fractions:

$$= ae^{ax} \left(\frac{(e^{ax} - 1) - (e^{ax} + 1)}{(e^{ax} + 1)(e^{ax} - 1)} \right) = ae^{ax} \left(\frac{-2}{e^{2ax} - 1} \right).$$

Thus,

$$f'(x) = -\frac{2ae^{ax}}{e^{2ax} - 1}.$$

(c) Domain of $f(x)$

The logarithm requires:

$$\frac{e^{ax} + 1}{e^{ax} - 1} > 0, \quad e^{ax} - 1 \neq 0.$$

Since $e^{ax} > 0$ for all x , the expression is positive only when

$$e^{ax} - 1 > 0 \quad \Rightarrow \quad x > 0.$$

Hence, the domain is:

$$\boxed{(0, \infty)}.$$

(d) Alternative Form

Multiply numerator and denominator of the derivative by e^{-ax} :

$$f'(x) = -\frac{2ae^{ax}}{e^{2ax} - 1} = -\frac{2a}{e^{ax} - e^{-ax}}.$$

(e) Monotonicity

For $x > 0$,

$$e^{ax} - e^{-ax} > 0, \quad a > 0.$$

Therefore,

$$f'(x) < 0 \quad \text{for all } x > 0.$$

The function $f(x)$ is strictly decreasing on its domain.

Question 10

Let $a > 0$ be a real constant and define

$$f(x) = x^{ax}, \quad x > 0.$$

- (a) Use logarithmic differentiation to find $f'(x)$.
- (b) Simplify your answer as much as possible.
- (c) Find the second derivative $f''(x)$.
- (d) Determine all stationary points of $f(x)$.
- (e) Hence, determine the intervals on which $f(x)$ is increasing and decreasing.

Question 10 - Solution

We are given

$$f(x) = x^{ax}, \quad x > 0, \quad a > 0.$$

(a) Logarithmic Differentiation

Let

$$y = x^{ax}.$$

Taking natural logarithms:

$$\ln y = ax \ln x.$$

(b) First Derivative

Differentiate both sides with respect to x :

$$\frac{1}{y} \frac{dy}{dx} = a \ln x + a.$$

Multiply through by y :

$$\frac{dy}{dx} = y(a \ln x + a).$$

Substitute back $y = x^{ax}$:

$$f'(x) = x^{ax} a(\ln x + 1).$$

$$\boxed{f'(x) = ax^{ax}(\ln x + 1)}$$

(c) Second Derivative

Differentiate $f'(x)$ using the product rule:

$$f'(x) = ax^{ax}(\ln x + 1).$$

Let

$$u = x^{ax}, \quad v = \ln x + 1.$$

Then:

$$u' = ax^{ax}(\ln x + 1), \quad v' = \frac{1}{x}.$$

Thus,

$$f''(x) = a[u'v + uv'] = a \left[x^{ax} a(\ln x + 1)^2 + x^{ax} \frac{1}{x} \right].$$

Factor out x^{ax} :

$$f''(x) = ax^{ax} \left[a(\ln x + 1)^2 + \frac{1}{x} \right].$$

(d) Stationary Points

Stationary points occur when

$$f'(x) = 0.$$

Since $a > 0$ and $x^{ax} > 0$ for $x > 0$, we require:

$$\ln x + 1 = 0.$$

Solving:

$$\ln x = -1 \quad \Rightarrow \quad x = e^{-1}.$$

Thus, the only stationary point occurs at:

$$\boxed{x = \frac{1}{e}}.$$

(e) Increasing and Decreasing Behaviour

From

$$f'(x) = ax^{ax}(\ln x + 1),$$

the sign of $f'(x)$ depends on $\ln x + 1$.

- If $0 < x < \frac{1}{e}$, then $\ln x + 1 < 0 \Rightarrow f'(x) < 0$.
- If $x > \frac{1}{e}$, then $\ln x + 1 > 0 \Rightarrow f'(x) > 0$.

Decreasing on $(0, e^{-1})$, Increasing on (e^{-1}, ∞) .

Question 1

Differentiate the function:

$$f(x) = x \cdot \arcsin(x)$$

Then evaluate $f'(x)$ at $x = \frac{1}{2}$.

Question 1 - Solution

We are given:

$$f(x) = x \cdot \arcsin(x)$$

Use the product rule:

$$f'(x) = \arcsin(x) + x \cdot \frac{1}{\sqrt{1-x^2}}$$

Answer:

$$f'(x) = \boxed{\arcsin(x) + \frac{x}{\sqrt{1-x^2}}}$$

Now evaluate at $x = \frac{1}{2}$:

$$f'\left(\frac{1}{2}\right) = \arcsin\left(\frac{1}{2}\right) + \frac{\frac{1}{2}}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = \frac{\pi}{6} + \frac{1/2}{\sqrt{3}/2} = \frac{\pi}{6} + \frac{1}{\sqrt{3}}$$

Answer: $f'\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{6} + \frac{1}{\sqrt{3}}}$

Question 2

Let

$$f(x) = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$$

- (a) Show that $f(x) = 2 \tan^{-1}(x)$ for all $x \in (-1, 1)$
- (b) Use the identity in part (a) to find $f'(x)$.

Question 2 - Solution

(a) Use an identity for tangent of double angle:

We recall:

$$\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

Let $\theta = \tan^{-1}(x) \Rightarrow \tan(\theta) = x$. Then:

$$\tan(2\theta) = \frac{2x}{1 - x^2} \Rightarrow 2\theta = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$$

So:

$$f(x) = \tan^{-1}\left(\frac{2x}{1 - x^2}\right) = 2 \tan^{-1}(x) \quad \text{for } x \in (-1, 1)$$

(b) Differentiate using identity:

From (a), $f(x) = 2 \tan^{-1}(x)$

Use the derivative:

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1 + x^2} \Rightarrow f'(x) = 2 \cdot \frac{1}{1 + x^2} = \boxed{\frac{2}{1 + x^2}}$$

Question 3

Let

$$f(x) = \arcsin\left(\frac{2x}{1+x^2}\right), \quad x > 0.$$

- (a) Express f in terms of $\arctan x$, separately for $0 < x \leq 1$ and $x > 1$.
- (b) Find $f'(x)$ wherever it exists, and analyze $x = 1$.

Question 3 - Solution

Put $t = \arctan x$. For $x > 0$, $0 < t < \pi/2$ and

$$\frac{2x}{1+x^2} = \sin(2t).$$

The principal range of $\arcsin$ is $[-\pi/2, \pi/2]$. Therefore

$$f(x) = \begin{cases} 2 \arctan x, & 0 < x \leq 1, \\ \pi - 2 \arctan x, & x > 1. \end{cases}$$

Differentiating on the two open intervals gives

$$f'(x) = \begin{cases} 2/(1+x^2), & 0 < x < 1, \\ -2/(1+x^2), & x > 1. \end{cases}$$

At $x = 1$ the left and right derivatives are 1 and -1 , respectively, so no derivative exists there.

Question 4

Let

$$f(x) = \tan^{-1}(\sqrt{1-x^2})$$

- (a) Determine the domain of $f(x)$.
- (b) Compute $f'(x)$ for $x \in (-1, 1)$.

Question 4 - Solution**(a) Domain:**

Since $f(x) = \tan^{-1}(\sqrt{1-x^2})$, we require the expression under the square root to be non-negative:

$$1 - x^2 \geq 0 \Rightarrow x^2 \leq 1 \Rightarrow -1 \leq x \leq 1$$

But the square root function $\sqrt{1-x^2}$ is only real for $x \in [-1, 1]$, and since it appears inside an inverse tangent (which is defined for all real inputs), the domain of f is:

$$\boxed{[-1, 1]}$$

(b) Compute the derivative:

Let's differentiate using the chain rule:

$$f(x) = \tan^{-1}(u), \quad \text{where } u = \sqrt{1-x^2} \Rightarrow f'(x) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

Now compute $\frac{du}{dx}$ where $u = \sqrt{1-x^2}$:

$$\frac{du}{dx} = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

Next, compute $1+u^2 = 1+(1-x^2) = 2-x^2$

So:

$$f'(x) = \frac{1}{2-x^2} \cdot \left(\frac{-x}{\sqrt{1-x^2}} \right) = \boxed{\frac{-x}{(2-x^2)\sqrt{1-x^2}}}$$

Question 5

Let

$$f(x) = \sec^{-1} \left(\frac{2x + 3}{x - 1} \right)$$

- (a) Determine the domain of $f(x)$.
- (b) Compute $f'(x)$.

Question 5 - Solution

(a) Domain:

The inverse secant function is defined for $|u| \geq 1$. Thus, we require

$$\left| \frac{2x+3}{x-1} \right| \geq 1$$

Squaring both sides:

$$\frac{(2x+3)^2}{(x-1)^2} \geq 1 \quad \Rightarrow \quad (2x+3)^2 \geq (x-1)^2$$

Expanding:

$$\begin{aligned} 4x^2 + 12x + 9 &\geq x^2 - 2x + 1 \\ 3x^2 + 14x + 8 &\geq 0 \end{aligned}$$

Factor:

$$(3x+2)(x+4) \geq 0$$

Critical points are $x = -4$ and $x = -\frac{2}{3}$. Since the quadratic opens upward, the inequality holds outside the interval between the roots:

$$x \leq -4 \quad \text{or} \quad x \geq -\frac{2}{3}$$

We must also exclude $x = 1$, where the denominator is zero.

Therefore, the domain of $f(x)$ is:

$$\boxed{(-\infty, -4] \cup [-\frac{2}{3}, 1) \cup (1, \infty)}$$

(b) Derivative:

Using the derivative formula:

$$\frac{d}{dx} (\sec^{-1}(u)) = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

Let

$$u = \frac{2x+3}{x-1}$$

Then

$$\frac{du}{dx} = \frac{(x-1)(2) - (2x+3)(1)}{(x-1)^2} = \frac{-5}{(x-1)^2}$$

Substitute into the formula:

$$f'(x) = \frac{1}{\left| \frac{2x+3}{x-1} \right| \sqrt{\left(\frac{2x+3}{x-1} \right)^2 - 1}} \cdot \frac{-5}{(x-1)^2}$$

$$\boxed{f'(x) = \frac{-5}{(x-1)^2 \left| \frac{2x+3}{x-1} \right| \sqrt{\left(\frac{2x+3}{x-1} \right)^2 - 1}}}$$

Question 6

Let

$$f(x) = \tan^{-1} \left(\sqrt{1 - x^2} \right)$$

- (a) Find the derivative $f'(x)$.
- (b) Determine the domain of $f'(x)$.

Question 6 - Solution

We are given:

$$f(x) = \tan^{-1}(\sqrt{1-x^2})$$

(a) Finding $f'(x)$:

Let $u(x) = \sqrt{1-x^2}$, so:

$$f(x) = \tan^{-1}(u) \Rightarrow f'(x) = \frac{1}{1+u^2} \cdot u'(x)$$

First, compute $u'(x)$:

$$u(x) = (1-x^2)^{1/2} \Rightarrow u'(x) = \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

Now compute:

$$f'(x) = \frac{1}{1+(1-x^2)} \cdot \frac{-x}{\sqrt{1-x^2}} = \frac{1}{2-x^2} \cdot \frac{-x}{\sqrt{1-x^2}}$$

Final Answer:

$$f'(x) = \frac{-x}{(2-x^2)\sqrt{1-x^2}}$$

(b) Domain of $f'(x)$:

We need: $\sqrt{1-x^2}$ to be real $1-x^2 \geq 0 \Rightarrow x \in [-1, 1]$, Denominator $\sqrt{1-x^2} \neq 0$ exclude $x = \pm 1$,
Also ensure $2-x^2 \neq 0 \Rightarrow x \neq \sqrt{2}, -\sqrt{2}$ (but these are already excluded)

So domain of $f'(x)$ is:

$$(-1, 1)$$

Question 7

Let

$$f(x) = \arcsin\left(\frac{2x}{\sqrt{1+x^2}+1}\right).$$

- (a) Determine the real domain.
- (b) Decide whether $f(x) = \arctan x$ holds on that domain. Justify your answer.
- (c) Compute $f'(x)$ wherever it exists.

Question 7 - Solution

Domain. Let $s = \sqrt{1+x^2} \geq 1$. The argument of arcsin must satisfy

$$\frac{4x^2}{(s+1)^2} = \frac{4(s-1)}{s+1} \leq 1.$$

This is equivalent to $s \leq 5/3$, hence

$$\boxed{D = [-4/3, 4/3]}.$$

Test the claimed identity. At $x = 4/3$, the arcsine is $\pi/2$, while $\arctan(4/3) < \pi/2$. Thus $f(x) = \arctan x$ is false.

Differentiate. For $|x| < 4/3$, put $u = 2x/(s+1)$. Then

$$u' = \frac{2}{s(s+1)}, \quad 1 - u^2 = \frac{5-3s}{s+1}.$$

Consequently

$$\boxed{f'(x) = \frac{2}{s\sqrt{s+1}\sqrt{5-3s}}}, \quad s = \sqrt{1+x^2}.$$

The derivative is defined on $(-4/3, 4/3)$ and grows without bound toward either endpoint.

Question 8

Let $f(x) = \tan^{-1}(\sqrt{1-x^2})$, where $-1 < x < 1$.

- (a) Compute $f'(x)$.
- (b) Simplify your result as much as possible.

Question 8 - Solution

We are given:

$$f(x) = \tan^{-1}(\sqrt{1-x^2})$$

(a) Differentiate using the chain rule:

Let $u = \sqrt{1-x^2}$, so:

$$f(x) = \tan^{-1}(u) \Rightarrow f'(x) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

Now compute each part:

$$u = (1-x^2)^{1/2}, \quad \frac{du}{dx} = \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

Now plug into the derivative:

$$f'(x) = \frac{1}{1+(1-x^2)} \cdot \frac{-x}{\sqrt{1-x^2}} = \frac{1}{2-x^2} \cdot \frac{-x}{\sqrt{1-x^2}}$$

$$f'(x) = \boxed{\frac{-x}{(2-x^2)\sqrt{1-x^2}}}$$

(b) Final simplified form:

$$f'(x) = \boxed{\frac{-x}{(2-x^2)\sqrt{1-x^2}}}$$

Question 9

Let $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, where $-1 < x < 1$.

- (a) Compute $f'(x)$.
- (b) Show that the result simplifies to a rational function.

Question 9 - Solution

We are given:

$$f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Let $u = \frac{2x}{1+x^2}$, so:

$$f'(x) = \frac{1}{\sqrt{1-u^2}} \cdot u'$$

Step 1: Differentiate $u = \frac{2x}{1+x^2}$:

Using the quotient rule:

$$u' = \frac{(1+x^2)(2) - 2x(2x)}{(1+x^2)^2} = \frac{2(1+x^2) - 4x^2}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}$$

Step 2: Simplify $\sqrt{1-u^2}$:

$$\begin{aligned} u = \frac{2x}{1+x^2} &\Rightarrow u^2 = \frac{4x^2}{(1+x^2)^2} \Rightarrow 1-u^2 = 1 - \frac{4x^2}{(1+x^2)^2} = \frac{(1+x^2)^2 - 4x^2}{(1+x^2)^2} \\ &= \frac{1+2x^2+x^4-4x^2}{(1+x^2)^2} = \frac{1-2x^2+x^4}{(1+x^2)^2} = \frac{(1-x^2)^2}{(1+x^2)^2} \Rightarrow \sqrt{1-u^2} = \frac{|1-x^2|}{1+x^2} \end{aligned}$$

Since $-1 < x < 1$, we have $1-x^2 > 0$, so:

$$\sqrt{1-u^2} = \frac{1-x^2}{1+x^2}$$

Step 3: Combine all parts:

$$f'(x) = \frac{1}{\sqrt{1-u^2}} \cdot u' = \frac{1}{\frac{1-x^2}{1+x^2}} \cdot \frac{2(1-x^2)}{(1+x^2)^2} = \frac{1+x^2}{1-x^2} \cdot \frac{2(1-x^2)}{(1+x^2)^2}$$

Cancel $1-x^2$ and one factor of $1+x^2$:

$$f'(x) = \frac{2}{1+x^2}$$

Final Answer:

$$\boxed{f'(x) = \frac{2}{1+x^2}}$$

Question 10

Let $f(x) = \tan^{-1}(\sqrt{1-x^2})$ for $-1 < x < 1$.

- (a) Find $f'(x)$.
- (b) Simplify your answer as much as possible.

Question 10 - Solution

We are given:

$$f(x) = \tan^{-1}(\sqrt{1-x^2})$$

Let $u = \sqrt{1-x^2}$, then by the chain rule:

$$f'(x) = \frac{1}{1+u^2} \cdot u'$$

Step 1: Compute u' :

$$u = (1-x^2)^{1/2} \Rightarrow u' = \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

Step 2: Compute $1+u^2$:

$$1+u^2 = 1+(1-x^2) = 2-x^2$$

Step 3: Combine everything:

$$f'(x) = \frac{1}{1+u^2} \cdot u' = \frac{1}{2-x^2} \cdot \frac{-x}{\sqrt{1-x^2}} = \boxed{\frac{-x}{(2-x^2)\sqrt{1-x^2}}}$$

Question 1

Differentiate the function:

$$f(x) = \sinh(x) \cdot \cosh(x)$$

Then express your answer in terms of a single hyperbolic function if possible.

Question 1 - Solution

We are given:

$$f(x) = \sinh(x) \cdot \cosh(x)$$

Use the product rule:

$$f'(x) = \cosh(x) \cdot \cosh(x) + \sinh(x) \cdot \sinh(x) = \cosh^2(x) + \sinh^2(x)$$

Recall the identity:

$$\cosh^2(x) - \sinh^2(x) = 1 \quad \Rightarrow \quad \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1$$

Thus:

$$f'(x) = \boxed{\cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1}$$

Answer: $f'(x) = \boxed{\cosh(2x)}$

Question 2

Let $f(x) = \frac{\sinh(3x)}{\cosh(x)}$.

- (a) Compute $f'(x)$ using differentiation rules.
- (b) Simplify the result as much as possible using hyperbolic identities.

Question 2 - Solution

The quotient rule gives

$$f'(x) = \frac{3 \cosh(3x) \cosh x - \sinh(3x) \sinh x}{\cosh^2 x}.$$

Use the product identities

$$\cosh(3x) \cosh x = \frac{1}{2}(\cosh 4x + \cosh 2x),$$

$$\sinh(3x) \sinh x = \frac{1}{2}(\cosh 4x - \cosh 2x).$$

Thus

$$f'(x) = \frac{\cosh 4x + 2 \cosh 2x}{\cosh^2 x}.$$

At zero the derivative is 3, agreeing with the original quotient-rule expression.

Question 3

Let $f(x) = \tanh(x) \sinh(x)$.

- (a) Compute $f'(x)$.
- (b) Simplify your result as much as possible using hyperbolic identities.

Question 3 - Solution

We are given:

$$f(x) = \tanh(x) \sinh(x)$$

This is a product of two functions, so we apply the product rule:

$$f'(x) = \tanh'(x) \sinh(x) + \tanh(x) \sinh'(x)$$

Recall the derivatives:

$$\tanh'(x) = \operatorname{sech}^2(x), \quad \sinh'(x) = \cosh(x)$$

Substitute:

$$f'(x) = \operatorname{sech}^2(x) \sinh(x) + \tanh(x) \cosh(x)$$

Now simplify. Recall:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \Rightarrow \tanh(x) \cosh(x) = \sinh(x)$$

So:

$$f'(x) = \operatorname{sech}^2(x) \sinh(x) + \sinh(x) = \sinh(x) (\operatorname{sech}^2(x) + 1)$$

Final Answer:

$$\boxed{f'(x) = \sinh(x) (\operatorname{sech}^2(x) + 1)}$$

Question 4

Let $f(x) = \frac{\cosh^2(x)-1}{\sinh(x)}$.

- (a) Simplify the expression before differentiating.
- (b) Compute $f'(x)$.
- (c) Identify any points where $f'(x)$ is undefined.

Question 4 - Solution**(a) Simplification:**

We start with:

$$f(x) = \frac{\cosh^2(x) - 1}{\sinh(x)}$$

Use the identity:

$$\cosh^2(x) - 1 = \sinh^2(x)$$

Thus:

$$f(x) = \frac{\sinh^2(x)}{\sinh(x)} = \sinh(x)$$

(b) Differentiate:

$$f'(x) = \frac{d}{dx} \sinh(x) = \cosh(x)$$

(c) Points of Non-Differentiability:

Since $f(x) = \sinh(x)$ is differentiable for all real x , and the simplification is valid as long as the denominator of the original function $\sinh(x) \neq 0$, we must consider the domain of the original function.

The original function is undefined at:

$$\sinh(x) = 0 \Rightarrow x = 0$$

So although $f(x) = \sinh(x)$ is differentiable everywhere, the original form $f(x) = \frac{\cosh^2(x)-1}{\sinh(x)}$ is undefined at $x = 0$, and thus f is not defined or differentiable there in its original form.

Conclusion:

$f'(x) = \cosh(x), \quad f'(x) \text{ undefined at } x = 0$

Question 5

Let $f(x) = \frac{\sinh(x)}{\cosh(x)}$.

- (a) Simplify the function if possible.
- (b) Find $f'(x)$.
- (c) Determine the intervals where $f(x)$ is increasing or decreasing.

Question 5 - Solution**(a) Simplify the Function:**

We have:

$$f(x) = \frac{\sinh(x)}{\cosh(x)}$$

This is the definition of the hyperbolic tangent function:

$$f(x) = \tanh(x)$$

(b) Differentiate $f(x) = \tanh(x)$:

The derivative of $\tanh(x)$ is:

$$f'(x) = \operatorname{sech}^2(x)$$

(c) Increasing/Decreasing Intervals:

Since $f'(x) = \operatorname{sech}^2(x) > 0$ for all real x , the function is strictly increasing on its entire domain.

Conclusion:

$f'(x) = \operatorname{sech}^2(x), \quad f(x) \text{ is increasing on } (-\infty, \infty)$
--

Question 6

Let $f(x) = x \cdot \cosh(x^2)$.

- (a) Find $f'(x)$.
- (b) Determine the value of the derivative at $x = 0$.
- (c) Is there a local minimum, maximum, or neither at $x = 0$? Justify using the derivative.

Question 6 - Solution

We are given:

$$f(x) = x \cdot \cosh(x^2)$$

(a) Find $f'(x)$:

This is a product of two functions: Let $u = x$, $v = \cosh(x^2)$

Then:

$$f'(x) = u'v + uv' = 1 \cdot \cosh(x^2) + x \cdot \frac{d}{dx}[\cosh(x^2)]$$

Differentiate $\cosh(x^2)$ using the chain rule:

$$\frac{d}{dx}[\cosh(x^2)] = \sinh(x^2) \cdot 2x$$

So:

$$f'(x) = \cosh(x^2) + x \cdot (2x \cdot \sinh(x^2)) = \cosh(x^2) + 2x^2 \sinh(x^2)$$

$$\boxed{f'(x) = \cosh(x^2) + 2x^2 \sinh(x^2)}$$

(b) Value at $x = 0$:

$$f'(0) = \cosh(0^2) + 2 \cdot 0^2 \cdot \sinh(0^2) = \cosh(0) = 1$$

$$\boxed{f'(0) = 1}$$

(c) Local Behavior at $x = 0$:

Since $f'(0) = 1 > 0$, the function is increasing at $x = 0$, so:

Conclusion: There is **no local minimum or maximum** at $x = 0$, because the derivative is positive and the function is increasing there.

Question 7

Let $f(x) = \tanh(x^2) \cdot \operatorname{sech}(x)$.

- (a) Differentiate $f(x)$ using the product rule and the chain rule.
- (b) Simplify the derivative as much as possible using hyperbolic identities.

Question 7 - Solution

We are given:

$$f(x) = \tanh(x^2) \cdot \operatorname{sech}(x)$$

(a) Use the product rule:

Let:

$$u = \tanh(x^2), \quad v = \operatorname{sech}(x)$$

Then:

$$f'(x) = u'v + uv'$$

Differentiate each part:

Step 1: $u = \tanh(x^2) \Rightarrow u' = \operatorname{sech}^2(x^2) \cdot 2x$ using chain rule

Step 2: $v = \operatorname{sech}(x) \Rightarrow v' = -\operatorname{sech}(x) \cdot \tanh(x)$

Now substitute into the product rule:

$$f'(x) = [\operatorname{sech}^2(x^2) \cdot 2x] \cdot \operatorname{sech}(x) + \tanh(x^2) \cdot [-\operatorname{sech}(x) \cdot \tanh(x)]$$

(b) Final expression:

$$f'(x) = 2x \operatorname{sech}^2(x^2) \operatorname{sech}(x) - \tanh(x^2) \operatorname{sech}(x) \tanh(x)$$

Question 8

Let $f(x) = \frac{\sinh(3x)}{x^2+1}$.

- (a) Find the derivative $f'(x)$ using the quotient rule.
- (b) Evaluate the derivative at $x = 0$.

Question 8 - Solution

We are given:

$$f(x) = \frac{\sinh(3x)}{x^2 + 1}$$

(a) Find $f'(x)$:

This is a quotient, so we use the quotient rule:

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$$

Let:

$$g(x) = \sinh(3x), \quad h(x) = x^2 + 1$$

Then:

$$g'(x) = 3 \cosh(3x), \quad h'(x) = 2x$$

So:

$$f'(x) = \frac{3 \cosh(3x)(x^2 + 1) - \sinh(3x)(2x)}{(x^2 + 1)^2}$$

$$f'(x) = \frac{3(x^2 + 1) \cosh(3x) - 2x \sinh(3x)}{(x^2 + 1)^2}$$

(b) Evaluate at $x = 0$:

First note:

$$\cosh(0) = 1, \quad \sinh(0) = 0$$

Then:

$$f'(0) = \frac{3(0^2 + 1) \cdot 1 - 0}{(0^2 + 1)^2} = \frac{3}{1} = 3$$

$$f'(0) = 3$$

Question 9

Let $f(x) = \tanh(x^2 + 1)$.

- (a) Find the derivative $f'(x)$ using the chain rule and hyperbolic identities.
- (b) Determine the value of the derivative at $x = 0$.

Question 9 - Solution

We are given:

$$f(x) = \tanh(x^2 + 1)$$

(a) Find $f'(x)$:

Use the chain rule:

$$f'(x) = \frac{d}{dx} [\tanh(u)] \cdot \frac{d}{dx} (x^2 + 1)$$

Recall:

$$\frac{d}{dx} [\tanh(u)] = \operatorname{sech}^2(u)$$

So let $u = x^2 + 1$, then:

$$f'(x) = \operatorname{sech}^2(x^2 + 1) \cdot 2x$$

$$\boxed{f'(x) = 2x \cdot \operatorname{sech}^2(x^2 + 1)}$$

(b) Evaluate at $x = 0$:

$$f'(0) = 2 \cdot 0 \cdot \operatorname{sech}^2(0^2 + 1) = 0$$

$$\boxed{f'(0) = 0}$$

Question 10

Let $f(x) = \frac{\sinh(3x)}{\cosh(x)}$.

- (a) Differentiate $f(x)$ using the quotient rule and known derivatives of hyperbolic functions.
- (b) Simplify the expression for $f'(x)$ as much as possible.

Question 10 - Solution

We are given:

$$f(x) = \frac{\sinh(3x)}{\cosh(x)}$$

(a) Apply the quotient rule:

The quotient rule is:

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Let:

$$u = \sinh(3x), \quad v = \cosh(x)$$

Differentiate:

$$u' = 3 \cosh(3x), \quad v' = \sinh(x)$$

Apply the rule:

$$f'(x) = \frac{3 \cosh(3x) \cosh(x) - \sinh(3x) \sinh(x)}{\cosh^2(x)}$$

(b) Final simplified expression:

$f'(x) = \frac{3 \cosh(3x) \cosh(x) - \sinh(3x) \sinh(x)}{\cosh^2(x)}$
--

Question 1

Differentiate the function:

$$f(x) = \sqrt{1 + e^{3x^2}}$$

Clearly show how the chain rule is used in multiple layers.

Question 1 - Solution

We are given:

$$f(x) = \sqrt{1 + e^{3x^2}} = (1 + e^{3x^2})^{1/2}$$

Use the chain rule:

$$\begin{aligned} f'(x) &= \frac{1}{2}(1 + e^{3x^2})^{-1/2} \cdot \frac{d}{dx}(1 + e^{3x^2}) \\ &= \frac{1}{2\sqrt{1 + e^{3x^2}}} \cdot \frac{d}{dx}(e^{3x^2}) \end{aligned}$$

Now apply the chain rule again to e^{3x^2} :

$$\frac{d}{dx}(e^{3x^2}) = e^{3x^2} \cdot \frac{d}{dx}(3x^2) = e^{3x^2} \cdot 6x$$

Putting it all together:

$$f'(x) = \frac{1}{2\sqrt{1 + e^{3x^2}}} \cdot e^{3x^2} \cdot 6x = \boxed{\frac{3x \cdot e^{3x^2}}{\sqrt{1 + e^{3x^2}}}}$$

Question 2

Let $f(x) = \sin(\sqrt{1+x^4})$.

- (a) Find $f'(x)$ using the chain rule.
- (b) Evaluate $f'(1)$ exactly (do not approximate numerically).

Question 2 - Solution

We are given:

$$f(x) = \sin\left(\sqrt{1+x^4}\right)$$

Let's apply the chain rule step by step.

Step 1: Let

$$u(x) = \sqrt{1+x^4} = (1+x^4)^{1/2} \quad \text{so that} \quad f(x) = \sin(u(x))$$

Step 2: Apply the chain rule:

$$f'(x) = \cos(u(x)) \cdot u'(x)$$

Step 3: Differentiate $u(x)$:

$$u(x) = (1+x^4)^{1/2} \Rightarrow u'(x) = \frac{1}{2}(1+x^4)^{-1/2} \cdot 4x^3 = \frac{2x^3}{\sqrt{1+x^4}}$$

Final derivative:

$$f'(x) = \cos\left(\sqrt{1+x^4}\right) \cdot \frac{2x^3}{\sqrt{1+x^4}}$$

$$f'(x) = \frac{2x^3 \cos\left(\sqrt{1+x^4}\right)}{\sqrt{1+x^4}}$$

(b) Evaluate at $x = 1$:

$$f'(1) = \frac{2(1)^3 \cdot \cos\left(\sqrt{1+1^4}\right)}{\sqrt{1+1^4}} = \frac{2 \cdot \cos(\sqrt{2})}{\sqrt{2}}$$

$$f'(1) = \frac{2 \cos(\sqrt{2})}{\sqrt{2}}$$

Question 3

Let $y = \ln(\tan^3(5x^2 + 1))$.

- (a) Differentiate y with respect to x using the chain rule.
- (b) Simplify the derivative as much as possible.

Question 3 - Solution

We are given:

$$y = \ln(\tan^3(5x^2 + 1))$$

Step 1: Use logarithmic identity:

$$\ln(\tan^3(5x^2 + 1)) = 3 \ln(\tan(5x^2 + 1))$$

Step 2: Differentiate using chain rule:

$$\frac{dy}{dx} = 3 \cdot \frac{1}{\tan(5x^2 + 1)} \cdot \sec^2(5x^2 + 1) \cdot (10x)$$

Step 3: Simplify the expression:

$$\frac{dy}{dx} = \frac{30x \sec^2(5x^2 + 1)}{\tan(5x^2 + 1)}$$

$\frac{dy}{dx} = \frac{30x \sec^2(5x^2 + 1)}{\tan(5x^2 + 1)}$

Question 4

Let $y = \sqrt{1 + \sin^2(3x)}$.

- (a) Find $\frac{dy}{dx}$ using the chain rule.
- (b) Simplify the result as much as possible.

Question 4 - Solution

We are given:

$$y = \sqrt{1 + \sin^2(3x)} = (1 + \sin^2(3x))^{1/2}$$

Step 1: Apply the chain rule:

Let: - Outer function: $u^{1/2}$ - Inner function: $u = 1 + \sin^2(3x)$

We compute:

$$\frac{dy}{dx} = \frac{1}{2}(1 + \sin^2(3x))^{-1/2} \cdot \frac{d}{dx}[\sin^2(3x)]$$

Now differentiate $\sin^2(3x)$ using the chain rule:

$$\frac{d}{dx}[\sin^2(3x)] = 2 \sin(3x) \cdot \cos(3x) \cdot 3 = 6 \sin(3x) \cos(3x)$$

Step 2: Combine terms:

$$\frac{dy}{dx} = \frac{1}{2}(1 + \sin^2(3x))^{-1/2} \cdot 6 \sin(3x) \cos(3x)$$

$$\frac{dy}{dx} = \frac{3 \sin(3x) \cos(3x)}{\sqrt{1 + \sin^2(3x)}}$$

$$\boxed{\frac{dy}{dx} = \frac{3 \sin(3x) \cos(3x)}{\sqrt{1 + \sin^2(3x)}}}$$

Question 5

Let $f(x) = e^{\tan(x^3+1)}$.

- (a) Use the chain rule to find $f'(x)$.
- (b) Clearly identify each function used in the composition and the order in which the chain rule is applied.

Question 5 - Solution

We are given:

$$f(x) = e^{\tan(x^3+1)}$$

This is a composition of multiple functions. Let's break it down:

- Outer function: $u = e^v$
- Middle function: $v = \tan(w)$
- Inner function: $w = x^3 + 1$

Using the chain rule:

$$f'(x) = \frac{d}{dx} \left[e^{\tan(x^3+1)} \right] = e^{\tan(x^3+1)} \cdot \frac{d}{dx} [\tan(x^3 + 1)]$$

Now,

$$\frac{d}{dx} [\tan(x^3 + 1)] = \sec^2(x^3 + 1) \cdot \frac{d}{dx} [x^3 + 1] = \sec^2(x^3 + 1) \cdot 3x^2$$

Putting it all together:

$$f'(x) = e^{\tan(x^3+1)} \cdot \sec^2(x^3 + 1) \cdot 3x^2$$

$$\boxed{f'(x) = 3x^2 \cdot \sec^2(x^3 + 1) \cdot e^{\tan(x^3+1)}}$$

Question 6

Let $f(x) = \ln\left(\sqrt{1 + \sin^2(5x)}\right)$.

- (a) Use the chain rule to compute $f'(x)$.
- (b) Identify the layers of composition involved and describe the differentiation process step-by-step.

Question 6 - Solution

We are given:

$$f(x) = \ln \left(\sqrt{1 + \sin^2(5x)} \right)$$

Let's simplify before differentiating:

$$f(x) = \ln \left((1 + \sin^2(5x))^{1/2} \right) = \frac{1}{2} \ln(1 + \sin^2(5x))$$

Now we differentiate:

$$f'(x) = \frac{1}{2} \cdot \frac{1}{1 + \sin^2(5x)} \cdot \frac{d}{dx}[\sin^2(5x)]$$

We apply the chain rule to $\sin^2(5x)$:

$$\frac{d}{dx}[\sin^2(5x)] = 2 \sin(5x) \cdot \cos(5x) \cdot 5 = 10 \sin(5x) \cos(5x)$$

Therefore:

$$f'(x) = \frac{1}{2} \cdot \frac{10 \sin(5x) \cos(5x)}{1 + \sin^2(5x)}$$

$$f'(x) = \frac{5 \sin(5x) \cos(5x)}{1 + \sin^2(5x)}$$

Question 7

Let $y = e^{\tan^3(2x)}$.

- (a) Use the chain rule to compute $\frac{dy}{dx}$.
- (b) Clearly outline the layers of composition and how each derivative is applied.

Question 7 - Solution

We are given:

$$y = e^{\tan^3(2x)} = e^{(\tan(2x))^3}$$

Let's identify the layers of composition:

- Outermost: e^u
- Middle: $u = v^3$
- Inner: $v = \tan(2x)$
- Innermost: $2x$

Apply the chain rule step-by-step:

$$\frac{dy}{dx} = \frac{d}{dx} \left(e^{(\tan(2x))^3} \right) = e^{(\tan(2x))^3} \cdot \frac{d}{dx} ((\tan(2x))^3)$$

Differentiate the power:

$$\frac{d}{dx} ((\tan(2x))^3) = 3(\tan(2x))^2 \cdot \frac{d}{dx} [\tan(2x)]$$

Differentiate $\tan(2x)$:

$$\frac{d}{dx} [\tan(2x)] = \sec^2(2x) \cdot 2$$

Now put it all together:

$$\frac{dy}{dx} = e^{(\tan(2x))^3} \cdot 3(\tan(2x))^2 \cdot \sec^2(2x) \cdot 2$$

Simplify:

$$\boxed{\frac{dy}{dx} = 6(\tan(2x))^2 \sec^2(2x) \cdot e^{(\tan(2x))^3}}$$

Question 8

Let $y = \ln\left(\sqrt{1 + e^{3x^2}}\right)$.

- (a) Use the chain rule to compute $\frac{dy}{dx}$.
- (b) Explain how the composition of functions influences the order of derivatives.

Question 8 - Solution

We are given:

$$y = \ln \left(\sqrt{1 + e^{3x^2}} \right)$$

We rewrite the square root as a power:

$$y = \ln \left((1 + e^{3x^2})^{1/2} \right)$$

Using the log rule:

$$y = \frac{1}{2} \ln(1 + e^{3x^2})$$

Differentiate:

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{1 + e^{3x^2}} \cdot \frac{d}{dx}[e^{3x^2}]$$

Differentiate the exponential:

$$\frac{d}{dx}[e^{3x^2}] = e^{3x^2} \cdot \frac{d}{dx}[3x^2] = e^{3x^2} \cdot 6x$$

Putting it all together:

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{1 + e^{3x^2}} \cdot e^{3x^2} \cdot 6x$$

Simplify:

$$\boxed{\frac{dy}{dx} = \frac{3xe^{3x^2}}{1 + e^{3x^2}}}$$

Explanation: This function has a nested composition: Innermost: $3x^2$, Exponential: e^{3x^2} , Added to 1, Square root $\rightarrow$ converted to a power, Then natural logarithm

At each step, the derivative is taken from the outside and multiplied by the derivative of the inside, following the chain rule.

Question 9

Let $f(x) = \sin^2(5x^3 + 1)$.

- (a) Use the chain rule to compute $f'(x)$.
- (b) Write down each layer of function composition clearly before differentiating.

Question 9 - Solution

We are given:

$$f(x) = \sin^2(5x^3 + 1)$$

This can be rewritten as:

$$f(x) = [\sin(5x^3 + 1)]^2$$

Layers of function composition:

1. Innermost function: $u(x) = 5x^3 + 1$ 2. Middle function: $v(u) = \sin(u)$ 3. Outer function: $f(v) = v^2$

Differentiate step by step:

Start from the outside:

$$f'(x) = 2 \cdot \sin(5x^3 + 1) \cdot \frac{d}{dx}[\sin(5x^3 + 1)]$$

Now differentiate the inner sine function:

$$\frac{d}{dx}[\sin(5x^3 + 1)] = \cos(5x^3 + 1) \cdot \frac{d}{dx}[5x^3 + 1] = \cos(5x^3 + 1) \cdot 15x^2$$

Putting it all together:

$$f'(x) = 2 \cdot \sin(5x^3 + 1) \cdot \cos(5x^3 + 1) \cdot 15x^2$$

$$\boxed{f'(x) = 30x^2 \cdot \sin(5x^3 + 1) \cdot \cos(5x^3 + 1)}$$

Alternatively, you may express the result using the identity:

$$2 \sin \theta \cos \theta = \sin(2\theta)$$

$$\Rightarrow f'(x) = 15x^2 \cdot \sin(2(5x^3 + 1)) = \boxed{15x^2 \cdot \sin(10x^3 + 2)}$$

Either boxed expression is valid.

Question 10

Let $y = \ln(\sqrt{1 + e^{4x}})$.

- (a) Use the chain rule to find $\frac{dy}{dx}$.
- (b) Clearly indicate the nested functions and show each step of differentiation.

Question 10 - Solution

We are given:

$$y = \ln \left(\sqrt{1 + e^{4x}} \right)$$

First, simplify the expression:

$$y = \ln \left[(1 + e^{4x})^{1/2} \right] = \frac{1}{2} \ln(1 + e^{4x})$$

Now differentiate:

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{1 + e^{4x}} \cdot \frac{d}{dx}(1 + e^{4x})$$

Differentiate the inner exponential:

$$\frac{d}{dx}(1 + e^{4x}) = 4e^{4x}$$

Putting it all together:

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{4e^{4x}}{1 + e^{4x}} = \boxed{\frac{2e^{4x}}{1 + e^{4x}}}$$

Nested Function Structure:

$$u = e^{4x}, v = 1 + u, w = \ln(\sqrt{v}) = \frac{1}{2} \ln(v)$$

Differentiation followed this path:

$$\frac{dw}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$

Question 1

Given the equation:

$$x^2 + xy + y^2 = 7$$

- (a) Use implicit differentiation to find $\frac{dy}{dx}$.
- (b) Find the slope of the tangent line at the point $(1, 2)$.

Question 1 - Solution

We are given:

$$x^2 + xy + y^2 = 7$$

Differentiate both sides with respect to x , using the product rule on xy :

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$

$$2x + \left(x \frac{dy}{dx} + y\right) + 2y \frac{dy}{dx} = 0$$

Group $\frac{dy}{dx}$ terms:

$$x \frac{dy}{dx} + 2y \frac{dy}{dx} = -2x - y \Rightarrow (x + 2y) \frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx} = \boxed{\frac{-2x - y}{x + 2y}}$$

(b) Evaluate at the point $(1, 2)$:

$$\frac{dy}{dx} = \frac{-2(1) - 2}{1 + 2(2)} = \frac{-4}{5}$$

Answer: Slope of the tangent line at $(1, 2)$ is $\boxed{-\frac{4}{5}}$

Question 2

Let the equation of a curve be defined implicitly by:

$$x^3 + y^3 = 6xy$$

- (a) Use implicit differentiation to find $\frac{dy}{dx}$.
- (b) Find the slope of the tangent line at the point $(3,3)$.

Question 2 - Solution

We are given:

$$x^3 + y^3 = 6xy$$

(a) Differentiate both sides with respect to x :

Left side:

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 3x^2 + 3y^2 \frac{dy}{dx}$$

Right side (Product Rule):

$$\frac{d}{dx}(6xy) = 6x \frac{dy}{dx} + 6y$$

Now equate both sides:

$$3x^2 + 3y^2 \frac{dy}{dx} = 6x \frac{dy}{dx} + 6y$$

Bring like terms together:

$$3y^2 \frac{dy}{dx} - 6x \frac{dy}{dx} = 6y - 3x^2$$

Factor out $\frac{dy}{dx}$:

$$(3y^2 - 6x) \frac{dy}{dx} = 6y - 3x^2$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$

Simplify:

$$\boxed{\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}}$$

(b) Evaluate the derivative at $(3, 3)$:

Substitute $x = 3$, $y = 3$:

$$\frac{dy}{dx} = \frac{2(3) - 3^2}{3^2 - 2(3)} = \frac{6 - 9}{9 - 6} = \frac{-3}{3} = \boxed{-1}$$

Question 3

Consider the equation:

$$x^2 \sin(y) + y^2 \cos(x) = 1$$

- (a) Use implicit differentiation to find $\frac{dy}{dx}$.
- (b) Find the slope of the tangent line to the curve at the point $(0, 1)$.

Question 3 - Solution

We are given:

$$x^2 \sin(y) + y^2 \cos(x) = 1$$

Differentiate both sides implicitly with respect to x :

$$\frac{d}{dx} [x^2 \sin(y)] + \frac{d}{dx} [y^2 \cos(x)] = 0$$

Using the product rule and chain rule:

$$2x \sin(y) + x^2 \cos(y) \frac{dy}{dx} + 2y \frac{dy}{dx} \cos(x) - y^2 \sin(x) = 0$$

Move the non- $\frac{dy}{dx}$ terms to the other side:

$$x^2 \cos(y) \frac{dy}{dx} + 2y \cos(x) \frac{dy}{dx} = y^2 \sin(x) - 2x \sin(y)$$

Factor out $\frac{dy}{dx}$:

$$(x^2 \cos(y) + 2y \cos(x)) \frac{dy}{dx} = y^2 \sin(x) - 2x \sin(y)$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{y^2 \sin(x) - 2x \sin(y)}{x^2 \cos(y) + 2y \cos(x)}$$

At $(0, 1)$:

$$\frac{dy}{dx} = \frac{(1)^2 \sin(0) - 2(0) \sin(1)}{(0)^2 \cos(1) + 2(1) \cos(0)} = \frac{0}{2} = \boxed{0}$$

Therefore, the slope of the tangent line at $(0, 1)$ is:

$$\boxed{0}$$

Question 4

Consider the equation:

$$\sin(xy) = x + y$$

- (a) Use implicit differentiation to find $\frac{dy}{dx}$.
- (b) Find the slope of the tangent line to the curve at the point $(0, 0)$.

Question 4 - Solution**(b) Tangent slope**

At $(0, 0)$, the original equation reads $\sin 0 = 0 + 0$, so the point belongs to the curve.

(a) Implicit derivative

Implicit differentiation gives

$$\cos(xy)(y + xy') = 1 + y'.$$

Collecting derivative terms,

$$(x \cos(xy) - 1)y' = 1 - y \cos(xy).$$

Thus, wherever the denominator is nonzero,

$$y' = \frac{1 - y \cos(xy)}{x \cos(xy) - 1}.$$

At the specified point the denominator is -1 , and the slope is $\boxed{-1}$.

Question 5

Consider the equation:

$$x^2y + e^{xy} = y^3 + 4x$$

- (a) Use implicit differentiation to find $\frac{dy}{dx}$.
- (b) Find the slope of the tangent line to the curve at the point $(0, 1)$.

Question 5 - Solution

We are given:

$$x^2y + e^{xy} = y^3 + 4x$$

Differentiate both sides implicitly with respect to x :

$$\frac{d}{dx}[x^2y] + \frac{d}{dx}[e^{xy}] = \frac{d}{dx}[y^3] + \frac{d}{dx}[4x]$$

Using the product rule and chain rule:

$$x^2 \frac{dy}{dx} + 2xy + e^{xy} \left(x \frac{dy}{dx} + y \right) = 3y^2 \frac{dy}{dx} + 4$$

Expand:

$$x^2 \frac{dy}{dx} + 2xy + xe^{xy} \frac{dy}{dx} + ye^{xy} = 3y^2 \frac{dy}{dx} + 4$$

Move the $\frac{dy}{dx}$ terms to one side:

$$x^2 \frac{dy}{dx} + xe^{xy} \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 4 - 2xy - ye^{xy}$$

Factor:

$$(x^2 + xe^{xy} - 3y^2) \frac{dy}{dx} = 4 - 2xy - ye^{xy}$$

Therefore,

$$\boxed{\frac{dy}{dx} = \frac{4 - 2xy - ye^{xy}}{x^2 + xe^{xy} - 3y^2}}$$

At $(0, 1)$, we have $xy = 0$, so $e^{xy} = e^0 = 1$. Thus,

$$\frac{dy}{dx} = \frac{4 - 2(0)(1) - (1)(1)}{0^2 + 0(1) - 3(1)^2} = \frac{3}{-3} = \boxed{-1}$$

Therefore, the slope of the tangent line at $(0, 1)$ is:

$$\boxed{-1}$$

Question 6

Consider the equation defined implicitly:

$$x^2 + y^2 + \ln(y) = 10$$

- (a) Use implicit differentiation to find $\frac{dy}{dx}$.
- (b) Find the slope of the tangent line at the point $(3, 1)$.

Question 6 - Solution

The logarithm requires $y > 0$. Differentiating gives

$$2x + 2yy' + y'/y = 0, \quad y' = \frac{-2x}{2y + 1/y}.$$

At $(3, 1)$, $3^2 + 1^2 + \ln 1 = 10$, so the point is on the curve.

The denominator is 3, giving $y'(3) = -2$.

Question 7

Given the curve defined implicitly by the equation:

$$\sin(xy) + x^2 = y$$

- (a) Use implicit differentiation to find $\frac{dy}{dx}$.
- (b) Find the value of $\frac{dy}{dx}$ at the point $(0, 0)$.

Question 7 - Solution

We are given:

$$\sin(xy) + x^2 = y$$

(a) Differentiate both sides implicitly with respect to x :

Use the chain rule on $\sin(xy)$, noting that xy is a product of functions:

$$\frac{d}{dx}[\sin(xy)] + \frac{d}{dx}[x^2] = \frac{d}{dx}[y]$$

Differentiate:

$$\cos(xy) \cdot \left(y + x \cdot \frac{dy}{dx} \right) + 2x = \frac{dy}{dx}$$

Now solve for $\frac{dy}{dx}$. Expand the left-hand side:

$$\cos(xy)y + \cos(xy)x \cdot \frac{dy}{dx} + 2x = \frac{dy}{dx}$$

Bring all $\frac{dy}{dx}$ terms to one side:

$$\cos(xy)x \cdot \frac{dy}{dx} - \frac{dy}{dx} = -\cos(xy)y - 2x$$

Factor $\frac{dy}{dx}$:

$$(\cos(xy)x - 1) \cdot \frac{dy}{dx} = -\cos(xy)y - 2x$$

Solve:

$$\boxed{\frac{dy}{dx} = \frac{-\cos(xy)y - 2x}{\cos(xy)x - 1}}$$

(b) Evaluate at the point $(0, 0)$:

Substitute $x = 0$, $y = 0$ into the derivative:

$$\cos(0) = 1$$

$$\frac{dy}{dx} = \frac{-1(0) - 2(0)}{1(0) - 1} = \frac{0}{-1} = 0$$

$$\boxed{\left. \frac{dy}{dx} \right|_{(0,0)} = 0}$$

Question 8

Consider the equation:

$$\ln(x + y) + x^2y = y^2 + 3x + \ln 2 - 3$$

- (a) Use implicit differentiation to find $\frac{dy}{dx}$.
- (b) Find the slope of the tangent line to the curve at the point $(1, 1)$.

Question 8 - Solution

At $(1, 1)$ both sides of the corrected equation equal $\ln 2 + 1$.

(a) Implicit derivative

Differentiating (with $x + y > 0$) gives

$$\frac{1 + y'}{x + y} + 2xy + x^2y' = 2yy' + 3.$$

Therefore

$$y' = \frac{3 - 2xy - 1/(x + y)}{1/(x + y) + x^2 - 2y}.$$

(b) Tangent slope

At $(1, 1)$ the numerator is $1/2$ and the denominator is $-1/2$, hence the slope is $\boxed{-1}$.

Question 9

Consider the equation:

$$x \sin(y) + y \cos(x) = 1$$

- (a) Use implicit differentiation to find $\frac{dy}{dx}$.
- (b) Find the slope of the tangent line to the curve at the point $(0, 1)$.

Question 9 - Solution

We are given:

$$x \sin(y) + y \cos(x) = 1$$

Differentiate both sides implicitly with respect to x :

$$\frac{d}{dx}[x \sin(y)] + \frac{d}{dx}[y \cos(x)] = 0$$

Using the product rule:

$$\sin(y) + x \cos(y) \frac{dy}{dx} + \cos(x) \frac{dy}{dx} - y \sin(x) = 0$$

Move the non- $\frac{dy}{dx}$ terms to the other side:

$$x \cos(y) \frac{dy}{dx} + \cos(x) \frac{dy}{dx} = y \sin(x) - \sin(y)$$

Factor out $\frac{dy}{dx}$:

$$(x \cos(y) + \cos(x)) \frac{dy}{dx} = y \sin(x) - \sin(y)$$

Solve for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = \frac{y \sin(x) - \sin(y)}{x \cos(y) + \cos(x)}}$$

At $(0, 1)$:

$$\frac{dy}{dx} = \frac{(1) \sin(0) - \sin(1)}{(0) \cos(1) + \cos(0)}$$

$$\frac{dy}{dx} = \frac{0 - \sin(1)}{0 + 1} = \boxed{-\sin(1)}$$

Therefore, the slope of the tangent line at $(0, 1)$ is:

$$\boxed{-\sin(1)}$$

Question 10

Consider the real relation

$$x^2 + y^2 = \sin(xy).$$

- (a) Derive the formal relation obtained by implicit differentiation. State when division would be valid.
- (b) Determine the real solution set and decide whether a tangent slope exists at $(0, 0)$.

Question 10 - Solution

Formal differentiation. If a differentiable branch existed, it would obey

$$2x + 2yy' = \cos(xy)(y + xy'),$$

so $(2y - x \cos(xy))y' = y \cos(xy) - 2x$. Division would require a nonzero coefficient.

Analyze the real point set first. For every real x, y ,

$$\sin(xy) \leq |xy| \leq \frac{x^2 + y^2}{2}.$$

Thus the given equation implies

$$0 \leq x^2 + y^2 \leq \frac{x^2 + y^2}{2},$$

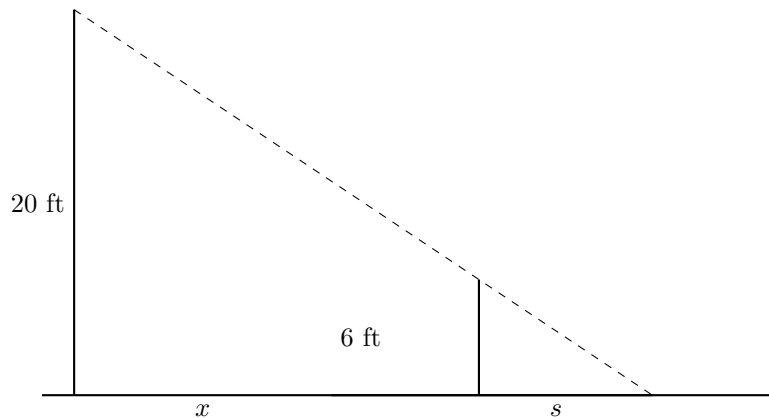
which forces $x = y = 0$. The real solution set is the singleton $\boxed{\{(0, 0)\}}$.

There is no curve branch through this isolated point and hence no tangent slope dy/dx there. The formal expression $0/0$ alone would not establish this conclusion.

Question 1

A streetlight is mounted at the top of a 20-foot pole. A person 6 feet tall walks directly away from the pole at a constant rate of 4 ft/sec.

- (a) How fast is the tip of the person's shadow moving along the ground when the person is 10 feet from the pole?
- (b) How fast is the length of the person's shadow changing at that instant?
- (c) Explain why the tip of the shadow moves faster than the person.



Question 1 - Solution

Let:

$x(t)$ = distance from the pole to the person

$s(t)$ = length of the shadow

The distance from the pole to the tip of the shadow is:

$$x + s$$

Step 1: Set up the relationship

Using similar triangles:

$$\frac{20}{x + s} = \frac{6}{s}$$

Cross-multiply:

$$20s = 6(x + s)$$

Simplify:

$$20s = 6x + 6s \Rightarrow 14s = 6x \Rightarrow s = \frac{3}{7}x$$

Step 2: Differentiate with respect to time

Differentiate both sides:

$$\frac{ds}{dt} = \frac{3}{7} \frac{dx}{dt}$$

Given:

$$\frac{dx}{dt} = 4$$

So:

$$\frac{ds}{dt} = \frac{12}{7}$$

(a) Speed of the tip of the shadow

$$\frac{d}{dt}(x + s) = \frac{dx}{dt} + \frac{ds}{dt} = 4 + \frac{12}{7} = \frac{40}{7}$$

Answer (a):

$$\boxed{\frac{40}{7} \text{ ft/sec}}$$

(b) Rate of change of the shadow length

$$\boxed{\frac{ds}{dt} = \frac{12}{7} \text{ ft/sec}}$$

(c) Explanation

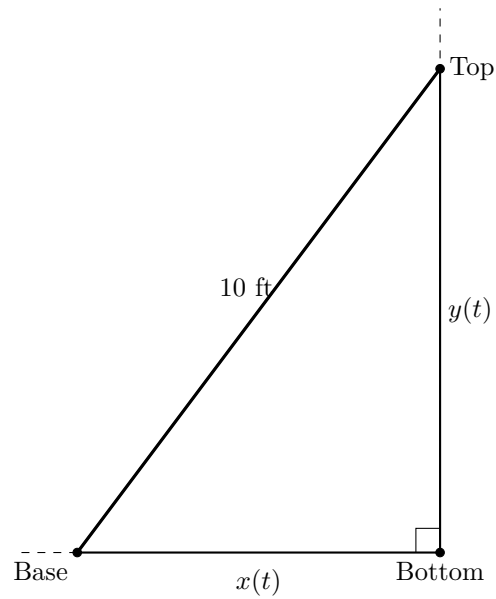
The shadow tip moves faster than the person because its motion includes both the person's movement and the changing length of the shadow.

Conclusion: The shadow tip moves faster than the person.

Question 2

A ladder 10 feet long is leaning against a vertical wall. The bottom of the ladder is sliding away from the wall at a rate of 2 ft/sec.

- (a) How fast is the top of the ladder sliding down the wall when the bottom is 6 feet from the wall?



Question 2 - Solution

Let: $x(t)$: distance from the wall to the bottom of the ladder (horizontal leg), $y(t)$: height of the ladder on the wall (vertical leg), Ladder: 10 ft, so by Pythagoras:

$$x^2 + y^2 = 10^2 = 100$$

Differentiate both sides w.r.t. time t :

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow x \frac{dx}{dt} + y \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

Given: $x = 6$, Ladder is 10 ft, so $y = \sqrt{100 - 36} = 8$, $\frac{dx}{dt} = 2$

$$\frac{dy}{dt} = -\frac{6}{8} \cdot 2 = -\frac{12}{8} = -\frac{3}{2}$$

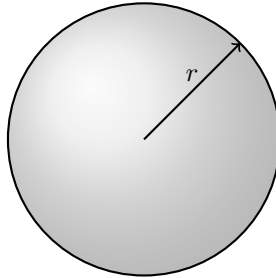
Answer:

$\frac{dy}{dt} = -\frac{3}{2} \text{ ft/sec}$

Question 3**Problem:**

A spherical balloon is being inflated such that its radius increases at a constant rate of $\frac{dr}{dt} = 3$ cm/min.

- (a) At what rate is the volume of the balloon increasing when the radius is 10 cm?
- (b) Include units and interpret the meaning of your result.



Spherical balloon

Question 3 - Solution

We are given:

$$\frac{dr}{dt} = 3 \text{ cm/min}, \quad r = 10 \text{ cm}$$

The volume of a sphere is given by:

$$V = \frac{4}{3}\pi r^3$$

Differentiate both sides with respect to t :

$$\frac{dV}{dt} = \frac{d}{dt} \left(\frac{4}{3}\pi r^3 \right) = 4\pi r^2 \frac{dr}{dt}$$

Substitute the known values:

$$\frac{dV}{dt} = 4\pi(10)^2(3) = 4\pi(100)(3) = 1200\pi$$

$$\frac{dV}{dt} \approx 3769.91 \text{ cm}^3/\text{min}$$

Answer: The volume is increasing at a rate of $\boxed{1200\pi \approx 3769.91 \text{ cm}^3/\text{min}}$ when the radius is 10 cm.

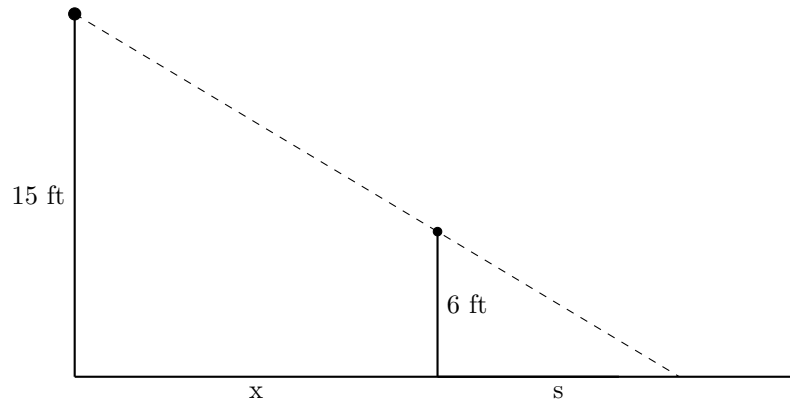
Interpretation: When the balloon's radius is 10 cm and growing at 3 cm per minute, the volume is expanding at approximately 3769.91 cubic centimeters per minute. This tells us how rapidly the space inside the balloon is increasing at that instant.

Question 4

Problem:

A 6-ft tall person walks away from a 15-ft tall streetlight at a rate of 3 ft/s.

- (a) How fast is the tip of the shadow moving when the person is 30 ft from the base of the light?
(b) At the same moment, how fast is the length of the shadow increasing?



Question 4 - Solution

Let: - $x(t)$ be the distance from the person to the light pole - $s(t)$ be the length of the shadow - The tip of the shadow is at a distance $x + s$ from the pole

Using similar triangles:

$$\frac{6}{s} = \frac{15}{x+s} \Rightarrow 6(x+s) = 15s \Rightarrow 6x + 6s = 15s \Rightarrow 6x = 9s \Rightarrow s = \frac{2}{3}x$$

Differentiate with respect to t :

$$\frac{ds}{dt} = \frac{2}{3} \frac{dx}{dt}$$

Given:

$$\frac{dx}{dt} = 3 \text{ ft/s} \Rightarrow \frac{ds}{dt} = \frac{2}{3}(3) = 2 \text{ ft/s}$$

Now, the tip of the shadow moves at:

$$\frac{d}{dt}(x+s) = \frac{dx}{dt} + \frac{ds}{dt} = 3 + 2 = \boxed{5 \text{ ft/s}}$$

Answers:

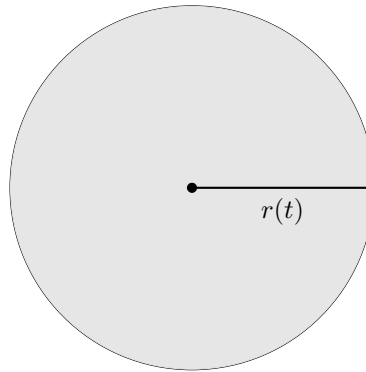
(a) The tip of the shadow is moving at $\boxed{5 \text{ ft/s}}$

(b) The shadow is lengthening at $\boxed{2 \text{ ft/s}}$

Question 5

A circular ripple is formed on the surface of a still pond when a pebble is dropped into it. The radius of the ripple increases at a constant rate of 0.05 m/s.

- (a) At what rate is the area enclosed by the ripple increasing when the radius is 0.3 m?
- (b) At what rate is the circumference of the ripple increasing at that moment?



Expanding ripple with changing radius

Question 5 - Solution

Let $r(t)$ be the radius of the ripple at time t , in meters.

(a) Rate of Change of Area

The area of a circle is:

$$A = \pi r^2$$

Differentiate both sides with respect to time t :

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

We're given:

$$\frac{dr}{dt} = 0.05 \text{ m/s}, \quad r = 0.3 \text{ m}$$

Substitute:

$$\frac{dA}{dt} = 2\pi(0.3)(0.05) = 0.03\pi \approx \boxed{0.0942 \text{ m}^2/\text{s}}$$

(b) Rate of Change of Circumference

Circumference is given by:

$$C = 2\pi r$$

Differentiate:

$$\frac{dC}{dt} = 2\pi \frac{dr}{dt}$$

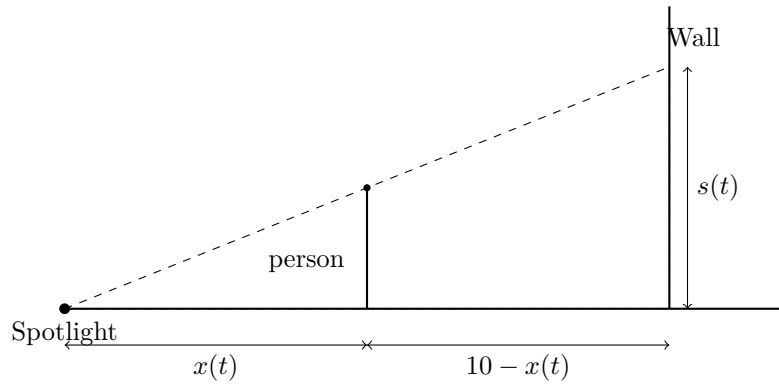
Substitute:

$$\frac{dC}{dt} = 2\pi(0.05) = \boxed{0.1\pi \approx 0.3142 \text{ m/s}}$$

Question 6

A spotlight is located on the ground 10 m away from a vertical wall. A person 2 m tall walks directly away from the wall at a speed of 1.2 m/s.

- (a) How fast is the length of the shadow on the wall changing when the person is 5 m from the wall?
- (b) At that moment, how fast is the tip of the shadow moving up the wall?



Question 6 - Solution

Let:

$x(t)$ = horizontal distance from the spotlight to the person (m)

$s(t)$ = height of the shadow on the wall (m)

From similar triangles:

$$\frac{s(t)}{10} = \frac{2}{x(t)} \Rightarrow s(t) = \frac{20}{x(t)}$$

Differentiate both sides:

$$\frac{ds}{dt} = -\frac{20}{x^2} \cdot \frac{dx}{dt}$$

Given:

$$x = 5 \text{ m}, \quad \frac{dx}{dt} = -1.2 \text{ m/s}$$

$$\frac{ds}{dt} = -\frac{20}{25}(-1.2) = 0.96 \text{ m/s}$$

(a) Rate of change of shadow height:

$\frac{ds}{dt} = 0.96 \text{ m/s}$

(b) Speed of the tip of the shadow:

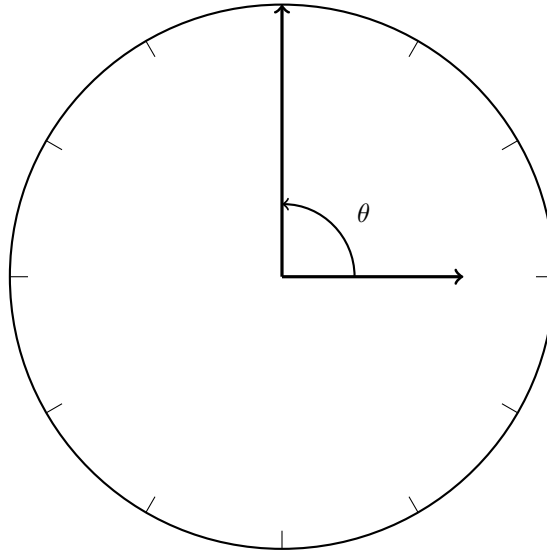
The tip moves upward at the same rate:

0.96 m/s upward

Question 7

The minute hand of a clock completes one full revolution every 60 minutes, and the hour hand completes one full revolution every 12 hours.

Question: At exactly 3:00, how fast is the *smaller angle* between the hour and minute hands changing?



Solution — Method 1 (Formal Related Rates)

Let $\theta(t)$ be the signed angle (in degrees) from the hour hand to the minute hand.

$$\omega_m = 360/60 = 6^\circ/\text{min}, \quad \omega_h = 360/(12 * 60) = 0.5^\circ/\text{min}$$

At 3:00, the hour hand is at 90° and the minute hand is at 0° .

$$\theta(t) = (6t) - (90 + 0.5t)$$

Differentiate:

$$\frac{d\theta}{dt} = 6 - 0.5 = 5.5$$

Since the smaller angle is initially decreasing,

$$\frac{d\theta}{dt} = -5.5^\circ/\text{min}$$

Question 7 - Solution)

The angle between the hands changes at the difference of their angular speeds.

$$\text{Minute hand speed} = 6^\circ/\text{min}$$

$$\text{Hour hand speed} = 0.5^\circ/\text{min}$$

So the angle closes at:

$$6 - 0.5 = 5.5^\circ/\text{min}$$

Since the hands are moving toward each other at 3:00, the angle is decreasing.

$$5.5^\circ/\text{min (decreasing)}$$

Final Answer

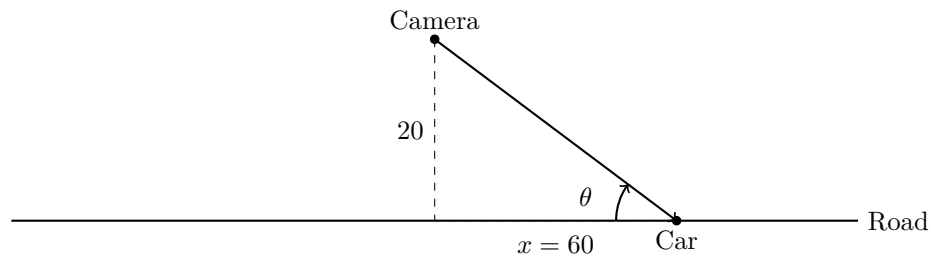
$$\frac{d\theta}{dt} = -5.5^\circ/\text{min}$$

The negative sign indicates the angle between the hands is decreasing.

Question 8

A camera is placed 20 meters from a straight road. A car is driving along the road away from the point closest to the camera at a speed of 30 m/s.

How fast is the angle between the camera's line of sight and the road changing when the car is 60 meters from the point on the road closest to the camera?



Question 8 - Solution

Let:

$x(t)$: horizontal distance from car to closest point on road to camera , $\theta(t)$: angle between the camera's line of sight and the road , The camera is 20 m from the road (fixed)

We have:

$$\tan(\theta) = \frac{20}{x}$$

Differentiate both sides with respect to time:

$$\sec^2(\theta) \cdot \frac{d\theta}{dt} = \frac{-20}{x^2} \cdot \frac{dx}{dt}$$

At the instant:

$x = 60$, $\frac{dx}{dt} = 30$ m/s (moving away) , $\tan(\theta) = \frac{20}{60} = \frac{1}{3} \Rightarrow \theta \approx 18.43^\circ$

Use identity:

$$\sec^2(\theta) = 1 + \tan^2(\theta) = 1 + \left(\frac{1}{3}\right)^2 = \frac{10}{9}$$

Now plug in:

$$\frac{10}{9} \cdot \frac{d\theta}{dt} = \frac{-20}{60^2} \cdot 30 = \frac{-20 \cdot 30}{3600} = \frac{-600}{3600} = -\frac{1}{6}$$

So:

$$\frac{d\theta}{dt} = \boxed{-\frac{9}{60} = -0.15 \text{ radians per second}}$$

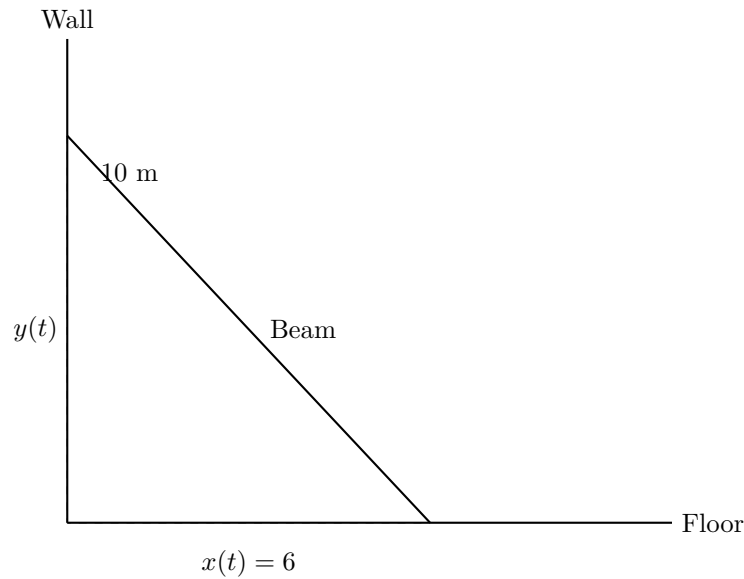
Conclusion: The angle between the camera's line of sight and the road is decreasing at a rate of:

$$\boxed{0.15 \text{ radians/second}}$$

Question 9

A 10-meter-long beam is leaning against a vertical wall and a horizontal floor, forming a right triangle. The bottom of the beam is sliding away from the wall along the floor at a rate of 1.5 m/s.

How fast is the top of the beam sliding down the wall when the bottom is 6 meters away from the wall?



Question 9 - Solution

Let:

$x(t)$: distance from the base of the wall to the bottom of the beam

$y(t)$: height of the beam on the wall

The beam is 10 m long $\Rightarrow x^2 + y^2 = 100$

Differentiate both sides with respect to time:

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow x \frac{dx}{dt} + y \frac{dy}{dt} = 0$$

Given: $x = 6$, $\frac{dx}{dt} = 1.5$

Find y from Pythagoras:

$$y = \sqrt{100 - 6^2} = \sqrt{64} = 8$$

Now solve for $\frac{dy}{dt}$:

$$6(1.5) + 8 \frac{dy}{dt} = 0 \Rightarrow 9 + 8 \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{9}{8}$$

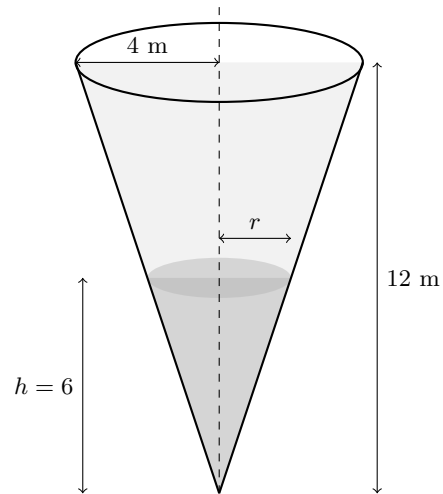
Conclusion: The top of the beam is sliding down at a rate of:

$\frac{dy}{dt} = -\frac{9}{8} = -1.125 \text{ m/s}$

Question 10

Water is poured into an inverted conical tank at a rate of $3 \text{ m}^3/\text{min}$. The tank is 12 meters deep and has a top radius of 4 meters.

How fast is the water level rising when the water is 6 meters deep?



Question 10 - Solution

Let:

V : volume of water

r : radius of water surface

h : depth of water

Given:

$$\frac{dV}{dt} = 3 \text{ m}^3/\text{min}$$

Tank dimensions: height = 12 m, top radius = 4 m

$$\text{Volume of cone: } V = \frac{1}{3}\pi r^2 h$$

Use similar triangles:

$$\frac{r}{h} = \frac{4}{12} = \frac{1}{3} \Rightarrow r = \frac{h}{3}$$

Substitute into volume formula:

$$V = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h = \frac{\pi}{27}h^3$$

Differentiate:

$$\frac{dV}{dt} = \frac{\pi}{9}h^2 \cdot \frac{dh}{dt}$$

Substitute known values:

$$3 = \frac{\pi}{9}(6)^2 \cdot \frac{dh}{dt} = \frac{\pi}{9} \cdot 36 \cdot \frac{dh}{dt} \Rightarrow 3 = 4\pi \cdot \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{3}{4\pi}$$

Final Answer:

$$\boxed{\frac{dh}{dt} = \frac{3}{4\pi} \text{ m/min} \approx 0.239 \text{ m/min}}$$

Question 1

Let:

$$f(x) = x^4 - 4x^3 + 6x^2$$

- (a) Find the first and second derivatives of $f(x)$.
- (b) Determine the intervals on which $f(x)$ is concave up and concave down.
- (c) Find all points of inflection.

Question 1 - Solution

We are given:

$$f(x) = x^4 - 4x^3 + 6x^2$$

(a) First derivative:

$$f'(x) = 4x^3 - 12x^2 + 12x$$

Second derivative:

$$f''(x) = 12x^2 - 24x + 12$$

Factor:

$$f''(x) = 12(x^2 - 2x + 1) = 12(x - 1)^2$$

Answer:

$$f'(x) = 4x^3 - 12x^2 + 12x, \quad f''(x) = 12(x - 1)^2$$

(b) Analyze sign of $f''(x)$:

Since $f''(x) = 12(x - 1)^2 \geq 0$ for all x , the second derivative is never negative.

$f''(x) > 0$ for $x \neq 1 \rightarrow$ concave up

$f''(1) = 0 \rightarrow$ possible inflection point

But since the sign does not change around $x = 1$, it is not an inflection point.

Answer: Concave up: $(-\infty, \infty)$ Concave down: None

(c) No change in concavity at $x = 1$, so:

Answer: No points of inflection

Question 2

Let $f(x) = x^2e^{-x}$.

- (a) Compute the first four derivatives of $f(x)$.
- (b) Identify a pattern in the derivatives.
- (c) Use the pattern to write a general formula for $f^{(n)}(x)$ for $n \geq 0$.

Question 2 - Solution

We are given $f(x) = x^2e^{-x}$. Let's compute derivatives step-by-step.

(a) First, compute the derivatives:

First derivative:

$$f'(x) = \frac{d}{dx}(x^2e^{-x}) = 2xe^{-x} + x^2(-e^{-x}) = (2x - x^2)e^{-x}$$

Second derivative:

$$f''(x) = \frac{d}{dx}[(2x - x^2)e^{-x}] = [2 - 2x]e^{-x} + (2x - x^2)(-e^{-x}) = [2 - 2x - 2x + x^2]e^{-x} = (x^2 - 4x + 2)e^{-x}$$

Third derivative:

$$f^{(3)}(x) = \frac{d}{dx}[(x^2 - 4x + 2)e^{-x}] = (2x - 4)e^{-x} + (x^2 - 4x + 2)(-e^{-x}) = (-x^2 + 6x - 6)e^{-x}$$

Fourth derivative:

$$f^{(4)}(x) = \frac{d}{dx}[(-x^2 + 6x - 6)e^{-x}] = (-2x + 6)e^{-x} + (-x^2 + 6x - 6)(-e^{-x}) = (x^2 - 8x + 12)e^{-x}$$

(b) Pattern:

Each derivative is a polynomial times e^{-x} . The polynomials alternate in sign and follow a recursive-like behavior:

$$\begin{aligned}f(x) &= x^2e^{-x} \\f'(x) &= (2x - x^2)e^{-x} \\f''(x) &= (x^2 - 4x + 2)e^{-x} \\f^{(3)}(x) &= (-x^2 + 6x - 6)e^{-x} \\f^{(4)}(x) &= (x^2 - 8x + 12)e^{-x}\end{aligned}$$

(c) General formula (structure):

Let:

$$f^{(n)}(x) = P_n(x)e^{-x}$$

Where $P_n(x)$ is a polynomial of degree 2, alternating in sign:

$$P_n(x) = (-1)^n(x^2 - 2nx + n(n-1))$$

Final boxed result:

$$\boxed{f^{(n)}(x) = (-1)^n(x^2 - 2nx + n(n-1))e^{-x}}$$

Question 3

Let $f(x) = \ln(x^2 + 1)$.

- (a) Compute the first three derivatives of $f(x)$.
- (b) Determine the value of $f^{(3)}(0)$.
- (c) Based on the derivatives at $x = 0$, write the third-degree Taylor polynomial centered at $x = 0$.

Question 3 - Solution

We are given:

$$f(x) = \ln(x^2 + 1)$$

(a) First three derivatives:

First derivative:

$$f'(x) = \frac{d}{dx} \ln(x^2 + 1) = \frac{2x}{x^2 + 1}$$

Second derivative:

$$f''(x) = \frac{d}{dx} \left(\frac{2x}{x^2 + 1} \right) = \frac{2(x^2 + 1) - 2x(2x)}{(x^2 + 1)^2} = \frac{2(x^2 + 1 - 2x^2)}{(x^2 + 1)^2} = \frac{2(1 - x^2)}{(x^2 + 1)^2}$$

Third derivative:

We apply the quotient rule to:

$$f''(x) = \frac{2(1 - x^2)}{(x^2 + 1)^2}$$

Let's compute:

$$f^{(3)}(x) = \frac{d}{dx} \left(\frac{2(1 - x^2)}{(x^2 + 1)^2} \right)$$

Numerator derivative:

$$\frac{d}{dx} [2(1 - x^2)] = -4x$$

Denominator derivative:

$$\frac{d}{dx} [(x^2 + 1)^2] = 2(x^2 + 1)(2x) = 4x(x^2 + 1)$$

Now apply quotient rule:

$$f^{(3)}(x) = \frac{(-4x)(x^2 + 1)^2 - 2(1 - x^2)(4x)(x^2 + 1)}{(x^2 + 1)^4}$$

Factor out $4x(x^2 + 1)$:

$$f^{(3)}(x) = \frac{-4x(x^2 + 1) [(x^2 + 1) + 2(1 - x^2)]}{(x^2 + 1)^4}$$

Simplify:

$$(x^2 + 1) + 2(1 - x^2) = x^2 + 1 + 2 - 2x^2 = 3 - x^2$$

So:

$$f^{(3)}(x) = \frac{-4x(x^2 + 1)(3 - x^2)}{(x^2 + 1)^4} = \frac{-4x(3 - x^2)}{(x^2 + 1)^3}$$

(b) Evaluate at $x = 0$:

$$f(0) = \ln(1) = 0, \quad f'(0) = 0, \quad f''(0) = 2, \quad f^{(3)}(0) = \frac{-4(0)(3 - 0)}{1^3} = 0$$

(c) Taylor polynomial of degree 3 at $x = 0$:

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 = 0 + 0 + \frac{2}{2}x^2 + 0 = x^2$$

$$\boxed{T_3(x) = x^2}$$

Question 4

Let $f(x) = e^x \cos(x)$.

- (a) Compute the first three derivatives: $f'(x)$, $f''(x)$, and $f^{(3)}(x)$.
- (b) Find the value of $f''(0)$.
- (c) Determine the third-degree Taylor polynomial of $f(x)$ centered at $x = 0$.

Question 4 - Solution

We are given:

$$f(x) = e^x \cos(x)$$

(a) First three derivatives:

Use product rule: $f(x) = u(x)v(x)$, where $u = e^x$, $v = \cos(x)$

First derivative:

$$f'(x) = u'v + uv' = e^x \cos(x) - e^x \sin(x) = e^x(\cos(x) - \sin(x))$$

Second derivative:

$$\begin{aligned} f''(x) &= \frac{d}{dx}[e^x(\cos(x) - \sin(x))] = e^x(\cos(x) - \sin(x)) + e^x(-\sin(x) - \cos(x)) \\ &= e^x[(\cos(x) - \sin(x)) + (-\sin(x) - \cos(x))] = e^x(-2\sin(x)) \end{aligned}$$

Third derivative:

$$f^{(3)}(x) = \frac{d}{dx}[e^x(-2\sin(x))] = e^x(-2\sin(x)) + e^x(-2\cos(x)) = -2e^x(\sin(x) + \cos(x))$$

(b) Evaluate $f''(0)$:

$$f''(0) = e^0(-2\sin(0)) = 1 \cdot 0 = \boxed{0}$$

(c) Taylor polynomial of degree 3 at $x = 0$:

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3$$

Compute values:

$$f(0) = e^0 \cos(0) = 1, \quad f'(0) = e^0(\cos(0) - \sin(0)) = 1, \quad f''(0) = 0$$

$$f^{(3)}(0) = -2e^0(\sin(0) + \cos(0)) = -2(0 + 1) = -2$$

So:

$$T_3(x) = 1 + x + 0 + \frac{-2}{6}x^3 = 1 + x - \frac{1}{3}x^3$$

$$\boxed{T_3(x) = 1 + x - \frac{1}{3}x^3}$$

Question 5

Let $f(x) = \sin(x^2)$.

- (a) Find $f'(x)$, $f''(x)$, and $f^{(3)}(x)$.
- (b) Compute $f''(0)$.
- (c) Use your results to write the second-degree Taylor polynomial of $f(x)$ centered at $x = 0$.

Question 5 - Solution

We are given:

$$f(x) = \sin(x^2)$$

(a) First three derivatives:

First derivative:

$$f'(x) = \frac{d}{dx} \sin(x^2) = \cos(x^2) \cdot 2x = 2x \cos(x^2)$$

Second derivative: Use product rule:

$$f''(x) = \frac{d}{dx} (2x \cos(x^2)) = 2 \cos(x^2) + 2x \cdot (-\sin(x^2)) \cdot 2x = 2 \cos(x^2) - 4x^2 \sin(x^2)$$

Third derivative: Differentiate $f''(x)$:

$$f^{(3)}(x) = \frac{d}{dx} (2 \cos(x^2) - 4x^2 \sin(x^2))$$

First term:

$$\frac{d}{dx} [2 \cos(x^2)] = -2 \sin(x^2) \cdot 2x = -4x \sin(x^2)$$

Second term (product rule):

$$\frac{d}{dx} [4x^2 \sin(x^2)] = 8x \sin(x^2) + 4x^2 \cdot \cos(x^2) \cdot 2x = 8x \sin(x^2) + 8x^3 \cos(x^2)$$

So:

$$f^{(3)}(x) = -4x \sin(x^2) - (8x \sin(x^2) + 8x^3 \cos(x^2)) = -12x \sin(x^2) - 8x^3 \cos(x^2)$$

(b) Compute $f''(0)$:

$$f''(x) = 2 \cos(x^2) - 4x^2 \sin(x^2) \Rightarrow f''(0) = 2 \cos(0) - 0 = 2$$

(c) Taylor Polynomial $T_2(x)$:

We use:

$$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

$$f(0) = \sin(0) = 0, \quad f'(0) = 2(0) \cos(0) = 0, \quad f''(0) = 2$$

So:

$$T_2(x) = 0 + 0 + \frac{2}{2}x^2 = x^2$$

$$\boxed{T_2(x) = x^2}$$

Question 6

Let $f(x) = \frac{1}{1-x}$.

- (a) Find the first four derivatives $f'(x), f''(x), f^{(3)}(x), f^{(4)}(x)$.
- (b) Find a general formula for $f^{(n)}(x)$.
- (c) Evaluate $f^{(n)}(0)$ and write the Taylor series for $f(x)$ centered at $x = 0$.

Question 6 - Solution

We are given:

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

(a) Compute derivatives:

Using the chain rule,

$$\frac{d}{dx}(1-x)^{-1} = -1(1-x)^{-2} \cdot (-1) = (1-x)^{-2}$$

$$f'(x) = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{d}{dx}(1-x)^{-2} = -2(1-x)^{-3} \cdot (-1) = \frac{2}{(1-x)^3}$$

$$f^{(3)}(x) = \frac{d}{dx} \left(\frac{2}{(1-x)^3} \right) = 2 \cdot 3(1-x)^{-4} = \frac{6}{(1-x)^4}$$

$$f^{(4)}(x) = \frac{d}{dx} \left(\frac{6}{(1-x)^4} \right) = 6 \cdot 4(1-x)^{-5} = \frac{24}{(1-x)^5}$$

(b) General formula:

Each derivative introduces a factorial factor and increases the power of the denominator by 1:

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$$

(c) Evaluate at $x = 0$:

$$f^{(n)}(0) = \frac{n!}{1^{n+1}} = n!$$

Thus the Taylor series centered at $x = 0$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} x^n$$

$$f(x) = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

Question 7

Let $f(x) = x^2e^x$.

- (a) Find the first three derivatives of $f(x)$.
- (b) Find a general expression for $f^{(n)}(x)$.
- (c) Evaluate $f^{(3)}(0)$.

Question 7 - Solution

We are given:

$$f(x) = x^2 e^x$$

(a) Compute Derivatives:

We use the product rule: If $u = x^2$, $v = e^x$, then $f = uv$

$$f'(x) = (x^2)'e^x + x^2(e^x) = 2xe^x + x^2e^x = e^x(2x + x^2)$$

$$f''(x) = \frac{d}{dx}[e^x(2x + x^2)] = e^x(2 + 2x + 2x + x^2) = e^x(x^2 + 4x + 2)$$

$$f^{(3)}(x) = \frac{d}{dx}[e^x(x^2 + 4x + 2)] = e^x(x^2 + 4x + 2 + 2x + 4 + 0) = e^x(x^2 + 6x + 6)$$

(b) General Formula:

We notice a pattern in each derivative:

$$f^{(n)}(x) = e^x \cdot P_n(x)$$

where $P_n(x)$ is a degree-2 polynomial (since the original function had x^2).

This suggests:

$$f^{(n)}(x) = e^x \cdot (x^2 + 2nx + n(n-1))$$

(This can be proven by induction.)

(c) Evaluate $f^{(3)}(0)$:

From (a):

$$f^{(3)}(0) = e^0(0^2 + 6 \cdot 0 + 6) = 1 \cdot 6 = \boxed{6}$$

Question 8

Let $f(x) = \frac{1}{x}$.

- (a) Find the first four derivatives of $f(x)$.
- (b) Based on the pattern, write a general formula for the n -th derivative $f^{(n)}(x)$.
- (c) Evaluate $f^{(6)}(x)$.

Question 8 - Solution

We are given:

$$f(x) = \frac{1}{x} = x^{-1}$$

(a) Compute derivatives:

$$f'(x) = \frac{d}{dx}(x^{-1}) = -x^{-2}$$

$$f''(x) = \frac{d}{dx}(-x^{-2}) = 2x^{-3}$$

$$f^{(3)}(x) = \frac{d}{dx}(2x^{-3}) = -6x^{-4}$$

$$f^{(4)}(x) = \frac{d}{dx}(-6x^{-4}) = 24x^{-5}$$

(b) General Pattern:

Observe the signs and coefficients:

$$f^{(n)}(x) = (-1)^n \cdot n! \cdot x^{-(n+1)}$$

Answer: $f^{(n)}(x) = (-1)^n \cdot n! \cdot x^{-(n+1)}$

(c) Evaluate $f^{(6)}(x)$:

Use the formula:

$$f^{(6)}(x) = (-1)^6 \cdot 6! \cdot x^{-7} = 720x^{-7}$$

Answer: $f^{(6)}(x) = \frac{720}{x^7}$

Question 9

Let $f(x) = \cos(2x)$.

- (a) Find the first four derivatives of $f(x)$.
- (b) Find a general formula for the n -th derivative $f^{(n)}(x)$.
- (c) Evaluate $f^{(15)}(x)$.

Question 9 - Solution

Successive differentiation yields

$$f' = -2 \sin 2x, \quad f'' = -4 \cos 2x, \quad f^{(3)} = 8 \sin 2x, \quad f^{(4)} = 16 \cos 2x.$$

Each derivative multiplies by 2 and advances the phase by $\pi/2$; induction therefore gives

$$\boxed{f^{(n)}(x) = 2^n \cos(2x + n\pi/2)}.$$

Since $15\pi/2$ is congruent to $3\pi/2$ modulo 2π ,

$$\boxed{f^{(15)}(x) = 32768 \cos(2x + 3\pi/2) = 32768 \sin 2x}.$$

Question 10

Let

$$f(x) = \ln(1 + x^2).$$

- (a) Show that all odd-order derivatives of $f(x)$ evaluated at $x = 0$ are zero.
- (b) Find a closed-form expression for the $2n$ -th derivative of $f(x)$ evaluated at $x = 0$, where $n \in \mathbb{N}$.
- (c) Using your result from part (b), write the Maclaurin series for $f(x)$ in summation notation.
- (d) Determine the radius of convergence of the Maclaurin series and justify your answer.
- (e) Use the Maclaurin series to approximate $\ln(1.1)$ to within an error of 10^{-4} , and explain how you control the error.

Question 10 - Solution

We are given

$$f(x) = \ln(1 + x^2).$$

(a) Odd-order derivatives at $x = 0$

Observe that $f(x)$ is an **even function**, since

$$f(-x) = \ln(1 + (-x)^2) = \ln(1 + x^2) = f(x).$$

The derivative of an even function is odd, and the derivative of an odd function is even. Therefore:

- $f'(x)$ is odd,
- $f''(x)$ is even,
- $f^{(3)}(x)$ is odd,
- and so on.

All odd functions evaluate to zero at $x = 0$. Hence,

$$f^{(2k+1)}(0) = 0 \quad \text{for all } k \geq 0.$$

(b) Closed form for $f^{(2n)}(0)$

We begin with the known Taylor expansion:

$$\ln(1 + u) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{u^n}{n}, \quad |u| < 1.$$

Substitute $u = x^2$:

$$\ln(1 + x^2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}.$$

Comparing this with the Maclaurin series definition:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

we see that only even powers appear. Matching coefficients of x^{2n} , we obtain:

$$\frac{f^{(2n)}(0)}{(2n)!} = (-1)^{n+1} \frac{1}{n}.$$

Solving for $f^{(2n)}(0)$:

$$f^{(2n)}(0) = (-1)^{n+1} \frac{(2n)!}{n}.$$

(c) Maclaurin series

Using the result above, the Maclaurin series for $f(x)$ is:

$$\ln(1 + x^2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}, \quad |x| < 1.$$

(d) Radius of convergence

The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}$$

is a power series in x^2 .

Using the Ratio Test:

$$\lim_{n \rightarrow \infty} \left| \frac{x^{2(n+1)}/(n+1)}{x^{2n}/n} \right| = |x|^2.$$

The series converges when:

$$|x|^2 < 1 \quad \Rightarrow \quad |x| < 1.$$

Thus, the radius of convergence is:

$$\boxed{R = 1.}$$

(e) Approximating $\ln(1.1)$

We write:

$$\ln(1.1) = \ln(1 + 0.1) = \ln(1 + x^2) \quad \text{with } x^2 = 0.1.$$

Using the alternating series:

$$\ln(1.1) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.1)^n}{n}.$$

Since this is an alternating series with decreasing terms, the error after N terms is bounded by the magnitude of the next term:

$$|R_N| \leq \frac{(0.1)^{N+1}}{N+1}.$$

We test values:

$$\frac{(0.1)^3}{3} \approx 0.00033 > 10^{-4}, \quad \frac{(0.1)^4}{4} = 0.000025 < 10^{-4}.$$

Thus, using the first four terms guarantees the desired accuracy.

$$\boxed{\ln(1.1) \approx 0.1 - \frac{0.1^2}{2} + \frac{0.1^3}{3} - \frac{0.1^4}{4}}$$

Question 1

Use logarithmic differentiation to find the derivative of:

$$f(x) = (x^2 + 1)^x$$

Simplify your answer as much as possible.

Question 1 - Solution

We are given:

$$f(x) = (x^2 + 1)^x$$

Take the natural log of both sides:

$$\ln f(x) = \ln((x^2 + 1)^x) = x \ln(x^2 + 1)$$

Differentiate both sides implicitly:

$$\frac{1}{f(x)} \cdot f'(x) = \ln(x^2 + 1) + x \cdot \frac{1}{x^2 + 1} \cdot 2x = \ln(x^2 + 1) + \frac{2x^2}{x^2 + 1}$$

Multiply both sides by $f(x) = (x^2 + 1)^x$:

$$f'(x) = (x^2 + 1)^x \left[\ln(x^2 + 1) + \frac{2x^2}{x^2 + 1} \right]$$

Final Answer:

$$f'(x) = \boxed{(x^2 + 1)^x \left[\ln(x^2 + 1) + \frac{2x^2}{x^2 + 1} \right]}$$

Question 2

Let the function be:

$$f(x) = \frac{(x^2 + 1)^5 \cdot \sqrt{x + 3}}{(x - 1)^4}$$

- (a) Use logarithmic differentiation to find $f'(x)$.
- (b) Simplify your final expression as much as possible.

Question 2 - Solution

The real domain is $x \geq -3$, $x \neq 1$. Where the function is nonzero, use absolute values in logarithmic differentiation:

$$\ln |f(x)| = 5 \ln(x^2 + 1) + \frac{1}{2} \ln(x + 3) - 4 \ln |x - 1|$$

Differentiating gives

$$f'(x) = f(x) \left(\frac{10x}{x^2 + 1} + \frac{1}{2(x + 3)} - \frac{4}{x - 1} \right).$$

This formula holds for $x > -3$, $x \neq 1$. At $x = -3$ there is no finite derivative because of the square-root factor.

Question 3

Let the function be:

$$f(x) = \frac{(3x+1)^4 \cdot \ln(x)}{(x^2-9)^{3/2}}$$

- (a) Use logarithmic differentiation to find $f'(x)$.
- (b) Clearly indicate how you apply the logarithmic properties before differentiating.

Question 3 - Solution

We are given:

$$f(x) = \frac{(3x+1)^4 \cdot \ln(x)}{(x^2-9)^{3/2}}$$

Step 1: Take the natural log of both sides:

$$\ln f(x) = \ln \left(\frac{(3x+1)^4 \cdot \ln(x)}{(x^2-9)^{3/2}} \right)$$

Apply logarithmic identities:

$$\ln f(x) = 4 \ln(3x+1) + \ln(\ln x) - \frac{3}{2} \ln(x^2-9)$$

Step 2: Differentiate both sides implicitly:

$$\frac{f'(x)}{f(x)} = \frac{4 \cdot 3}{3x+1} + \frac{1}{\ln x} \cdot \frac{1}{x} - \frac{3}{2} \cdot \frac{2x}{x^2-9}$$

Simplify:

$$\frac{f'(x)}{f(x)} = \frac{12}{3x+1} + \frac{1}{x \ln x} - \frac{3x}{x^2-9}$$

Step 3: Multiply both sides by $f(x)$ to isolate $f'(x)$:

$$f'(x) = f(x) \left(\frac{12}{3x+1} + \frac{1}{x \ln x} - \frac{3x}{x^2-9} \right)$$

Substitute back the original $f(x)$:

$$f'(x) = \frac{(3x+1)^4 \cdot \ln(x)}{(x^2-9)^{3/2}} \left(\frac{12}{3x+1} + \frac{1}{x \ln x} - \frac{3x}{x^2-9} \right)$$

$$\boxed{f'(x) = \frac{(3x+1)^4 \cdot \ln(x)}{(x^2-9)^{3/2}} \left(\frac{12}{3x+1} + \frac{1}{x \ln x} - \frac{3x}{x^2-9} \right)}$$

Question 4

For $x > 1$ (so the base is positive), let the function be:

$$f(x) = \left(\frac{x^2 \cdot \sqrt{x+1}}{(x^2+4)^3 \cdot \ln x} \right)^{\sin x}$$

- (a) Use logarithmic differentiation to find $f'(x)$.
- (b) Simplify your answer as much as possible.

Question 4 - Solution

For $x > 1$, define the positive base

$$B(x) = \frac{x^2\sqrt{x+1}}{(x^2+4)^3 \ln x}, \quad f(x) = B(x)^{\sin x}.$$

Set

$$A(x) = \ln B(x) = 2 \ln x + \frac{1}{2} \ln(x+1) - 3 \ln(x^2+4) - \ln(\ln x).$$

Then $\ln f(x) = \sin x A(x)$ and

$$A'(x) = \frac{2}{x} + \frac{1}{2(x+1)} - \frac{6x}{x^2+4} - \frac{1}{x \ln x}.$$

Logarithmic differentiation yields

$$\boxed{f'(x) = B(x)^{\sin x} (\cos x A(x) + \sin x A'(x)), \quad x > 1.}$$

Here B, A, A' are the explicit expressions above.

Question 5

Let the function be:

$$f(x) = \frac{(x^2 + 1)^5 \cdot \sqrt[3]{\ln(x)}}{e^{x^3} \cdot (x - 4)^2}$$

- (a) Use logarithmic differentiation to find $f'(x)$.
- (b) Simplify your answer as much as possible.

Question 5 - Solution

The real domain is $x > 0$, $x \neq 4$. Where the function is nonzero, use absolute values in logarithmic differentiation:

$$\ln |f(x)| = 5 \ln(x^2 + 1) + \frac{1}{3} \ln |\ln x| - x^3 - 2 \ln |x - 4|$$

Differentiating gives

$$f'(x) = f(x) \left(\frac{10x}{x^2 + 1} + \frac{1}{3x \ln x} - 3x^2 - \frac{2}{x - 4} \right).$$

This formula holds for $x > 0$, $x \neq 1, 4$. At $x = 1$, $f(1) = 0$, but the cube-root factor gives no finite derivative.

Question 6

Let the function be:

$$f(x) = \frac{(x^2 + 3x + 5)^4 \cdot \sqrt{x^2 - 1}}{(x + 2)^5 \cdot (3x - 1)^{1/3}}$$

- (a) Use logarithmic differentiation to find $f'(x)$.
- (b) Simplify the result as much as possible.

Question 6 - Solution

The real domain is $|x| \geq 1$, $x \neq -2$. Where the function is nonzero, use absolute values in logarithmic differentiation:

$$\ln |f(x)| = 4 \ln(x^2 + 3x + 5) + \frac{1}{2} \ln(x^2 - 1) - 5 \ln |x + 2| - \frac{1}{3} \ln |3x - 1|$$

Differentiating gives

$$f'(x) = f(x) \left(\frac{4(2x+3)}{x^2+3x+5} + \frac{x}{x^2-1} - \frac{5}{x+2} - \frac{1}{3x-1} \right).$$

This formula holds for $|x| > 1$, $x \neq -2$. At $x = \pm 1$ there is no finite derivative. The cube root is the real cube root.

Question 7

Let the function be:

$$y = \left(\frac{(2x^2 + 1)^5}{\sqrt{5x - 3} \cdot (x^2 + 4)^3} \right)^2$$

- (a) Use logarithmic differentiation to find $\frac{dy}{dx}$.
- (b) Simplify the result as much as possible.

Question 7 - Solution

We are given:

$$y = \left(\frac{(2x^2 + 1)^5}{\sqrt{5x - 3} \cdot (x^2 + 4)^3} \right)^2$$

Take the natural logarithm of both sides:

$$\ln y = \ln \left[\left(\frac{(2x^2 + 1)^5}{(5x - 3)^{1/2} \cdot (x^2 + 4)^3} \right)^2 \right]$$

Bring the exponent outside:

$$\ln y = 2 \cdot \ln \left(\frac{(2x^2 + 1)^5}{(5x - 3)^{1/2} \cdot (x^2 + 4)^3} \right)$$

Use log rules:

$$\ln y = 2 \left[5 \ln(2x^2 + 1) - \frac{1}{2} \ln(5x - 3) - 3 \ln(x^2 + 4) \right]$$

Differentiate both sides:

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 \left[\frac{5 \cdot 4x}{2x^2 + 1} - \frac{1}{2} \cdot \frac{5}{5x - 3} - \frac{3 \cdot 2x}{x^2 + 4} \right]$$

Simplify:

$$\frac{dy}{dx} = y \cdot \left(\frac{40x}{2x^2 + 1} - \frac{5}{5x - 3} - \frac{12x}{x^2 + 4} \right)$$

Substitute y back:

$$\frac{dy}{dx} = \left(\frac{(2x^2 + 1)^5}{\sqrt{5x - 3} \cdot (x^2 + 4)^3} \right)^2 \cdot \left(\frac{40x}{2x^2 + 1} - \frac{5}{5x - 3} - \frac{12x}{x^2 + 4} \right)$$

Final Answer:

$$\boxed{\frac{dy}{dx} = \left(\frac{(2x^2 + 1)^5}{\sqrt{5x - 3} \cdot (x^2 + 4)^3} \right)^2 \cdot \left(\frac{40x}{2x^2 + 1} - \frac{5}{5x - 3} - \frac{12x}{x^2 + 4} \right)}$$

Question 8

Let the function be:

$$y = (x^2 + 3x + 1)^4 \cdot \sqrt{x^2 - 1} \cdot \frac{1}{(2x + 5)^3}$$

- (a) Use logarithmic differentiation to find $\frac{dy}{dx}$.
- (b) Simplify your answer as much as possible.

Question 8 - Solution

Put $A = x^2 + 3x + 1$. The real domain is $|x| \geq 1$, $x \neq -5/2$.

For $A \neq 0$ and $|x| > 1$, logarithmic differentiation gives

$$\ln |y| = 4 \ln |A| + \frac{1}{2} \ln(x^2 - 1) - 3 \ln |2x + 5|,$$

$$\frac{y'}{y} = \frac{4(2x + 3)}{A} + \frac{x}{x^2 - 1} - \frac{6}{2x + 5}.$$

Cancel before evaluating at zeros of A :

$$y' = \frac{A^3 [4(2x + 3)(x^2 - 1)(2x + 5) + Ax(2x + 5) - 6A(x^2 - 1)]}{\sqrt{x^2 - 1}(2x + 5)^4}.$$

This formula holds for $|x| > 1$, $x \neq -5/2$, including $x = (-3 - \sqrt{5})/2$, where $y' = 0$. The other root of A is outside the domain.

There is no finite derivative at $x = \pm 1$.

Question 9

Let

$$y = \frac{\sqrt{(x^2 + 4)^3(x - 1)^2}}{(x^2 - 1)^4}$$

- (a) Use logarithmic differentiation to find $\frac{dy}{dx}$.
- (b) Simplify your final answer fully.

Question 9 - Solution

For $x \neq \pm 1$, the square root simplifies to

$$y = \frac{(x^2 + 4)^{3/2}|x - 1|}{(x^2 - 1)^4} > 0.$$

Hence

$$\ln y = \frac{3}{2} \ln(x^2 + 4) + \ln|x - 1| - 4 \ln|x^2 - 1|.$$

Differentiating and multiplying by y gives

$$y' = \frac{(x^2 + 4)^{3/2}|x - 1|}{(x^2 - 1)^4} \left(\frac{3x}{x^2 + 4} + \frac{1}{x - 1} - \frac{8x}{x^2 - 1} \right), \quad x \neq \pm 1.$$

Question 10

Let

$$y = \frac{(x^2 + 1)^5 \cdot \sqrt[3]{x^2 - 4}}{\sqrt{2x + 1} \cdot (x - 3)^2}$$

- (a) Use logarithmic differentiation to find $\frac{dy}{dx}$.
- (b) Simplify your answer fully.

Question 10 - Solution

The real domain is $x > -\frac{1}{2}$, $x \neq 3$. Where the function is nonzero, use absolute values in logarithmic differentiation:

$$\ln |y| = 5 \ln(x^2 + 1) + \frac{1}{3} \ln |x^2 - 4| - \frac{1}{2} \ln(2x + 1) - 2 \ln |x - 3|$$

Differentiating gives

$$y' = y \left(\frac{10x}{x^2 + 1} + \frac{2x}{3(x^2 - 4)} - \frac{1}{2x + 1} - \frac{2}{x - 3} \right).$$

This formula holds for $x > -\frac{1}{2}$, $x \neq 2, 3$. At $x = 2$, $y = 0$, but there is no finite derivative because of the cube-root factor.