

Question 1

Let $f(x) = \sqrt{x+4}$. Find the slope of the secant line through the points where $x = 0$ and $x = 5$, and compare it with the slope of the tangent line at $x = 2$. Then, write the equation of the tangent line at $x = 2$.

Question 1 - Solution

We are given:

$$f(x) = \sqrt{x+4}$$

Step 1: Slope of secant line from $x = 0$ to $x = 5$.

$$f(0) = \sqrt{4} = 2, \quad f(5) = \sqrt{9} = 3$$

$$\text{Secant slope} = \frac{3-2}{5-0} = \frac{1}{5}$$

Step 2: Derivative of $f(x)$.

$$f'(x) = \frac{1}{2\sqrt{x+4}}$$

$$f'(2) = \frac{1}{2\sqrt{6}} \approx 0.204$$

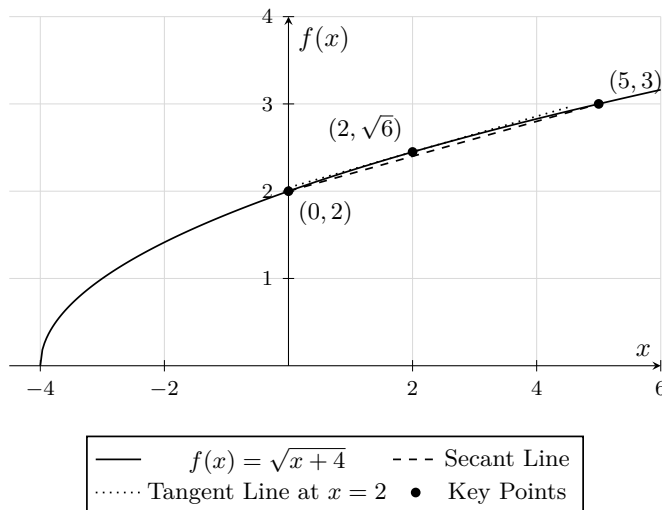
Step 3: Point at $x = 2$.

$$f(2) = \sqrt{6}$$

Step 4: Tangent line at $x = 2$:

$$y - \sqrt{6} = \frac{1}{2\sqrt{6}}(x - 2)$$

Comparison: The slope of the tangent line at $x = 2$ is slightly greater than the secant slope over the interval, showing that the graph is concave down.



Question 2

The displacement of a car traveling along a straight road is given by

$$s(t) = 10t - 0.5t^2$$

where s is in meters and t is in seconds.

- (a) Find the average velocity of the car over the interval $[2, 6]$.
- (b) Find the instantaneous velocity at $t = 2$.
- (c) Sketch the displacement function and the secant and tangent lines described above.

Question 2 - Solution

We are given:

$$s(t) = 10t - 0.5t^2$$

(a) **Average velocity on** $[2, 6]$:

$$s(2) = 10(2) - 0.5(4) = 20 - 2 = 18$$

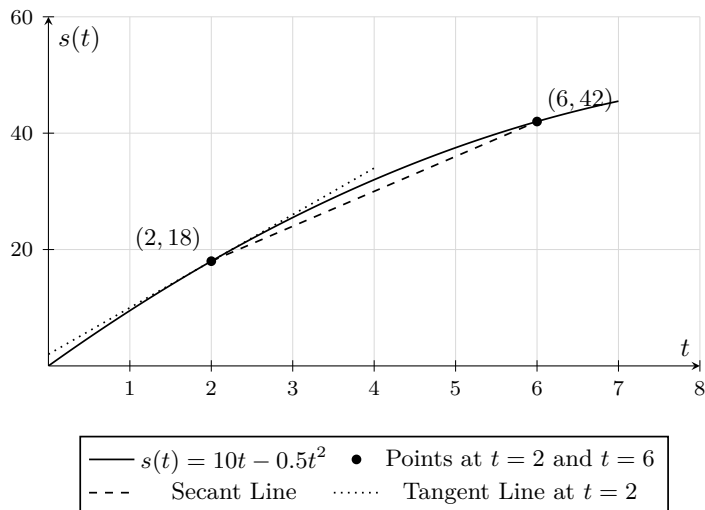
$$s(6) = 10(6) - 0.5(36) = 60 - 18 = 42$$

$$\text{Average velocity} = \frac{42 - 18}{6 - 2} = \frac{24}{4} = 6 \text{ m/s}$$

(b) **Instantaneous velocity at** $t = 2$:

$$s'(t) = 10 - t \Rightarrow s'(2) = 10 - 2 = 8 \text{ m/s}$$

(c) **Interpretation:** The average velocity between $t = 2$ and $t = 6$ is 6 m/s. At $t = 2$, the car is moving faster than average at 8 m/s. We now sketch the curve with the secant line between $t = 2$ and $t = 6$, and the tangent line at $t = 2$.



Question 3

A population of bacteria in a lab experiment is modeled by the function

$$P(t) = 200e^{0.3t},$$

where t is the time in hours since the start of the experiment and $P(t)$ is the number of bacteria.

- (a) Estimate the average rate of growth of the population from $t = 2$ to $t = 5$.
- (b) Find the instantaneous rate of growth at $t = 2$.
- (c) Sketch the population function along with the secant and tangent lines.

Question 3 - Solution

We are given:

$$P(t) = 200e^{0.3t}$$

(a) Average rate of change on $[2, 5]$:

$$P(2) = 200e^{0.6} \approx 200(1.8221) = 364.42$$

$$P(5) = 200e^{1.5} \approx 200(4.4817) = 896.34$$

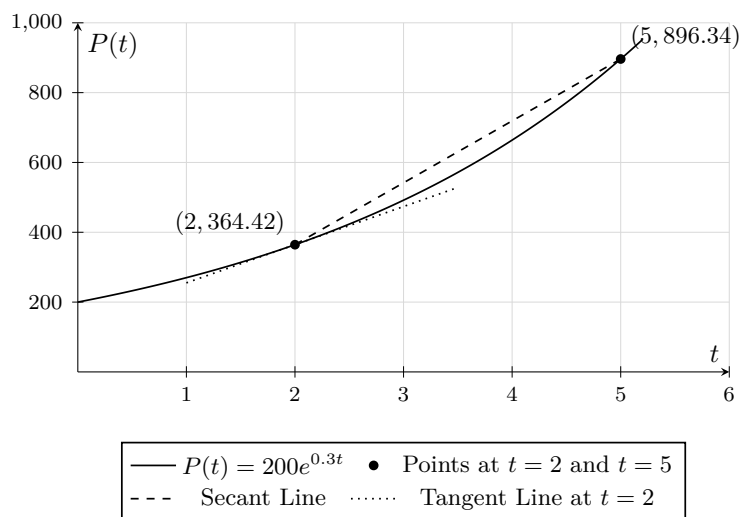
$$\text{Average rate} = \frac{896.34 - 364.42}{5 - 2} \approx \frac{531.92}{3} \approx 177.31 \text{ bacteria/hr}$$

(b) Instantaneous rate at $t = 2$:

$$P'(t) = 200 \cdot 0.3e^{0.3t} = 60e^{0.3t}$$

$$P'(2) = 60e^{0.6} \approx 60(1.8221) = 109.33 \text{ bacteria/hr}$$

(c) **Interpretation:** The population grows faster over the longer interval than it does exactly at $t = 2$, illustrating exponential acceleration.



Question 4

A ball is thrown vertically upward from the ground with an initial velocity of $v_0 = 20$ m/s. Its height at time t seconds is modeled by the function

$$h(t) = 20t - 4.9t^2.$$

- (a) Compute the average velocity over the interval $[1, 2]$.
- (b) Find the instantaneous velocity at $t = 1$.
- (c) Sketch the height function and both the secant and tangent lines involved.

Question 4 - Solution

Given:

$$h(t) = 20t - 4.9t^2$$

(a) Average velocity on $[1, 2]$:

$$h(1) = 20(1) - 4.9(1)^2 = 20 - 4.9 = 15.1$$

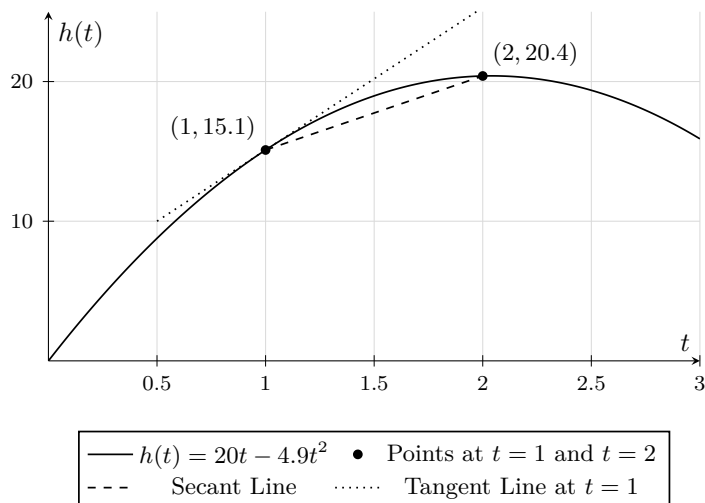
$$h(2) = 20(2) - 4.9(4) = 40 - 19.6 = 20.4$$

$$\text{Average velocity} = \frac{20.4 - 15.1}{2 - 1} = 5.3 \text{ m/s}$$

(b) Instantaneous velocity at $t = 1$:

$$h'(t) = 20 - 9.8t \Rightarrow h'(1) = 20 - 9.8 = 10.2 \text{ m/s}$$

(c) **Interpretation:** The ball's velocity at exactly $t = 1$ is greater than the average velocity over the interval, indicating it is still on the way up. Below is the sketch of the graph.



Question 5

The position of a particle moving along a straight path is given by the function

$$s(t) = t^3 - 6t^2 + 9t,$$

where $s(t)$ is in meters and t is in seconds.

- (a) Compute the average velocity of the particle on the interval $[1, 3]$.
- (b) Find the instantaneous velocity at $t = 1$.
- (c) Sketch the position function and include both the secant line over $[1, 3]$ and the tangent line at $t = 1$.

Question 5 - Solution

Given:

$$s(t) = t^3 - 6t^2 + 9t$$

(a) Average velocity on $[1, 3]$:

$$s(1) = 1^3 - 6(1)^2 + 9(1) = 1 - 6 + 9 = 4$$

$$s(3) = 27 - 54 + 27 = 0$$

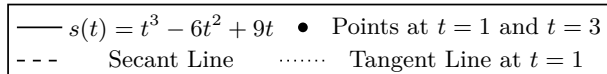
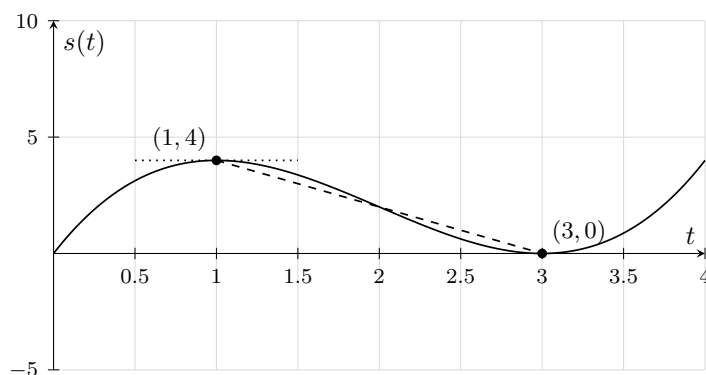
$$\text{Average velocity} = \frac{0 - 4}{3 - 1} = \frac{-4}{2} = -2 \text{ m/s}$$

(b) Instantaneous velocity at $t = 1$:

$$s'(t) = 3t^2 - 12t + 9$$

$$s'(1) = 3(1)^2 - 12(1) + 9 = 3 - 12 + 9 = 0 \text{ m/s}$$

(c) **Interpretation:** The particle is momentarily at rest at $t = 1$ (instantaneous velocity is 0), even though its average velocity between $t = 1$ and $t = 3$ is negative, indicating motion reversal.



Question 6

The height h (in meters) of a rocket launched vertically is given by the function

$$h(t) = 100 + 40t - 5t^2,$$

where t is time in seconds after launch.

- (a) Compute the average velocity of the rocket over the time interval $[2, 4]$.
- (b) Find the instantaneous velocity at $t = 2$.
- (c) Sketch the height function, the secant line over $[2, 4]$, and the tangent line at $t = 2$.

Question 6 - Solution

Given:

$$h(t) = 100 + 40t - 5t^2$$

(a) Average velocity on $[2, 4]$:

$$h(2) = 100 + 80 - 20 = 160$$

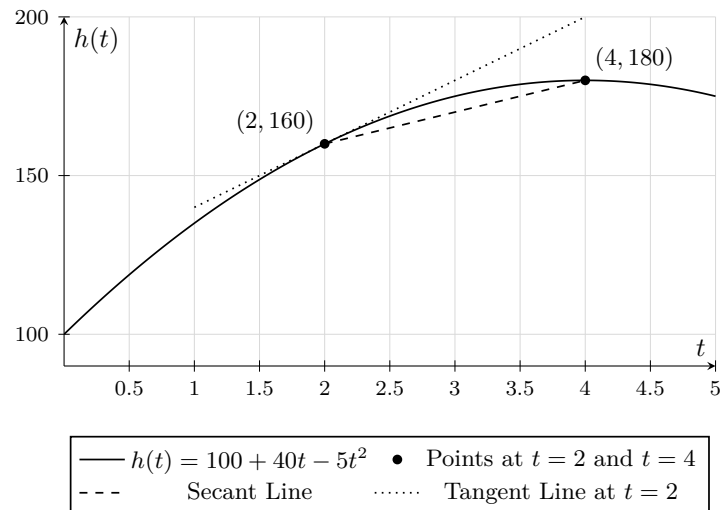
$$h(4) = 100 + 160 - 80 = 180$$

$$\text{Average velocity} = \frac{180 - 160}{4 - 2} = \frac{20}{2} = 10 \text{ m/s}$$

(b) Instantaneous velocity at $t = 2$:

$$h'(t) = 40 - 10t \Rightarrow h'(2) = 40 - 20 = 20 \text{ m/s}$$

(c) **Interpretation:** At $t = 2$, the rocket is climbing at a rate of 20 m/s, which is faster than its average velocity between $t = 2$ and $t = 4$.



Question 7

The temperature T (in degrees Celsius) of a cooling object is modeled by

$$T(t) = 25 + 60e^{-0.4t},$$

where t is time in minutes.

- (a) Compute the average rate of change of the temperature over the interval $[1, 4]$.
- (b) Find the instantaneous rate of change of the temperature at $t = 1$.
- (c) Sketch the temperature function, the secant line over $[1, 4]$, and the tangent line at $t = 1$.

Question 7 - Solution

Given:

$$T(t) = 25 + 60e^{-0.4t}$$

(a) Average rate of change on $[1, 4]$:

$$T(1) = 25 + 60e^{-0.4} \approx 25 + 60(0.6703) \approx 65.22$$

$$T(4) = 25 + 60e^{-1.6} \approx 25 + 60(0.2019) \approx 37.11$$

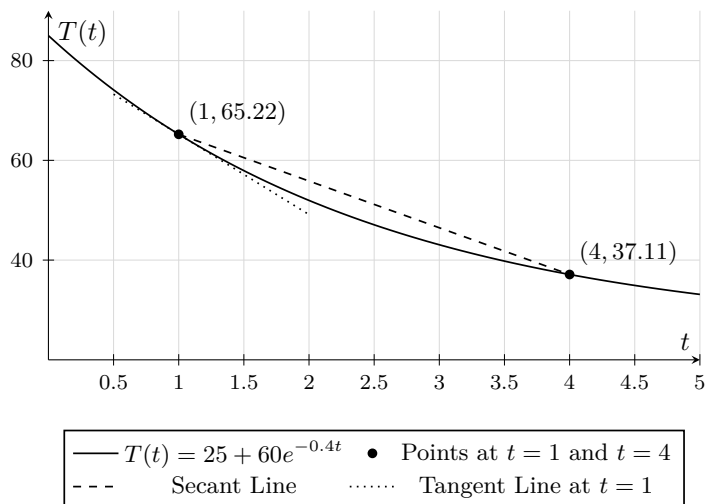
$$\text{Average rate} = \frac{37.11 - 65.22}{4 - 1} \approx \frac{-28.11}{3} \approx -9.37 \text{ }^{\circ}\text{C/min}$$

(b) Instantaneous rate of change at $t = 1$:

$$T'(t) = 60(-0.4)e^{-0.4t} = -24e^{-0.4t}$$

$$T'(1) = -24e^{-0.4} \approx -24(0.6703) \approx -16.09 \text{ }^{\circ}\text{C/min}$$

(c) **Interpretation:** At $t = 1$, the object is cooling at a faster rate than the average cooling over the interval $[1, 4]$, consistent with exponential decay.



Question 8

The population P (in thousands) of a bacteria culture after t hours is modeled by

$$P(t) = \frac{100t}{1 + 0.5t}.$$

- (a) Compute the average rate of change of the population on the interval $[1, 3]$.
- (b) Find the instantaneous rate of change at $t = 1$.
- (c) Sketch the population function, the secant line over $[1, 3]$, and the tangent line at $t = 1$.

Question 8 - Solution

Given:

$$P(t) = \frac{100t}{1 + 0.5t}$$

(a) Average rate of change on $[1, 3]$:

$$P(1) = \frac{100(1)}{1 + 0.5(1)} = \frac{100}{1.5} \approx 66.67$$

$$P(3) = \frac{100(3)}{1 + 0.5(3)} = \frac{300}{2.5} = 120$$

$$\text{Average rate} = \frac{120 - 66.67}{3 - 1} = \frac{53.33}{2} = 26.67 \text{ thousand/hour}$$

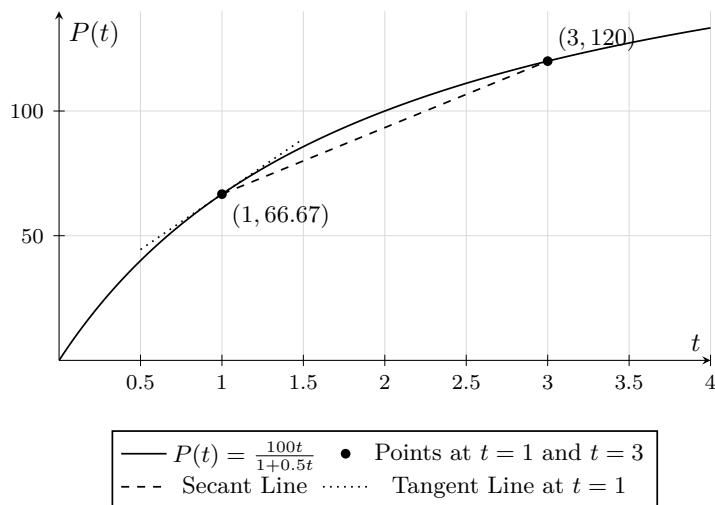
(b) Instantaneous rate of change at $t = 1$:

Use the quotient rule:

$$P'(t) = \frac{(1 + 0.5t)(100) - 100t(0.5)}{(1 + 0.5t)^2} = \frac{100 + 50t - 50t}{(1 + 0.5t)^2} = \frac{100}{(1 + 0.5t)^2}$$

$$P'(1) = \frac{100}{(1 + 0.5)^2} = \frac{100}{2.25} \approx 44.44 \text{ thousand/hour}$$

(c) **Interpretation:** The population is increasing more rapidly at $t = 1$ than the average over the interval. The tangent line shows the instantaneous trend, and the secant line reflects the average change.



Question 9

The revenue R (in dollars) generated by selling t hundred items is given by the function

$$R(t) = 200t - 3t^2.$$

- (a) Compute the average rate of change of revenue on the interval $[2, 6]$.
- (b) Find the instantaneous rate of change at $t = 2$.
- (c) Sketch the revenue function, the secant line over $[2, 6]$, and the tangent line at $t = 2$.

Question 9 - Solution

Given:

$$R(t) = 200t - 3t^2$$

(a) Average rate of change on $[2, 6]$:

$$R(2) = 200(2) - 3(2)^2 = 400 - 12 = 388$$

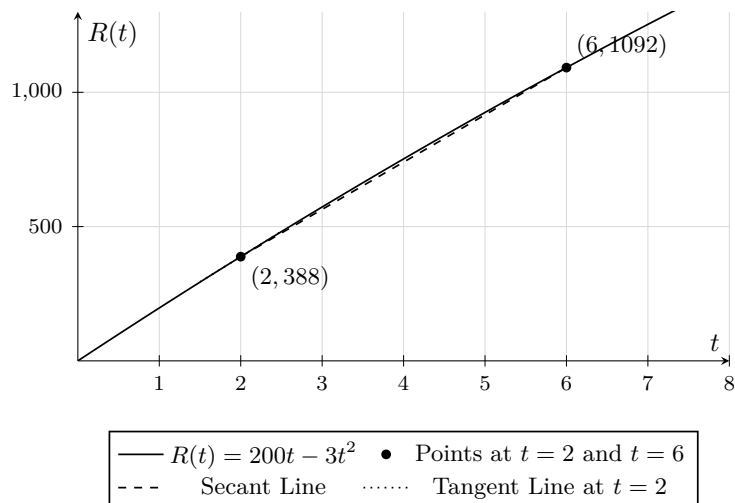
$$R(6) = 200(6) - 3(6)^2 = 1200 - 108 = 1092$$

$$\text{Average rate} = \frac{1092 - 388}{6 - 2} = \frac{704}{4} = 176 \text{ dollars per hundred items}$$

(b) Instantaneous rate of change at $t = 2$:

$$R'(t) = 200 - 6t \Rightarrow R'(2) = 200 - 12 = 188$$

(c) **Interpretation:** At $t = 2$, revenue is increasing faster than the average over the interval $[2, 6]$. The tangent line shows this steeper slope.



Question 10

The displacement $s(t)$ (in meters) of a particle moving along a straight line is given by

$$s(t) = t^3 - 6t^2 + 9t,$$

where t is time in seconds.

- (a) Find the average velocity of the particle over the interval $[1, 4]$.
- (b) Find the instantaneous velocity of the particle at $t = 1$.
- (c) Sketch the displacement function, the secant line over $[1, 4]$, and the tangent line at $t = 1$.

Question 10 - Solution

Given:

$$s(t) = t^3 - 6t^2 + 9t$$

(a) Average velocity on $[1, 4]$:

$$s(1) = 1^3 - 6(1)^2 + 9(1) = 1 - 6 + 9 = 4$$

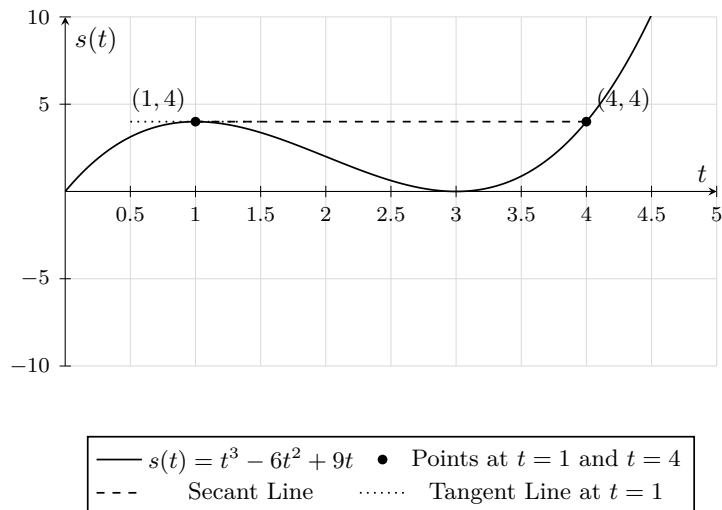
$$s(4) = 64 - 96 + 36 = 4$$

$$\text{Average velocity} = \frac{4 - 4}{4 - 1} = 0 \text{ m/s}$$

(b) Instantaneous velocity at $t = 1$:

$$s'(t) = 3t^2 - 12t + 9 \Rightarrow s'(1) = 3(1)^2 - 12(1) + 9 = 3 - 12 + 9 = 0 \text{ m/s}$$

(c) **Interpretation:** The average and instantaneous velocities are both zero, indicating the particle returns to its starting displacement at $t = 4$, and is momentarily at rest at $t = 1$.



Question 1

Evaluate the following limit, and explain your reasoning clearly:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{\sqrt{x} - 1}$$

Question 1 - Solution

We are asked to evaluate:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{\sqrt{x} - 1}$$

This limit gives an indeterminate form $\frac{0}{0}$, so we must simplify.
Start by factoring the numerator:

$$x^2 - 1 = (x - 1)(x + 1)$$

Now rewrite the expression:

$$\frac{(x - 1)(x + 1)}{\sqrt{x} - 1}$$

To simplify, multiply numerator and denominator by the ****conjugate**** of the denominator:

$$\frac{(x - 1)(x + 1)}{\sqrt{x} - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1}$$

This gives:

$$\frac{(x - 1)(x + 1)(\sqrt{x} + 1)}{x - 1}$$

Now cancel $x - 1$ (which is nonzero near $x = 1$):

$$(x + 1)(\sqrt{x} + 1)$$

Now substitute $x = 1$:

$$(1 + 1)(\sqrt{1} + 1) = 2 \cdot 2 = \boxed{4}$$

Question 2

Let the function be defined as:

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x < 2 \\ 3x - 2, & x \geq 2 \end{cases}$$

Determine whether $\lim_{x \rightarrow 2} f(x)$ exists. If it does, find its value. Justify your answer.

Question 2 - Solution

We are asked to evaluate:

$$\lim_{x \rightarrow 2} f(x)$$

where

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x < 2 \\ 3x - 2, & x \geq 2 \end{cases}$$

Step 1: Evaluate the Left-Hand Limit

For $x < 2$,

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2}$$

As long as $x \neq 2$, we can cancel $x - 2$:

$$f(x) = x + 2$$

So:

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x + 2) = 4$$

Step 2: Evaluate the Right-Hand Limit

For $x \geq 2$, $f(x) = 3x - 2$, so:

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (3x - 2) = 3(2) - 2 = 6 - 2 = 4$$

Step 3: Compare One-Sided Limits

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = 4 \Rightarrow \lim_{x \rightarrow 2} f(x) = \boxed{4}$$

Conclusion: The limit exists and is equal to $\boxed{4}$.

Question 3

Evaluate the limit, if it exists:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+3x} - \sqrt{1-x}}{x}$$

Explain your reasoning clearly.

Question 3 - Solution

We are asked to evaluate:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+3x} - \sqrt{1-x}}{x}$$

Direct substitution gives an indeterminate form $\frac{0}{0}$, so we simplify.

Multiply the numerator and denominator by the conjugate of the numerator:

$$\frac{\sqrt{1+3x} - \sqrt{1-x}}{x} \cdot \frac{\sqrt{1+3x} + \sqrt{1-x}}{\sqrt{1+3x} + \sqrt{1-x}}$$

This yields:

$$\frac{(1+3x) - (1-x)}{x(\sqrt{1+3x} + \sqrt{1-x})} = \frac{4x}{x(\sqrt{1+3x} + \sqrt{1-x})}$$

Cancel x :

$$\frac{4}{\sqrt{1+3x} + \sqrt{1-x}}$$

Now take the limit as $x \rightarrow 0$:

$$\frac{4}{\sqrt{1} + \sqrt{1}} = \frac{4}{2} = \boxed{2}$$

Question 4

Evaluate the following limit:

$$\lim_{x \rightarrow 5} \frac{x^2 - 25}{\sqrt{x+1} - \sqrt{6}}$$

Provide a clear justification for each step.

Question 4 - Solution

We want to evaluate:

$$\lim_{x \rightarrow 5} \frac{x^2 - 25}{\sqrt{x+1} - \sqrt{6}}$$

First, try direct substitution:

$$\frac{5^2 - 25}{\sqrt{5+1} - \sqrt{6}} = \frac{25 - 25}{\sqrt{6} - \sqrt{6}} = \frac{0}{0}$$

This is an indeterminate form, so we need to simplify the expression before evaluating the limit.

Factor the numerator:

$$x^2 - 25 = (x - 5)(x + 5)$$

So the limit becomes:

$$\lim_{x \rightarrow 5} \frac{(x - 5)(x + 5)}{\sqrt{x+1} - \sqrt{6}}$$

The denominator contains square roots, so we multiply by the conjugate:

$$\frac{(x - 5)(x + 5)}{\sqrt{x+1} - \sqrt{6}} \cdot \frac{\sqrt{x+1} + \sqrt{6}}{\sqrt{x+1} + \sqrt{6}}$$

This does not change the value of the expression because we are multiplying by 1.

Now simplify the denominator using the difference of squares formula:

$$(a - b)(a + b) = a^2 - b^2$$

Here,

$$a = \sqrt{x+1} \quad \text{and} \quad b = \sqrt{6}$$

Therefore,

$$(\sqrt{x+1} - \sqrt{6})(\sqrt{x+1} + \sqrt{6}) = (\sqrt{x+1})^2 - (\sqrt{6})^2$$

$$= x + 1 - 6$$

$$= x - 5$$

So the expression becomes:

$$\lim_{x \rightarrow 5} \frac{(x - 5)(x + 5)(\sqrt{x+1} + \sqrt{6})}{x - 5}$$

Since we are evaluating the limit as x approaches 5, we may cancel the common factor $x - 5$ for $x \neq 5$:

$$\lim_{x \rightarrow 5} (x + 5)(\sqrt{x+1} + \sqrt{6})$$

Now substitute $x = 5$:

$$(5 + 5)(\sqrt{5+1} + \sqrt{6})$$

$$= 10(\sqrt{6} + \sqrt{6})$$

$$= 10(2\sqrt{6})$$

$$= 20\sqrt{6}$$

Therefore,

$$\boxed{20\sqrt{6}}$$

Question 5

Evaluate the following limit:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{\tan(5x)}$$

Give a full explanation of your steps and reasoning.

Question 5 - Solution

We want to evaluate:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{\tan(5x)}$$

First, try direct substitution:

$$\frac{\sin(3 \cdot 0)}{\tan(5 \cdot 0)} = \frac{\sin(0)}{\tan(0)} = \frac{0}{0}$$

This is an indeterminate form, so we need to rewrite the expression before evaluating the limit.

We will use the standard trigonometric limits:

$$\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$$

and

$$\lim_{u \rightarrow 0} \frac{\tan u}{u} = 1$$

Since $3x \rightarrow 0$ and $5x \rightarrow 0$ as $x \rightarrow 0$, we can apply these limits to $\sin(3x)$ and $\tan(5x)$.

Start with the original expression:

$$\frac{\sin(3x)}{\tan(5x)}$$

We want to create the forms

$$\frac{\sin(3x)}{3x} \quad \text{and} \quad \frac{\tan(5x)}{5x}$$

To do this, multiply and divide by the needed factors:

$$\frac{\sin(3x)}{\tan(5x)} = \frac{\sin(3x)}{3x} \cdot \frac{5x}{\tan(5x)} \cdot \frac{3x}{5x}$$

This rewriting is valid because the factors we introduced balance each other out.

Now take the limit of each factor:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} = 1$$

Also,

$$\lim_{x \rightarrow 0} \frac{5x}{\tan(5x)} = 1$$

This is because

$$\lim_{x \rightarrow 0} \frac{\tan(5x)}{5x} = 1$$

so its reciprocal also approaches 1.

Finally,

$$\frac{3x}{5x} = \frac{3}{5}$$

for $x \neq 0$. Since limits only depend on values near 0, not necessarily at 0, this cancellation is allowed.

Therefore,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin(3x)}{\tan(5x)} &= 1 \cdot 1 \cdot \frac{3}{5} \\ &= \boxed{\frac{3}{5}} \end{aligned}$$

Question 6

Evaluate the limit:

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^3 - 8}$$

Provide detailed steps and justification for each transformation you make.

Question 6 - Solution

We want to evaluate:

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^3 - 8}$$

First, try direct substitution:

$$\frac{2^2 - 4}{2^3 - 8} = \frac{4 - 4}{8 - 8} = \frac{0}{0}$$

This is an indeterminate form, so we need to simplify the expression before evaluating the limit.

Now factor the numerator:

$$x^2 - 4 = (x - 2)(x + 2)$$

This is a difference of squares because:

$$x^2 - 4 = x^2 - 2^2$$

Next, factor the denominator:

$$x^3 - 8 = x^3 - 2^3$$

This is a difference of cubes. Recall the formula:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Here,

$$a = x \quad \text{and} \quad b = 2$$

So:

$$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$$

Now rewrite the original limit using these factorizations:

$$\lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{(x - 2)(x^2 + 2x + 4)}$$

Both the numerator and denominator contain the factor $x - 2$. Since we are finding the limit as x approaches 2, we only need to consider values of x near 2, but not equal to 2. Therefore, for $x \neq 2$, we may cancel the common factor $x - 2$:

$$\lim_{x \rightarrow 2} \frac{x + 2}{x^2 + 2x + 4}$$

Now the expression is no longer indeterminate, so we can substitute $x = 2$:

$$\begin{aligned} & \frac{2 + 2}{2^2 + 2(2) + 4} \\ &= \frac{4}{4 + 4 + 4} \\ &= \frac{4}{12} \\ &= \frac{1}{3} \end{aligned}$$

Therefore,

$$\boxed{\frac{1}{3}}$$

Question 7

Let

$$f(x) = \begin{cases} \frac{\sin(5x)}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

Determine the value of k that makes $f(x)$ continuous at $x = 0$.

Question 7 - Solution

We are given a piecewise function:

$$f(x) = \begin{cases} \frac{\sin(5x)}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

To make $f(x)$ continuous at $x = 0$, we must have:

$$\lim_{x \rightarrow 0} f(x) = f(0) = k$$

We now compute:

$$\lim_{x \rightarrow 0} \frac{\sin(5x)}{x}$$

We use the identity:

$$\lim_{x \rightarrow 0} \frac{\sin(ax)}{x} = a$$

So:

$$\lim_{x \rightarrow 0} \frac{\sin(5x)}{x} = 5$$

Thus, for continuity at $x = 0$, we must have:

$$k = \boxed{5}$$

Conclusion: The function is continuous at $x = 0$ if $\boxed{k = 5}$.

Question 8

Evaluate the following limit:

$$\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\sin(3x)}$$

Question 8 - Solution

We are asked to compute:

$$\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\sin(3x)}$$

First, substitute $x = 0$ into the numerator:

$$e^{2(0)} - 1 = e^0 - 1 = 1 - 1 = 0.$$

Now substitute $x = 0$ into the denominator:

$$\sin(3 \cdot 0) = \sin(0) = 0.$$

So the expression has the indeterminate form

$$\frac{0}{0}.$$

Since both the numerator and denominator approach 0, we can use L'Hopital's Rule.

Differentiate the numerator:

$$\frac{d}{dx} (e^{2x} - 1) = 2e^{2x}.$$

Differentiate the denominator:

$$\frac{d}{dx} (\sin(3x)) = 3 \cos(3x).$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\sin(3x)} = \lim_{x \rightarrow 0} \frac{2e^{2x}}{3 \cos(3x)}.$$

Now substitute $x = 0$:

$$\frac{2e^{2(0)}}{3 \cos(3 \cdot 0)} = \frac{2e^0}{3 \cos(0)}.$$

Since

$$e^0 = 1$$

and

$$\cos(0) = 1,$$

we get

$$\frac{2e^0}{3 \cos(0)} = \frac{2(1)}{3(1)} = \frac{2}{3}.$$

Thus,

$$\boxed{\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\sin(3x)} = \frac{2}{3}}$$

Question 9

Evaluate the limit:

$$\lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2}$$

Question 9 - Solution

We are given:

$$\lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2}$$

First, observe that:

$$x \rightarrow 4 \Rightarrow \text{numerator } x - 4 \rightarrow 0, \quad \text{denominator } \sqrt{x} - 2 \rightarrow 0$$

So, this is an indeterminate form $\frac{0}{0}$. We rationalize the denominator to resolve it. Multiply numerator and denominator by the conjugate of the denominator:

$$\frac{x - 4}{\sqrt{x} - 2} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2} = \frac{(x - 4)(\sqrt{x} + 2)}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \frac{(x - 4)(\sqrt{x} + 2)}{x - 4}$$

Now cancel the common factor $x - 4$, assuming $x \neq 4$:

$$= \sqrt{x} + 2$$

Now take the limit:

$$\lim_{x \rightarrow 4} \sqrt{x} + 2 = \sqrt{4} + 2 = 2 + 2 = \boxed{4}$$

Question 10

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x}$$

Question 10 - Solution

We are given:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x}$$

We want to simplify the expression to match a known limit. Recall the standard limit:

$$\lim_{u \rightarrow 0} \frac{\sin(u)}{u} = 1$$

Let $u = 3x$. As $x \rightarrow 0$, we also have $u \rightarrow 0$.

So:

$$\frac{\sin(3x)}{x} = \frac{\sin(3x)}{3x} \cdot 3$$

Now apply the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = \lim_{x \rightarrow 0} \left(\frac{\sin(3x)}{3x} \cdot 3 \right) = \left(\lim_{u \rightarrow 0} \frac{\sin(u)}{u} \right) \cdot 3 = 1 \cdot 3 = \boxed{3}$$

Question 1

Let the function f be defined as:

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & \text{if } x < 2 \\ x + 2, & \text{if } x \geq 2 \end{cases}$$

Evaluate:

$$\lim_{x \rightarrow 2^-} f(x) \quad \text{and} \quad \lim_{x \rightarrow 2^+} f(x)$$

State whether $\lim_{x \rightarrow 2} f(x)$ exists.

Question 1 - Solution

We analyze the left-hand and right-hand limits separately.

Left-hand limit:

For $x < 2$,

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2}$$

For $x \neq 2$, the $x - 2$ cancels:

$$f(x) = x + 2$$

So:

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x + 2) = 4$$

Right-hand limit:

For $x \geq 2$, we use the second piece:

$$f(x) = x + 2 \Rightarrow \lim_{x \rightarrow 2^+} f(x) = 4$$

Conclusion:

Since both one-sided limits exist and are equal:

$$\lim_{x \rightarrow 2} f(x) = \boxed{4}$$

Question 2

Let the function f be defined as follows:

$$f(x) = \begin{cases} 3x + 1, & x < 1 \\ x^2, & x \geq 1 \end{cases}$$

- (a) Compute $\lim_{x \rightarrow 1^-} f(x)$
- (b) Compute $\lim_{x \rightarrow 1^+} f(x)$
- (c) Does $\lim_{x \rightarrow 1} f(x)$ exist? Justify your answer.

Question 2 - Solution

We are given:

$$f(x) = \begin{cases} 3x + 1, & x < 1 \\ x^2, & x \geq 1 \end{cases}$$

(a) **Left-hand limit:** For $x < 1$, use $f(x) = 3x + 1$:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (3x + 1) = 3(1) + 1 = 4$$

(b) **Right-hand limit:** For $x \geq 1$, use $f(x) = x^2$:

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x^2 = 1^2 = 1$$

(c) **Two-sided limit:** Since the left and right limits are not equal:

$$\lim_{x \rightarrow 1^-} f(x) = 4 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = 1 \Rightarrow \lim_{x \rightarrow 1} f(x) \text{ does not exist}$$

Final Answer:

$\lim_{x \rightarrow 1^-} f(x) = 4, \quad \lim_{x \rightarrow 1^+} f(x) = 1, \quad \lim_{x \rightarrow 1} f(x) \text{ does not exist}$
--

Question 3

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x < 2 \\ x + 1, & x \geq 2 \end{cases}$$

- (a) Compute $\lim_{x \rightarrow 2^-} f(x)$
- (b) Compute $\lim_{x \rightarrow 2^+} f(x)$
- (c) Is $f(x)$ continuous at $x = 2$? Explain.

Question 3 - Solution

We are given:

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x < 2 \\ x + 1, & x \geq 2 \end{cases}$$

(a) **Left-hand limit:** First, simplify the expression for $x < 2$:

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2 \quad \text{for } x \neq 2$$

So,

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x + 2) = 2 + 2 = 4$$

(b) **Right-hand limit:**

$$f(x) = x + 1 \quad \text{for } x \geq 2$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x + 1) = 2 + 1 = 3$$

(c) **Continuity at $x = 2$:** Since

$$\lim_{x \rightarrow 2^-} f(x) = 4, \quad \lim_{x \rightarrow 2^+} f(x) = 3 \Rightarrow \lim_{x \rightarrow 2} f(x) \text{ does not exist}$$

And because the limit does not exist, the function is not continuous at $x = 2$.

Final Answer:

$\lim_{x \rightarrow 2^-} f(x) = 4, \quad \lim_{x \rightarrow 2^+} f(x) = 3, \quad \text{Not continuous at } x = 2$

Question 4

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} x^2 - 4, & x < 1 \\ 2x + 1, & x \geq 1 \end{cases}$$

- (a) Find $\lim_{x \rightarrow 1^-} f(x)$
- (b) Find $\lim_{x \rightarrow 1^+} f(x)$
- (c) Does $\lim_{x \rightarrow 1} f(x)$ exist? Explain.

Question 4 - Solution

We are given:

$$f(x) = \begin{cases} x^2 - 4, & x < 1 \\ 2x + 1, & x \geq 1 \end{cases}$$

(a) Left-hand limit:

For $x < 1$, we use $f(x) = x^2 - 4$:

$$\lim_{x \rightarrow 1^-} f(x) = 1^2 - 4 = -3$$

$$\boxed{\lim_{x \rightarrow 1^-} f(x) = -3}$$

(b) Right-hand limit:

For $x \geq 1$, we use $f(x) = 2x + 1$:

$$\lim_{x \rightarrow 1^+} f(x) = 2(1) + 1 = 3$$

$$\boxed{\lim_{x \rightarrow 1^+} f(x) = 3}$$

(c) Two-sided limit:

$$\lim_{x \rightarrow 1^-} f(x) = -3 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = 3$$

Since the one-sided limits are not equal, the overall limit does not exist:

$$\boxed{\lim_{x \rightarrow 1} f(x) \text{ does not exist}}$$

Question 5

Consider the function:

$$f(x) = \begin{cases} \frac{1}{x-3}, & x < 3 \\ \sqrt{x-3}, & x \geq 3 \end{cases}$$

- (a) Find $\lim_{x \rightarrow 3^-} f(x)$
- (b) Find $\lim_{x \rightarrow 3^+} f(x)$
- (c) Does $\lim_{x \rightarrow 3} f(x)$ exist? Justify your answer.

Question 5 - Solution

We are given:

$$f(x) = \begin{cases} \frac{1}{x-3}, & x < 3 \\ \sqrt{x-3}, & x \geq 3 \end{cases}$$

(a) Left-hand limit:

As $x \rightarrow 3^-$, we are approaching 3 from values less than 3. The expression is:

$$f(x) = \frac{1}{x-3}$$

As $x \rightarrow 3^-$, $x-3 \rightarrow 0^- \Rightarrow \frac{1}{x-3} \rightarrow -\infty$

$$\boxed{\lim_{x \rightarrow 3^-} f(x) = -\infty}$$

(b) Right-hand limit:

As $x \rightarrow 3^+$, the function becomes:

$$f(x) = \sqrt{x-3}$$

As $x \rightarrow 3^+$, $x-3 \rightarrow 0^+ \Rightarrow \sqrt{x-3} \rightarrow 0$

$$\boxed{\lim_{x \rightarrow 3^+} f(x) = 0}$$

(c) Does the limit exist?

Since

$$\lim_{x \rightarrow 3^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow 3^+} f(x) = 0$$

the two one-sided limits are not equal (and one diverges), so the limit $\lim_{x \rightarrow 3} f(x)$ does not exist.

$$\boxed{\text{Limit does not exist at } x = 3}$$

Question 6

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} \frac{\sin(\pi x)}{x}, & x < 0 \\ \frac{1}{x+1}, & x \geq 0 \end{cases}$$

- (a) Compute $\lim_{x \rightarrow 0^-} f(x)$
- (b) Compute $\lim_{x \rightarrow 0^+} f(x)$
- (c) Does $\lim_{x \rightarrow 0} f(x)$ exist? Explain.

Question 6 - Solution

We are given:

$$f(x) = \begin{cases} \frac{\sin(\pi x)}{x}, & x < 0 \\ \frac{1}{x+1}, & x \geq 0 \end{cases}$$

(a) Left-hand limit:

For $x < 0$, use:

$$\lim_{x \rightarrow 0^-} \frac{\sin(\pi x)}{x}$$

Using the identity $\lim_{x \rightarrow 0} \frac{\sin(\pi x)}{x} = \pi$ (from standard limit $\lim_{u \rightarrow 0} \frac{\sin(u)}{u} = 1$):

$$\lim_{x \rightarrow 0^-} \frac{\sin(\pi x)}{x} = \pi$$

$$\lim_{x \rightarrow 0^-} f(x) = \pi$$

(b) Right-hand limit:

For $x \geq 0$, use:

$$\lim_{x \rightarrow 0^+} \frac{1}{x+1} = \frac{1}{1} = 1$$

$$\lim_{x \rightarrow 0^+} f(x) = 1$$

(c) Two-sided limit:

Since:

$$\lim_{x \rightarrow 0^-} f(x) = \pi \quad \text{and} \quad \lim_{x \rightarrow 0^+} f(x) = 1$$

These are not equal, so the limit does not exist:

$$\lim_{x \rightarrow 0} f(x) \text{ does not exist}$$

Question 7

Consider the piecewise-defined function:

$$f(x) = \begin{cases} 3x + 1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$$

- (a) Compute $\lim_{x \rightarrow 2^-} f(x)$.
- (b) Compute $\lim_{x \rightarrow 2^+} f(x)$.
- (c) Is $\lim_{x \rightarrow 2} f(x)$ defined? Explain.
- (d) What is the value of $f(2)$?

Question 7 - Solution

We are given:

$$f(x) = \begin{cases} 3x + 1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$$

(a) Left-hand limit as $x \rightarrow 2^-$:

Use the definition for $x < 2$:

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (3x + 1) = 3(2) + 1 = 7$$

$$\boxed{\lim_{x \rightarrow 2^-} f(x) = 7}$$

(b) Right-hand limit as $x \rightarrow 2^+$:

Use the definition for $x \geq 2$:

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x^2 - 4) = 2^2 - 4 = 0$$

$$\boxed{\lim_{x \rightarrow 2^+} f(x) = 0}$$

(c) Two-sided limit:

$$\lim_{x \rightarrow 2^-} f(x) = 7, \quad \lim_{x \rightarrow 2^+} f(x) = 0$$

Since the one-sided limits are not equal:

$$\boxed{\lim_{x \rightarrow 2} f(x) \text{ does not exist}}$$

(d) Value of the function at 2:

Since $x = 2 \geq 2$, use the second case:

$$f(2) = 2^2 - 4 = 0$$

$$\boxed{f(2) = 0}$$

Question 8

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{if } x < 2 \\ 3x - 4, & \text{if } x \geq 2 \end{cases}$$

- (a) Find $\lim_{x \rightarrow 2^-} f(x)$.
- (b) Find $\lim_{x \rightarrow 2^+} f(x)$.
- (c) Determine whether $\lim_{x \rightarrow 2} f(x)$ exists.
- (d) Is f continuous at $x = 2$? Justify your answer.

Question 8 - Solution

We are given:

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{if } x < 2 \\ 3x-4, & \text{if } x \geq 2 \end{cases}$$

(a) Left-hand limit:

Note: $x^2 - 4 = (x - 2)(x + 2)$, so for $x \neq 2$,

$$\frac{x^2 - 4}{x - 2} = x + 2$$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x + 2) = 4$$

$\lim_{x \rightarrow 2^-} f(x) = 4$

(b) Right-hand limit:

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (3x - 4) = 6 - 4 = 2$$

$\lim_{x \rightarrow 2^+} f(x) = 2$

(c) Two-sided limit:

The left-hand and right-hand limits are not equal:

$$\lim_{x \rightarrow 2^-} f(x) = 4 \neq 2 = \lim_{x \rightarrow 2^+} f(x)$$

$\lim_{x \rightarrow 2} f(x) \text{ does not exist}$

(d) Continuity at $x = 2$:

$\lim_{x \rightarrow 2} f(x)$ does not exist, $f(2) = 3(2) - 4 = 6 - 4 = 2$ (defined)

But since the limit does not exist, f is not continuous at $x = 2$.

$f \text{ is not continuous at } x = 2$

Question 9

Consider the piecewise-defined function:

$$f(x) = \begin{cases} \frac{\sin(x)}{x}, & \text{if } x < 0 \\ 2x + 1, & \text{if } x \geq 0 \end{cases}$$

- (a) Evaluate $\lim_{x \rightarrow 0^-} f(x)$.
- (b) Evaluate $\lim_{x \rightarrow 0^+} f(x)$.
- (c) Does $\lim_{x \rightarrow 0} f(x)$ exist? Explain.
- (d) Find $f(0)$, if it exists.

Question 9 - Solution

We are given:

$$f(x) = \begin{cases} \frac{\sin(x)}{x}, & \text{if } x < 0 \\ 2x + 1, & \text{if } x \geq 0 \end{cases}$$

(a) Left-hand limit as $x \rightarrow 0^-$:

Since $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$, and this applies as $x \rightarrow 0^-$ as well:

$$\lim_{x \rightarrow 0^-} f(x) = 1$$

(b) Right-hand limit as $x \rightarrow 0^+$:

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (2x + 1) = 1$$

$$\lim_{x \rightarrow 0^+} f(x) = 1$$

(c) Two-sided limit:

Since both one-sided limits are equal:

$$\lim_{x \rightarrow 0} f(x) = 1$$

(d) Value at the point:

Use the definition for $x \geq 0$:

$$f(0) = 2(0) + 1 = 1$$

$$f(0) = 1$$

Question 10

Consider the piecewise-defined function:

$$f(x) = \begin{cases} x^2 - 3x + 2, & \text{if } x < 1 \\ \sqrt{x+3} - 2, & \text{if } x \geq 1 \end{cases}$$

- (a) Find $\lim_{x \rightarrow 1^-} f(x)$.
- (b) Find $\lim_{x \rightarrow 1^+} f(x)$.
- (c) Determine whether $\lim_{x \rightarrow 1} f(x)$ exists.
- (d) Is f continuous at $x = 1$? Justify your answer.

Question 10 - Solution

We are given:

$$f(x) = \begin{cases} x^2 - 3x + 2, & \text{if } x < 1 \\ \sqrt{x+3} - 2, & \text{if } x \geq 1 \end{cases}$$

(a) Left-hand limit:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^2 - 3x + 2) = 1^2 - 3(1) + 2 = 0$$

$$\boxed{\lim_{x \rightarrow 1^-} f(x) = 0}$$

(b) Right-hand limit:

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (\sqrt{x+3} - 2) = \sqrt{4} - 2 = 2 - 2 = 0$$

$$\boxed{\lim_{x \rightarrow 1^+} f(x) = 0}$$

(c) Two-sided limit:

Since both one-sided limits exist and are equal:

$$\boxed{\lim_{x \rightarrow 1} f(x) = 0}$$

(d) Continuity at $x = 1$:

We already have:

$$\lim_{x \rightarrow 1} f(x) = 0$$

Now evaluate the function at $x = 1$:

$$f(1) = \sqrt{1+3} - 2 = \sqrt{4} - 2 = 2 - 2 = 0$$

Since: $f(1)$ is defined , $\lim_{x \rightarrow 1} f(x) = f(1)$

We conclude:

$$\boxed{f \text{ is continuous at } x = 1}$$

Question 1

Let the functions $f(x)$, $g(x)$, and $h(x)$ be defined by the following limits:

$$\lim_{x \rightarrow 2} f(x) = 4, \quad \lim_{x \rightarrow 2} g(x) = -3, \quad \lim_{x \rightarrow 2} h(x) = 0$$

Evaluate the following limit expressions using limit laws and properties:

(a) $\lim_{x \rightarrow 2} [2f(x) + 5g(x)]$

(b) $\lim_{x \rightarrow 2} \left[\frac{f(x) \cdot g(x)}{h(x)} \right]$

(c) Explain whether the given information determines a signed infinite limit in part (b).

Assume the quotient denominator is nonzero in some deleted neighborhood of the approach point. Distinguish finite limits from signed infinite limits.

Question 1 - Solution**(a) Sum and constant-multiple laws**

Using the given limits,

$$\begin{aligned}\lim_{x \rightarrow 2} [2f(x) + 5g(x)] &= 2 \lim_{x \rightarrow 2} f(x) + 5 \lim_{x \rightarrow 2} g(x) \\ &= 2(4) + 5(-3) = \boxed{-7}.\end{aligned}$$

(b) Quotient with a zero denominator limit

The product law gives

$$\lim_{x \rightarrow 2} f(x)g(x) = 4(-3) = -12, \quad \lim_{x \rightarrow 2} h(x) = 0.$$

The quotient law does not apply because the denominator tends to zero. Since the numerator tends to a nonzero value, its magnitude stays bounded away from zero near $x = 2$. Therefore,

$$\left| \frac{f(x)g(x)}{h(x)} \right| \rightarrow \infty.$$

The quotient has no finite limit.

(c) Is the sign of an infinite limit determined?

No. Take $f(x) = 4$ and $g(x) = -3$. Each denominator below satisfies the given limit and is nonzero for $x \neq 2$:

Denominator $h(x)$	Behavior of $-12/h(x)$
$(x - 2)^2$	$\rightarrow -\infty$
$-(x - 2)^2$	$\rightarrow +\infty$
$x - 2$	opposite one-sided infinities

Thus more information about the sign of $h(x)$ is needed.

A signed infinite limit is not determined.

Question 2

Suppose that

$$\lim_{x \rightarrow 3} f(x) = 5 \quad \text{and} \quad \lim_{x \rightarrow 3} g(x) = 0.$$

Define

$$h(x) = \frac{f(x)^2 - 25}{g(x)}.$$

- (a) Explain why direct substitution does not determine $\lim_{x \rightarrow 3} h(x)$.
- (b) Use algebraic manipulation to rewrite $h(x)$.
- (c) State precisely what additional information would be required to evaluate $\lim_{x \rightarrow 3} h(x)$.

Question 2 - Solution

We are given:

$$\lim_{x \rightarrow 3} f(x) = 5, \quad \lim_{x \rightarrow 3} g(x) = 0, \quad h(x) = \frac{f(x)^2 - 25}{g(x)}.$$

(a) Indeterminate form

Substituting the known limits gives:

$$\lim_{x \rightarrow 3} h(x) = \frac{5^2 - 25}{0} = \frac{0}{0},$$

which is an indeterminate form. Therefore, the limit cannot be evaluated by direct substitution.

(b) Algebraic simplification

Factor the numerator:

$$f(x)^2 - 25 = (f(x) - 5)(f(x) + 5).$$

Then,

$$h(x) = \frac{(f(x) - 5)(f(x) + 5)}{g(x)} = \left(\frac{f(x) - 5}{g(x)} \right) (f(x) + 5).$$

Taking limits where possible,

$$\lim_{x \rightarrow 3} (f(x) + 5) = 10.$$

(c) Required additional information

The limit

$$\lim_{x \rightarrow 3} h(x)$$

exists if and only if the limit

$$\lim_{x \rightarrow 3} \frac{f(x) - 5}{g(x)}$$

exists.

If we let

$$L = \lim_{x \rightarrow 3} \frac{f(x) - 5}{g(x)},$$

then

$$\lim_{x \rightarrow 3} h(x) = 10L.$$

Without information about how fast $f(x) \rightarrow 5$ relative to $g(x) \rightarrow 0$, the value of the limit cannot be determined.

The limit cannot be evaluated without additional information.

Question 3

Given the following limits:

$$\lim_{x \rightarrow 1} f(x) = 4, \quad \lim_{x \rightarrow 1} g(x) = -2, \quad \lim_{x \rightarrow 1} h(x) = 0$$

Define:

$$k(x) = \frac{f(x) \cdot h(x) + g(x)}{h(x)}$$

Evaluate $\lim_{x \rightarrow 1} k(x)$, or explain why it does not exist.

Assume the quotient denominator is nonzero in some deleted neighborhood of the approach point. Distinguish finite limits from signed infinite limits.

Question 3 - Solution

Step 1: Apply the product and sum laws

For the numerator,

$$\lim_{x \rightarrow 1} (f(x)h(x) + g(x)) = 4(0) + (-2) = -2.$$

The denominator satisfies $\lim_{x \rightarrow 1} h(x) = 0$.

Step 2: Determine whether a finite limit exists

The numerator tends to a nonzero number, so its magnitude stays bounded away from zero near $x = 1$. Since $h(x)$ is nonzero in a deleted neighborhood,

$$|k(x)| = \left| \frac{f(x)h(x) + g(x)}{h(x)} \right| \rightarrow \infty.$$

There is no finite limit.

Step 3: Examine the sign

The sign depends on the denominator. For example, choose $f(x) = 4$ and $g(x) = -2$, so that $k(x) = 4 - 2/h(x)$.

Denominator $h(x)$	Behavior of $k(x)$
$(x - 1)^2$	$\rightarrow -\infty$
$-(x - 1)^2$	$\rightarrow +\infty$
$x - 1$	opposite one-sided infinities

All three choices satisfy the given limits. Therefore,

A signed infinite limit is not determined.

Division by a zero limiting value is not a valid use of the quotient law.

Question 4

Let

$$f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$

- (a) Find $\lim_{x \rightarrow 3} f(x)$ using limit properties.
- (b) Determine the value of k such that $f(x)$ is continuous at $x = 3$.

Question 4 - Solution

We are given:

$$f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$

(a) Find the limit:

First, factor the numerator:

$$x^2 - 9 = (x - 3)(x + 3)$$

So for $x \neq 3$, we can simplify:

$$f(x) = \frac{(x - 3)(x + 3)}{x - 3} = x + 3$$

Thus,

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} (x + 3) = 6$$

Answer: $\boxed{\lim_{x \rightarrow 3} f(x) = 6}$

(b) Value of k for Continuity:

For $f(x)$ to be continuous at $x = 3$, we must have:

$$\lim_{x \rightarrow 3} f(x) = f(3)$$

From (a), the limit is 6. Therefore, set $f(3) = k = 6$

Answer: $\boxed{k = 6}$

Question 5

Let $f(x) = \sqrt{4x + 5}$ and $g(x) = \frac{x^2 + 2x - 3}{x - 1}$, and assume $\lim_{x \rightarrow 1} f(x) = 3$.

(a) Find $\lim_{x \rightarrow 1} g(x)$.

(b) Use the limit properties to find $\lim_{x \rightarrow 1} [f(x) \cdot g(x)]$.

Question 5 - Solution

We are given:

$$f(x) = \sqrt{4x + 5}, \quad g(x) = \frac{x^2 + 2x - 3}{x - 1}, \quad \lim_{x \rightarrow 1} f(x) = 3$$

(a) Find $\lim_{x \rightarrow 1} g(x)$:

First, factor the numerator of $g(x)$:

$$x^2 + 2x - 3 = (x + 3)(x - 1)$$

$$g(x) = \frac{(x + 3)(x - 1)}{x - 1}$$

For $x \neq 1$, cancel common factors:

$$g(x) = x + 3$$

So,

$$\lim_{x \rightarrow 1} g(x) = \lim_{x \rightarrow 1} (x + 3) = 4$$

Answer: 4

(b) Find $\lim_{x \rightarrow 1} [f(x) \cdot g(x)]$:

Use the limit product law:

$$\lim_{x \rightarrow 1} [f(x) \cdot g(x)] = \lim_{x \rightarrow 1} f(x) \cdot \lim_{x \rightarrow 1} g(x) = 3 \cdot 4 = 12$$

Answer: 12

Question 6

Let $f(x)$ and $g(x)$ be functions such that:

$$\lim_{x \rightarrow 2} f(x) = 5, \quad \lim_{x \rightarrow 2} g(x) = -3$$

Use limit laws to compute the following:

(a) $\lim_{x \rightarrow 2} [3f(x) - 2g(x)]$

(b) $\lim_{x \rightarrow 2} \left[\frac{f(x)}{g(x)} \right]$

Question 6 - Solution

We are given:

$$\lim_{x \rightarrow 2} f(x) = 5, \quad \lim_{x \rightarrow 2} g(x) = -3$$

(a) Use limit laws for scalar multiplication and addition/subtraction:

$$\begin{aligned} \lim_{x \rightarrow 2} [3f(x) - 2g(x)] &= 3 \cdot \lim_{x \rightarrow 2} f(x) - 2 \cdot \lim_{x \rightarrow 2} g(x) \\ &= 3(5) - 2(-3) = 15 + 6 = \boxed{21} \end{aligned}$$

(b) Use the quotient limit law:

$$\lim_{x \rightarrow 2} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow 2} f(x)}{\lim_{x \rightarrow 2} g(x)} = \frac{5}{-3} = \boxed{-\frac{5}{3}}$$

Question 7

Given that

$$\lim_{x \rightarrow 1} f(x) = 4, \quad \lim_{x \rightarrow 1} g(x) = 0, \quad \lim_{x \rightarrow 1} h(x) = -2$$

compute the following limits using limit laws:

(a) $\lim_{x \rightarrow 1} [f(x)^2 + 3h(x)]$

(b) $\lim_{x \rightarrow 1} \left[\frac{f(x) + h(x)}{g(x)^2 + 1} \right]$

Question 7 - Solution

We are given:

$$\lim_{x \rightarrow 1} f(x) = 4, \quad \lim_{x \rightarrow 1} g(x) = 0, \quad \lim_{x \rightarrow 1} h(x) = -2$$

(a) Use the limit laws for powers and linear combinations:

$$\lim_{x \rightarrow 1} [f(x)^2 + 3h(x)] = (\lim_{x \rightarrow 1} f(x))^2 + 3 \cdot \lim_{x \rightarrow 1} h(x) = (4)^2 + 3(-2) = 16 - 6 = \boxed{10}$$

(b) Use limit laws for addition, powers, and quotients (denominator never zero):

$$\lim_{x \rightarrow 1} \left[\frac{f(x) + h(x)}{g(x)^2 + 1} \right] = \frac{\lim_{x \rightarrow 1} (f(x) + h(x))}{\lim_{x \rightarrow 1} (g(x)^2 + 1)} = \frac{4 + (-2)}{0^2 + 1} = \frac{2}{1} = \boxed{2}$$

Question 8

Assume the following limits exist:

$$\lim_{x \rightarrow -1} f(x) = 2, \quad \lim_{x \rightarrow -1} g(x) = 3$$

Evaluate the limit:

$$\lim_{x \rightarrow -1} [(f(x) - g(x))^2 + 4f(x)]$$

Use limit laws to justify each step.

Question 8 - Solution

We are given:

$$\lim_{x \rightarrow -1} f(x) = 2, \quad \lim_{x \rightarrow -1} g(x) = 3$$

We evaluate:

$$\lim_{x \rightarrow -1} [(f(x) - g(x))^2 + 4f(x)]$$

Step 1: Apply limit laws to each part

First, consider the difference:

$$\lim_{x \rightarrow -1} (f(x) - g(x)) = \lim_{x \rightarrow -1} f(x) - \lim_{x \rightarrow -1} g(x) = 2 - 3 = -1$$

Now square the result:

$$\lim_{x \rightarrow -1} (f(x) - g(x))^2 = (-1)^2 = 1$$

Next, evaluate the linear term:

$$\lim_{x \rightarrow -1} 4f(x) = 4 \cdot \lim_{x \rightarrow -1} f(x) = 4 \cdot 2 = 8$$

Step 2: Combine results

Using the sum rule for limits:

$$\lim_{x \rightarrow -1} [(f(x) - g(x))^2 + 4f(x)] = 1 + 8 = \boxed{9}$$

Question 9

Let $\lim_{x \rightarrow 3} f(x) = 5$ and $\lim_{x \rightarrow 3} g(x) = -2$.

Evaluate the following limit using limit laws:

$$\lim_{x \rightarrow 3} \left(\frac{f(x)^2 + 3g(x)}{g(x) - f(x)} \right)$$

Justify each step using properties of limits.

Question 9 - Solution

We are given:

$$\lim_{x \rightarrow 3} f(x) = 5, \quad \lim_{x \rightarrow 3} g(x) = -2$$

We need to evaluate:

$$\lim_{x \rightarrow 3} \left(\frac{f(x)^2 + 3g(x)}{g(x) - f(x)} \right)$$

Step 1: Use limit laws in numerator and denominator.

Numerator:

$$\lim_{x \rightarrow 3} f(x)^2 = \left(\lim_{x \rightarrow 3} f(x) \right)^2 = 5^2 = 25$$

$$\lim_{x \rightarrow 3} 3g(x) = 3 \cdot \lim_{x \rightarrow 3} g(x) = 3 \cdot (-2) = -6$$

$$\text{So, } \lim_{x \rightarrow 3} (f(x)^2 + 3g(x)) = 25 - 6 = 19$$

Denominator:

$$\lim_{x \rightarrow 3} (g(x) - f(x)) = \lim_{x \rightarrow 3} g(x) - \lim_{x \rightarrow 3} f(x) = -2 - 5 = -7$$

Step 2: Final Answer

$$\lim_{x \rightarrow 3} \left(\frac{f(x)^2 + 3g(x)}{g(x) - f(x)} \right) = \frac{19}{-7} = \boxed{-\frac{19}{7}}$$

Question 10

Let the functions $f(x)$, $g(x)$, and $h(x)$ satisfy the inequalities:

$$g(x) \leq f(x) \leq h(x)$$

for all x near 2 (but not necessarily at 2), and suppose:

$$\lim_{x \rightarrow 2} g(x) = 4 \quad \text{and} \quad \lim_{x \rightarrow 2} h(x) = 4$$

Use this information to determine:

$$\lim_{x \rightarrow 2} f(x)$$

Explain the reasoning behind your conclusion.

Question 10 - Solution

We are told that for all x near 2:

$$g(x) \leq f(x) \leq h(x)$$

and

$$\lim_{x \rightarrow 2} g(x) = 4, \quad \lim_{x \rightarrow 2} h(x) = 4$$

This is a perfect scenario to apply the "Squeeze Theorem", which states:

If $g(x) \leq f(x) \leq h(x)$ near $x = a$, and

$$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} f(x) = L$$

Applying this here:

$$\lim_{x \rightarrow 2} f(x) = \boxed{4}$$

Conclusion: By the Squeeze Theorem, since both bounding functions tend to 4 as $x \rightarrow 2$, so must $f(x)$.

Question 1

Evaluate the following limit, showing all necessary steps and justification:

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 4} - 2}{x}$$

Question 1 - Solution

We are given:

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 4} - 2}{x}$$

At $x = 0$, this becomes:

$$\frac{\sqrt{0+4}-2}{0} = \frac{2-2}{0} = \frac{0}{0}$$

This is an indeterminate form, so we must simplify the expression.

Step 1: Multiply numerator and denominator by the conjugate.

Multiply by:

$$\frac{\sqrt{x^2 + 4} + 2}{\sqrt{x^2 + 4} + 2}$$
$$\frac{\sqrt{x^2 + 4} - 2}{x} \cdot \frac{\sqrt{x^2 + 4} + 2}{\sqrt{x^2 + 4} + 2} = \frac{(\sqrt{x^2 + 4} - 2)(\sqrt{x^2 + 4} + 2)}{x(\sqrt{x^2 + 4} + 2)}$$

The numerator is a difference of squares:

$$(\sqrt{x^2 + 4})^2 - 2^2 = x^2 + 4 - 4 = x^2$$

So the entire expression becomes:

$$\frac{x^2}{x(\sqrt{x^2 + 4} + 2)} = \frac{x}{\sqrt{x^2 + 4} + 2}$$

Step 2: Evaluate the limit.

Now:

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{x^2 + 4} + 2} = \frac{0}{\sqrt{0+4} + 2} = \frac{0}{4} = 0$$

Final Answer:

$$\boxed{0}$$

Question 2

Evaluate the limit:

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$$

Question 2 - Solution

We are given:

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$$

Step 1: Check for indeterminate form.

At $x = 3$, the numerator becomes:

$$x^2 - 9 = 9 - 9 = 0$$

The denominator becomes:

$$x - 3 = 0$$

So we have the indeterminate form $\frac{0}{0}$. We must simplify.

Step 2: Factor the numerator.

$$x^2 - 9 = (x - 3)(x + 3)$$

$$\Rightarrow \frac{(x - 3)(x + 3)}{x - 3}$$

Cancel the common factor $x - 3$ (note: this is valid for $x \neq 3$):

$$= x + 3$$

Step 3: Take the limit of the simplified expression.

$$\lim_{x \rightarrow 3} x + 3 = 3 + 3 = \boxed{6}$$

Final Answer: $\boxed{6}$

Question 3

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$$

Question 3 - Solution

We are given:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$$

Step 1: Check for indeterminate form.

At $x = 0$, the expression becomes:

$$\frac{\sqrt{1+0} - 1}{0} = \frac{1 - 1}{0} = \frac{0}{0}$$

So we have the indeterminate form $\frac{0}{0}$. To resolve this, we rationalize the numerator.

Step 2: Multiply numerator and denominator by the conjugate of the numerator.

$$\frac{\sqrt{1+x} - 1}{x} \cdot \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1} = \frac{(1+x-1)}{x(\sqrt{1+x} + 1)} = \frac{x}{x(\sqrt{1+x} + 1)}$$

Cancel x from numerator and denominator:

$$= \frac{1}{\sqrt{1+x} + 1}$$

Step 3: Take the limit of the simplified expression.

$$\lim_{x \rightarrow 0} \frac{1}{\sqrt{1+x} + 1} = \frac{1}{\sqrt{1+1} + 1} = \frac{1}{2}$$

Final Answer:

$$\boxed{\frac{1}{2}}$$

Question 4

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x}$$

Question 4 - Solution

We are asked to evaluate:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x}$$

Step 1: Use standard limit identity

Recall the standard identity:

$$\lim_{x \rightarrow 0} \frac{\sin(kx)}{x} = k$$

This applies directly here with $k = 3$.

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = 3$$

Answer: 3

Question 5

Evaluate the following limit:

$$\lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2}$$

Question 5 - Solution

We are given:

$$\lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-2}$$

Step 1: Direct Substitution

Substitute $x = 4$:

$$\frac{4-4}{\sqrt{4}-2} = \frac{0}{0}$$

This is an indeterminate form, so we need to simplify.

Step 2: Use Algebraic Manipulation

We can rationalize the denominator. Multiply the numerator and denominator by the conjugate of the denominator:

$$\frac{x-4}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \frac{(x-4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)}$$

Simplify the denominator:

$$(\sqrt{x}-2)(\sqrt{x}+2) = x-4$$

So the expression becomes:

$$\frac{(x-4)(\sqrt{x}+2)}{x-4}$$

Cancel the common factor $x-4$ (note: valid as $x \rightarrow 4$, not at $x = 4$):

$$= \sqrt{x}+2$$

Now take the limit:

$$\lim_{x \rightarrow 4} \sqrt{x}+2 = \sqrt{4}+2 = 2+2 = \boxed{4}$$

Question 6

Evaluate the limit:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$$

Question 6 - Solution

We are asked to evaluate:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$$

Step 1: Factor the numerator

Recall that $x^2 - 1 = (x - 1)(x + 1)$. So,

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1}$$

For $x \neq 1$, we can cancel $x - 1$ from numerator and denominator:

$$= x + 1$$

Step 2: Take the limit

Now that the expression simplifies to $x + 1$, we compute:

$$\lim_{x \rightarrow 1} (x + 1) = 2$$

Answer: 2

Question 7

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+3x} - \sqrt{1+x}}{x}.$$

Question 7 - Solution

We evaluate

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+3x} - \sqrt{1+x}}{x}.$$

Step 1: Identify the Indeterminate Form

Substituting $x = 0$ gives

$$\frac{\sqrt{1} - \sqrt{1}}{0} = \frac{0}{0},$$

which is an indeterminate form. Algebraic simplification is required.

Step 2: Multiply by the Conjugate

Multiply the numerator and denominator by the conjugate of the numerator:

$$\frac{\sqrt{1+3x} - \sqrt{1+x}}{x} \cdot \frac{\sqrt{1+3x} + \sqrt{1+x}}{\sqrt{1+3x} + \sqrt{1+x}}.$$

This yields

$$\frac{(1+3x) - (1+x)}{x(\sqrt{1+3x} + \sqrt{1+x})} = \frac{2x}{x(\sqrt{1+3x} + \sqrt{1+x})}.$$

Cancel the common factor of x :

$$\frac{2}{\sqrt{1+3x} + \sqrt{1+x}}.$$

Step 3: Evaluate the Limit

Now substitute $x = 0$:

$$\lim_{x \rightarrow 0} \frac{2}{\sqrt{1+3x} + \sqrt{1+x}} = \frac{2}{1+1} = 1.$$

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Question 8

Evaluate the limit:

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x} - x)$$

Question 8 - Solution

We are asked to evaluate:

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x} - x)$$

Step 1: Indeterminate Form

As $x \rightarrow \infty$, both $\sqrt{x^2 + 5x}$ and x grow large. The expression becomes $\infty - \infty$, an indeterminate form. We simplify.

Step 2: Multiply by the Conjugate

Multiply numerator and denominator by the conjugate:

$$\begin{aligned} & (\sqrt{x^2 + 5x} - x) \cdot \frac{\sqrt{x^2 + 5x} + x}{\sqrt{x^2 + 5x} + x} \\ &= \frac{(x^2 + 5x) - x^2}{\sqrt{x^2 + 5x} + x} = \frac{5x}{\sqrt{x^2 + 5x} + x} \end{aligned}$$

Step 3: Divide numerator and denominator by x

$$= \frac{5}{\sqrt{1 + \frac{5}{x}} + 1}$$

As $x \rightarrow \infty$, $\frac{5}{x} \rightarrow 0$, so:

$$\lim_{x \rightarrow \infty} \frac{5}{\sqrt{1 + \frac{5}{x}} + 1} = \frac{5}{1 + 1} = \frac{5}{2}$$

Answer:

$$\boxed{\frac{5}{2}}$$

Question 9

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{\tan(5x)}.$$

Question 9 - Solution

We evaluate

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{\tan(5x)}.$$

Step 1: Express tangent in terms of sine and cosine

Recall that

$$\tan(5x) = \frac{\sin(5x)}{\cos(5x)}.$$

Thus,

$$\frac{\sin(3x)}{\tan(5x)} = \frac{\sin(3x) \cos(5x)}{\sin(5x)}.$$

Step 2: Apply standard trigonometric limits

Rewrite the expression as

$$\frac{\sin(3x)}{3x} \cdot \frac{5x}{\sin(5x)} \cdot \frac{3}{5} \cdot \cos(5x).$$

Now take limits term by term as $x \rightarrow 0$:

$$\frac{\sin(3x)}{3x} \rightarrow 1, \quad \frac{5x}{\sin(5x)} \rightarrow 1, \quad \cos(5x) \rightarrow 1.$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{\tan(5x)} = 1 \cdot 1 \cdot \frac{3}{5} \cdot 1 = \frac{3}{5}.$$

$$\boxed{\frac{3}{5}}$$

Question 10

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{1 - \cos(4x)}{x^2}$$

Question 10 - Solution

We are given the limit:

$$\lim_{x \rightarrow 0} \frac{1 - \cos(4x)}{x^2}$$

Step 1: Use the identity

$$1 - \cos(\theta) = 2 \sin^2\left(\frac{\theta}{2}\right)$$

Apply this identity to the numerator:

$$1 - \cos(4x) = 2 \sin^2(2x)$$

So the expression becomes:

$$\lim_{x \rightarrow 0} \frac{2 \sin^2(2x)}{x^2}$$

Step 2: Rewrite the sine term

$$= \lim_{x \rightarrow 0} 2 \cdot \left(\frac{\sin(2x)}{x}\right)^2$$

We can write:

$$\frac{\sin(2x)}{x} = \frac{\sin(2x)}{2x} \cdot 2 \rightarrow 1 \cdot 2 = 2 \quad \text{as } x \rightarrow 0$$

So:

$$\left(\frac{\sin(2x)}{x}\right)^2 \rightarrow 4 \Rightarrow 2 \cdot 4 = 8$$

Final Answer: 8

Question 1

Evaluate the following limit and describe the behavior of the function near the point of discontinuity:

$$\lim_{x \rightarrow 1^-} \frac{3}{(x-1)^2}$$

Question 1 - Solution

We are asked to evaluate:

$$\lim_{x \rightarrow 1^-} \frac{3}{(x-1)^2}$$

Step 1: Observe the behavior of the denominator

As $x \rightarrow 1^-$, the value of $x - 1$ becomes a small negative number. Squaring it makes it a small positive number:

$$(x-1)^2 > 0 \quad \text{but very small}$$

So the expression becomes:

$$\frac{3}{\text{small positive number}} \rightarrow +\infty$$

Conclusion:

The function grows without bound as x approaches 1 from the left.

$$\boxed{\lim_{x \rightarrow 1^-} \frac{3}{(x-1)^2} = \infty}$$

Question 2

Evaluate the limit:

$$\lim_{x \rightarrow 0^+} \frac{\tan x}{x}.$$

Question 2 - Solution

We evaluate

$$\lim_{x \rightarrow 0^+} \frac{\tan x}{x}.$$

Method 1: Local Approximation

As $x \rightarrow 0$, the Taylor expansion of $\tan x$ is

$$\tan x = x + \frac{x^3}{3} + \cdots.$$

Thus,

$$\frac{\tan x}{x} = \frac{x + \frac{x^3}{3} + \cdots}{x} = 1 + \frac{x^2}{3} + \cdots.$$

As $x \rightarrow 0^+$, the higher-order terms approach 0, so

$$\lim_{x \rightarrow 0^+} \frac{\tan x}{x} = 1.$$

Method 2: Rewrite Using Sine and Cosine

Recall that

$$\tan x = \frac{\sin x}{\cos x}.$$

Then,

$$\frac{\tan x}{x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x}.$$

Now evaluate each factor as $x \rightarrow 0^+$:

$$\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1, \quad \lim_{x \rightarrow 0^+} \cos x = 1.$$

Therefore,

$$\lim_{x \rightarrow 0^+} \frac{\tan x}{x} = 1 \cdot \frac{1}{1} = 1.$$

Conclusion

$$\boxed{\lim_{x \rightarrow 0^+} \frac{\tan x}{x} = 1}$$

Question 3

Evaluate the following limit:

$$\lim_{x \rightarrow 3^-} \frac{1}{(x-3)^2}$$

Question 3 - Solution

We are given:

$$\lim_{x \rightarrow 3^-} \frac{1}{(x-3)^2}$$

Step 1: Analyze the behavior of the denominator

As $x \rightarrow 3^-$, we are approaching 3 from the left. The expression $x - 3$ becomes a small negative number. But since it is squared:

$$(x-3)^2 \rightarrow 0^+ \quad (\text{very small positive number})$$

Step 2: Behavior of the function

Since we are dividing by a very small positive number, the entire expression becomes very large:

$$\frac{1}{(x-3)^2} \rightarrow +\infty$$

Conclusion:

$$\boxed{\lim_{x \rightarrow 3^-} \frac{1}{(x-3)^2} = \infty}$$

Question 4

Evaluate the following limit:

$$\lim_{x \rightarrow 1^+} \frac{5}{x - 1}$$

Question 4 - Solution

We are given:

$$\lim_{x \rightarrow 1^+} \frac{5}{x-1}$$

Step 1: Understand the behavior of the denominator

As $x \rightarrow 1^+$, we are approaching 1 from the right. This means $x - 1$ is a small positive number.

Step 2: Behavior of the whole expression

$$\frac{5}{x-1} \rightarrow \frac{5}{\text{small positive number}} \rightarrow +\infty$$

Conclusion:

$$\boxed{\lim_{x \rightarrow 1^+} \frac{5}{x-1} = \infty}$$

Question 5

Evaluate the following limit:

$$\lim_{x \rightarrow 0^-} \frac{1}{x^2}$$

Question 5 - Solution

We are given:

$$\lim_{x \rightarrow 0^-} \frac{1}{x^2}$$

Step 1: Analyze the expression

As $x \rightarrow 0^-$, x is approaching 0 from the left. So x is negative, but very close to 0. Now, consider the square of x :

$$x^2 > 0 \text{ for all } x \neq 0$$

Even though x is negative, x^2 is positive. As $x \rightarrow 0^-$, $x^2 \rightarrow 0^+$

Step 2: Behavior of the expression

$$\frac{1}{x^2} \rightarrow \frac{1}{\text{small positive number}} \rightarrow \infty$$

Conclusion:

$$\boxed{\lim_{x \rightarrow 0^-} \frac{1}{x^2} = \infty}$$

Question 6

Evaluate the limit:

$$\lim_{x \rightarrow -2^+} \frac{4}{(x+2)^3}$$

Question 6 - Solution

We are asked to evaluate:

$$\lim_{x \rightarrow -2^+} \frac{4}{(x+2)^3}$$

Step 1: Analyze the denominator

As $x \rightarrow -2^+$, we approach -2 from the right, so:

$$x + 2 \rightarrow 0^+ \quad (\text{a small positive number})$$

Now consider the cube:

$$(x + 2)^3 \rightarrow 0^+ \quad (\text{still positive, but very small})$$

Step 2: Behavior of the expression

Since the numerator is positive and the denominator approaches a very small positive number:

$$\frac{4}{(x+2)^3} \rightarrow +\infty$$

Conclusion:

$$\boxed{\lim_{x \rightarrow -2^+} \frac{4}{(x+2)^3} = \infty}$$

Question 7

Evaluate the limit:

$$\lim_{x \rightarrow 1^-} \frac{x^2 + 3x - 4}{x - 1}$$

Question 7 - Solution

We are given:

$$\lim_{x \rightarrow 1^-} \frac{x^2 + 3x - 4}{x - 1}$$

Step 1: Factor the numerator

Factor the quadratic:

$$x^2 + 3x - 4 = (x + 4)(x - 1)$$

So the expression becomes:

$$\frac{(x + 4)(x - 1)}{x - 1}$$

We can cancel $x - 1$ when $x \neq 1$:

$$= x + 4 \quad \text{for } x \neq 1$$

So the original expression simplifies to $x + 4$, and we can evaluate the limit directly.

Step 2: Evaluate the limit

Since we simplified the expression:

$$\lim_{x \rightarrow 1^-} (x + 4) = 1 + 4 = 5$$

Conclusion:

$$\boxed{\lim_{x \rightarrow 1^-} \frac{x^2 + 3x - 4}{x - 1} = 5}$$

Question 8

Evaluate the limit:

$$\lim_{x \rightarrow 2^+} \frac{1}{x - 2}$$

Question 8 - Solution

We are given the limit:

$$\lim_{x \rightarrow 2^+} \frac{1}{x-2}$$

Step 1: Analyze the expression near $x = 2^+$

As x approaches 2 from the right, the value of $x - 2$ becomes a very small positive number. Therefore, $\frac{1}{x-2}$ becomes a very large positive number.

$$x \rightarrow 2^+ \quad \Rightarrow \quad x - 2 \rightarrow 0^+ \quad \Rightarrow \quad \frac{1}{x-2} \rightarrow +\infty$$

Conclusion:

$$\boxed{\lim_{x \rightarrow 2^+} \frac{1}{x-2} = +\infty}$$

This indicates that the function grows without bound as x approaches 2 from the right.

Question 9

Evaluate the limit:

$$\lim_{x \rightarrow 0^-} \frac{2x + 1}{x}$$

Question 9 - Solution

We are asked to evaluate:

$$\lim_{x \rightarrow 0^-} \frac{2x + 1}{x}$$

Step 1: Behavior near zero from the left

As $x \rightarrow 0^-$, the numerator $2x + 1 \rightarrow 1$, since $2x \rightarrow 0^-$ and 1 is a constant.

The denominator $x \rightarrow 0^-$, which is a small negative number.

So the entire fraction behaves like:

$$\frac{2x + 1}{x} \approx \frac{1}{\text{tiny negative}} = -\infty$$

Step 2: More formal analysis (optional)

We can write:

$$\frac{2x + 1}{x} = 2 + \frac{1}{x}$$

As $x \rightarrow 0^-$, $\frac{1}{x} \rightarrow -\infty$, and the constant 2 becomes negligible in comparison.

Conclusion:

$$\lim_{x \rightarrow 0^-} \frac{2x + 1}{x} = -\infty$$

Question 10

Evaluate the limit:

$$\lim_{x \rightarrow 3^+} \frac{5}{x - 3}$$

Question 10 - Solution

We are given:

$$\lim_{x \rightarrow 3^+} \frac{5}{x-3}$$

Step 1: Analyze the denominator

As $x \rightarrow 3^+$, the value of x is slightly greater than 3, so $x - 3$ is a small positive number.

$$x - 3 \rightarrow 0^+ \Rightarrow \frac{1}{x-3} \rightarrow +\infty$$

Multiplying by 5, we have:

$$\frac{5}{x-3} \rightarrow +\infty$$

Conclusion:

$$\boxed{\lim_{x \rightarrow 3^+} \frac{5}{x-3} = +\infty}$$

Question 1

Evaluate the limit:

$$\lim_{x \rightarrow \infty} \frac{6x^3 - 4x^2 + 10}{3x^3 + 5x - 7}$$

Question 1 - Solution

We are given:

$$\lim_{x \rightarrow \infty} \frac{6x^3 - 4x^2 + 10}{3x^3 + 5x - 7}$$

Step 1: Identify the dominant terms.

As $x \rightarrow \infty$, the highest degree terms dominate: Numerator: $6x^3$, Denominator: $3x^3$

So we focus on:

$$\frac{6x^3}{3x^3} = 2$$

Step 2: Divide all terms by x^3 :

$$\lim_{x \rightarrow \infty} \frac{6 - \frac{4}{x} + \frac{10}{x^3}}{3 + \frac{5}{x^2} - \frac{7}{x^3}} = \frac{6}{3} = 2$$

Conclusion:

$\lim_{x \rightarrow \infty} \frac{6x^3 - 4x^2 + 10}{3x^3 + 5x - 7} = 2$
--

Question 2

Evaluate the limit:

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x} - x)$$

Question 2 - Solution

We are given:

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x} - x)$$

This expression is of the indeterminate form $\infty - \infty$, so we need to rationalize:

Multiply by the conjugate:

$$(\sqrt{x^2 + 5x} - x) \cdot \frac{\sqrt{x^2 + 5x} + x}{\sqrt{x^2 + 5x} + x}$$

This gives:

$$\frac{(\sqrt{x^2 + 5x})^2 - x^2}{\sqrt{x^2 + 5x} + x} = \frac{x^2 + 5x - x^2}{\sqrt{x^2 + 5x} + x} = \frac{5x}{\sqrt{x^2 + 5x} + x}$$

Now divide numerator and denominator by x :

$$\frac{5}{\sqrt{1 + \frac{5}{x}} + 1}$$

As $x \rightarrow \infty$, $\frac{5}{x} \rightarrow 0$, so:

$$\frac{5}{\sqrt{1+0}+1} = \frac{5}{1+1} = \frac{5}{2}$$

Final Answer:

$$\boxed{\frac{5}{2}}$$

Question 3

Evaluate the limit:

$$\lim_{x \rightarrow \infty} \frac{3x^2 - x + 4}{5x^2 + 7x - 9}$$

Question 3 - Solution

We are given:

$$\lim_{x \rightarrow \infty} \frac{3x^2 - x + 4}{5x^2 + 7x - 9}$$

To evaluate this limit, we divide both numerator and denominator by x^2 , the highest power of x :

$$\lim_{x \rightarrow \infty} \frac{3 - \frac{1}{x} + \frac{4}{x^2}}{5 + \frac{7}{x} - \frac{9}{x^2}}$$

As $x \rightarrow \infty$, the terms $\frac{1}{x}$, $\frac{4}{x^2}$, $\frac{7}{x}$, $\frac{9}{x^2} \rightarrow 0$, so we get:

$$\frac{3 - 0 + 0}{5 + 0 - 0} = \frac{3}{5}$$

Final Answer:

$$\boxed{\frac{3}{5}}$$

Question 4

Evaluate the limit:

$$\lim_{x \rightarrow -\infty} \frac{6x^3 + 4x - 1}{-2x^3 + 7x^2 + 5}$$

Question 4 - Solution

We are given:

$$\lim_{x \rightarrow -\infty} \frac{6x^3 + 4x - 1}{-2x^3 + 7x^2 + 5}$$

We divide both the numerator and denominator by x^3 , the highest power of x :

$$= \lim_{x \rightarrow -\infty} \frac{6 + \frac{4}{x^2} - \frac{1}{x^3}}{-2 + \frac{7}{x} + \frac{5}{x^3}}$$

As $x \rightarrow -\infty$, all terms with negative exponents approach zero:

$$= \frac{6}{-2} = -3$$

Final Answer:

$$\boxed{-3}$$

Question 5

Determine the limit:

$$\lim_{x \rightarrow \infty} \frac{5x^4 - 3x^2 + x + 1}{2x^3 + 4x - 7}$$

Question 5 - Solution

We are given:

$$\lim_{x \rightarrow \infty} \frac{5x^4 - 3x^2 + x + 1}{2x^3 + 4x - 7}$$

Step 1: Compare the degrees of the numerator and denominator.

Degree of numerator: 4 , Degree of denominator: 3

Since the degree of the numerator is greater than the degree of the denominator, the limit is infinite (the function diverges).

Step 2: Determine the sign of the infinity.

As $x \rightarrow \infty$, both numerator and denominator are positive:

$$\frac{5x^4 + \text{smaller terms}}{2x^3 + \text{smaller terms}} > 0$$

So the limit tends to:

$$\boxed{+\infty}$$

Question 6

Evaluate the limit:

$$\lim_{x \rightarrow \infty} \frac{3x^5 - 2x^2 + 1}{x^5 + 4x^3 - x}$$

Question 6 - Solution

We are given:

$$\lim_{x \rightarrow \infty} \frac{3x^5 - 2x^2 + 1}{x^5 + 4x^3 - x}$$

Step 1: Divide numerator and denominator by the highest power of x , which is x^5 :

$$= \lim_{x \rightarrow \infty} \frac{3 - \frac{2}{x^3} + \frac{1}{x^5}}{1 + \frac{4}{x^2} - \frac{1}{x^4}}$$

Step 2: As $x \rightarrow \infty$, all terms with negative powers of x go to zero:

$$= \frac{3 - 0 + 0}{1 + 0 - 0} = \boxed{3}$$

Question 7

Evaluate the limit:

$$\lim_{x \rightarrow -\infty} \frac{5x^3 + x^2 - 7x + 1}{-2x^3 + 3x - 9}$$

Question 7 - Solution

We are given:

$$\lim_{x \rightarrow -\infty} \frac{5x^3 + x^2 - 7x + 1}{-2x^3 + 3x - 9}$$

Step 1: Factor out the highest degree of x from both numerator and denominator:

$$= \lim_{x \rightarrow -\infty} \frac{x^3 \left(5 + \frac{1}{x} - \frac{7}{x^2} + \frac{1}{x^3} \right)}{x^3 \left(-2 + \frac{3}{x^2} - \frac{9}{x^3} \right)}$$

Cancel the common factor of x^3 :

$$= \lim_{x \rightarrow -\infty} \frac{5 + \frac{1}{x} - \frac{7}{x^2} + \frac{1}{x^3}}{-2 + \frac{3}{x^2} - \frac{9}{x^3}}$$

Step 2: Let $x \rightarrow -\infty$. All terms with negative powers vanish:

$$= \frac{5 + 0 - 0 + 0}{-2 + 0 - 0} = \boxed{-\frac{5}{2}}$$

Question 8

Determine the limit:

$$\lim_{x \rightarrow \infty} \frac{6x^2 - x + 8}{\sqrt{x^4 + 3x^2 + 1}}$$

Question 8 - Solution

We are given:

$$\lim_{x \rightarrow \infty} \frac{6x^2 - x + 8}{\sqrt{x^4 + 3x^2 + 1}}$$

Step 1: Rewrite the denominator using properties of radicals:

$$\sqrt{x^4 + 3x^2 + 1} = x^2 \sqrt{1 + \frac{3}{x^2} + \frac{1}{x^4}}$$

So the entire expression becomes:

$$\frac{6x^2 - x + 8}{x^2 \sqrt{1 + \frac{3}{x^2} + \frac{1}{x^4}}} = \frac{x^2 (6 - \frac{1}{x} + \frac{8}{x^2})}{x^2 \sqrt{1 + \frac{3}{x^2} + \frac{1}{x^4}}}$$

Cancel x^2 :

$$= \frac{6 - \frac{1}{x} + \frac{8}{x^2}}{\sqrt{1 + \frac{3}{x^2} + \frac{1}{x^4}}}$$

Step 2: Let $x \rightarrow \infty$:

$$= \frac{6 - 0 + 0}{\sqrt{1 + 0 + 0}} = \frac{6}{1} = \boxed{6}$$

Question 9

Evaluate the limit:

$$\lim_{x \rightarrow -\infty} \frac{2x^3 + x^2 - 5}{x^3 - 4x + 1}$$

Question 9 - Solution

We are given:

$$\lim_{x \rightarrow -\infty} \frac{2x^3 + x^2 - 5}{x^3 - 4x + 1}$$

Step 1: Factor out the highest power of x (which is x^3) from both numerator and denominator:

$$= \lim_{x \rightarrow -\infty} \frac{x^3 \left(2 + \frac{1}{x} - \frac{5}{x^3}\right)}{x^3 \left(1 - \frac{4}{x^2} + \frac{1}{x^3}\right)}$$

Cancel the common x^3 terms:

$$= \lim_{x \rightarrow -\infty} \frac{2 + \frac{1}{x} - \frac{5}{x^3}}{1 - \frac{4}{x^2} + \frac{1}{x^3}}$$

Step 2: Take the limit as $x \rightarrow -\infty$. All terms with $\frac{1}{x^n} \rightarrow 0$:

$$= \frac{2 + 0 - 0}{1 - 0 + 0} = \boxed{2}$$

Question 10

Evaluate the limit:

$$\lim_{x \rightarrow \infty} \frac{5x^2 + 3x - 7}{2x^2 - x + 4}$$

Question 10 - Solution

We are given:

$$\lim_{x \rightarrow \infty} \frac{5x^2 + 3x - 7}{2x^2 - x + 4}$$

Step 1: Factor out the highest power of x (which is x^2) from both the numerator and denominator:

$$= \lim_{x \rightarrow \infty} \frac{x^2 \left(5 + \frac{3}{x} - \frac{7}{x^2}\right)}{x^2 \left(2 - \frac{1}{x} + \frac{4}{x^2}\right)}$$

Step 2: Cancel the common factor x^2 :

$$= \lim_{x \rightarrow \infty} \frac{5 + \frac{3}{x} - \frac{7}{x^2}}{2 - \frac{1}{x} + \frac{4}{x^2}}$$

Step 3: Take the limit as $x \rightarrow \infty$. Terms with $\frac{1}{x^n} \rightarrow 0$:

$$= \frac{5 + 0 - 0}{2 - 0 + 0} = \boxed{\frac{5}{2}}$$

Question 1

Evaluate the limit:

$$\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 5x + 7} - \sqrt{x^2 + x} \right).$$

Question 1 - Solution

We evaluate

$$\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 5x + 7} - \sqrt{x^2 + x} \right).$$

This expression has the indeterminate form $\infty - \infty$. To resolve it, we multiply by the conjugate.

Step 1: Multiply by the Conjugate

$$= \lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 5x + 7} - \sqrt{x^2 + x} \right) \cdot \frac{\sqrt{x^2 + 5x + 7} + \sqrt{x^2 + x}}{\sqrt{x^2 + 5x + 7} + \sqrt{x^2 + x}}.$$

This simplifies to

$$= \lim_{x \rightarrow \infty} \frac{(x^2 + 5x + 7) - (x^2 + x)}{\sqrt{x^2 + 5x + 7} + \sqrt{x^2 + x}}.$$

Step 2: Simplify the Numerator

$$(x^2 + 5x + 7) - (x^2 + x) = 4x + 7,$$

so the limit becomes

$$\lim_{x \rightarrow \infty} \frac{4x + 7}{\sqrt{x^2 + 5x + 7} + \sqrt{x^2 + x}}.$$

Step 3: Factor Out x

Factor x from each square root:

$$= \lim_{x \rightarrow \infty} \frac{4x + 7}{x \left(\sqrt{1 + \frac{5}{x} + \frac{7}{x^2}} + \sqrt{1 + \frac{1}{x}} \right)}.$$

Divide numerator and denominator by x :

$$= \lim_{x \rightarrow \infty} \frac{4 + \frac{7}{x}}{\sqrt{1 + \frac{5}{x} + \frac{7}{x^2}} + \sqrt{1 + \frac{1}{x}}}.$$

Step 4: Take the Limit

As $x \rightarrow \infty$,

$$\frac{7}{x} \rightarrow 0, \quad \frac{5}{x} \rightarrow 0, \quad \frac{7}{x^2} \rightarrow 0, \quad \frac{1}{x} \rightarrow 0.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{4 + \frac{7}{x}}{\sqrt{1 + \frac{5}{x} + \frac{7}{x^2}} + \sqrt{1 + \frac{1}{x}}} = \frac{4}{1 + 1} = 2.$$

Final Answer

2

Question 2

Evaluate the limit:

$$\lim_{x \rightarrow -\infty} \frac{3x^2 + 7x - 5}{\sqrt{x^4 + 2x^2 + 1}}.$$

Question 2 - Solution

We evaluate

$$\lim_{x \rightarrow -\infty} \frac{3x^2 + 7x - 5}{\sqrt{x^4 + 2x^2 + 1}}.$$

Step 1: Rewrite the denominator carefully

Note that

$$\sqrt{x^4 + 2x^2 + 1} = \sqrt{(x^2 + 1)^2} = |x^2 + 1|.$$

Since $x^2 + 1 > 0$ for all real x ,

$$|x^2 + 1| = x^2 + 1.$$

Thus, the expression becomes

$$\frac{3x^2 + 7x - 5}{x^2 + 1}.$$

Step 2: Divide by the highest power of x

Divide both numerator and denominator by x^2 :

$$\frac{3 + \frac{7}{x} - \frac{5}{x^2}}{1 + \frac{1}{x^2}}.$$

Step 3: Take the limit

As $x \rightarrow -\infty$,

$$\frac{7}{x} \rightarrow 0, \quad \frac{5}{x^2} \rightarrow 0, \quad \frac{1}{x^2} \rightarrow 0.$$

Therefore,

$$\lim_{x \rightarrow -\infty} \frac{3 + \frac{7}{x} - \frac{5}{x^2}}{1 + \frac{1}{x^2}} = \frac{3}{1} = 3.$$

Final Answer

3

Question 3

Evaluate the limit:

$$\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 6x + 5} - \sqrt{x^2 + 2x + 1} \right).$$

Question 3 - Solution

We evaluate

$$\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 6x + 5} - \sqrt{x^2 + 2x + 1} \right).$$

This expression has the indeterminate form $\infty - \infty$, so we rationalize.

Step 1: Multiply by the Conjugate

$$= \lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 6x + 5} - \sqrt{x^2 + 2x + 1} \right) \cdot \frac{\sqrt{x^2 + 6x + 5} + \sqrt{x^2 + 2x + 1}}{\sqrt{x^2 + 6x + 5} + \sqrt{x^2 + 2x + 1}}.$$

This simplifies to

$$= \lim_{x \rightarrow \infty} \frac{(x^2 + 6x + 5) - (x^2 + 2x + 1)}{\sqrt{x^2 + 6x + 5} + \sqrt{x^2 + 2x + 1}}.$$

Step 2: Simplify the Numerator

$$(x^2 + 6x + 5) - (x^2 + 2x + 1) = 4x + 4,$$

so the limit becomes

$$\lim_{x \rightarrow \infty} \frac{4x + 4}{\sqrt{x^2 + 6x + 5} + \sqrt{x^2 + 2x + 1}}.$$

Step 3: Factor Out x

Factor x from each square root:

$$= \lim_{x \rightarrow \infty} \frac{4x + 4}{x \left(\sqrt{1 + \frac{6}{x} + \frac{5}{x^2}} + \sqrt{1 + \frac{2}{x} + \frac{1}{x^2}} \right)}.$$

Divide numerator and denominator by x :

$$= \lim_{x \rightarrow \infty} \frac{4 + \frac{4}{x}}{\sqrt{1 + \frac{6}{x} + \frac{5}{x^2}} + \sqrt{1 + \frac{2}{x} + \frac{1}{x^2}}}.$$

Step 4: Take the Limit

As $x \rightarrow \infty$, all terms involving $\frac{1}{x}$ and higher powers go to 0. Therefore,

$$\lim_{x \rightarrow \infty} \frac{4 + \frac{4}{x}}{\sqrt{1 + \frac{6}{x} + \frac{5}{x^2}} + \sqrt{1 + \frac{2}{x} + \frac{1}{x^2}}} = \frac{4}{1 + 1} = 2.$$

Final Answer

2

Question 4

Evaluate the limit:

$$\lim_{x \rightarrow -\infty} \frac{x^3 + 2x^2 - 5x + 7}{x^3 - 4x + 1}$$

Question 4 - Solution

We are asked to evaluate:

$$\lim_{x \rightarrow -\infty} \frac{x^3 + 2x^2 - 5x + 7}{x^3 - 4x + 1}$$

Step 1: Compare the degrees of numerator and denominator

Both numerator and denominator are degree 3 polynomials.

When the degrees are the same, the limit as $x \rightarrow \pm\infty$ is the ratio of the leading coefficients.

Step 2: Extract leading terms

Numerator leading term: x^3 , Denominator leading term: x^3

So,

$$\lim_{x \rightarrow -\infty} \frac{x^3 + 2x^2 - 5x + 7}{x^3 - 4x + 1} = \lim_{x \rightarrow -\infty} \frac{1 + \frac{2}{x} - \frac{5}{x^2} + \frac{7}{x^3}}{1 - \frac{4}{x^2} + \frac{1}{x^3}}$$

As $x \rightarrow -\infty$, all terms with $\frac{1}{x^k} \rightarrow 0$ for $k > 0$

Step 3: Evaluate the limit

$$\lim_{x \rightarrow -\infty} \frac{1 + 0 - 0 + 0}{1 - 0 + 0} = \frac{1}{1} = 1$$

Final Answer:

$$\boxed{1}$$

Question 5

Evaluate the limit:

$$\lim_{x \rightarrow \infty} \left(\frac{x^2 + 3x - 7}{\sqrt{x^4 + 2x^3 + 5}} \right)$$

Question 5 - Solution

We are given:

$$\lim_{x \rightarrow \infty} \left(\frac{x^2 + 3x - 7}{\sqrt{x^4 + 2x^3 + 5}} \right)$$

Step 1: Understand the growth of numerator and denominator

As $x \rightarrow \infty$, both numerator and denominator tend to infinity, so we need to simplify.

Step 2: Factor out highest power of x

In the numerator: highest power is x^2 In the denominator (inside square root): highest power is x^4

Factor numerator:

$$x^2 \left(1 + \frac{3}{x} - \frac{7}{x^2} \right)$$

Factor denominator inside square root:

$$\sqrt{x^4 \left(1 + \frac{2}{x} + \frac{5}{x^4} \right)} = x^2 \sqrt{1 + \frac{2}{x} + \frac{5}{x^4}}$$

Step 3: Substitute factored forms into the expression**

$$\frac{x^2 \left(1 + \frac{3}{x} - \frac{7}{x^2} \right)}{x^2 \sqrt{1 + \frac{2}{x} + \frac{5}{x^4}}}$$

Cancel x^2 :

$$\frac{1 + \frac{3}{x} - \frac{7}{x^2}}{\sqrt{1 + \frac{2}{x} + \frac{5}{x^4}}}$$

Step 4: Take the limit as $x \rightarrow \infty$

As $x \rightarrow \infty$, all terms with $\frac{1}{x}$ go to 0:

$$\lim_{x \rightarrow \infty} \frac{1 + \frac{3}{x} - \frac{7}{x^2}}{\sqrt{1 + \frac{2}{x} + \frac{5}{x^4}}} = \frac{1 + 0 - 0}{\sqrt{1 + 0 + 0}} = \frac{1}{1} = 1$$

Final Answer:

$$\boxed{1}$$

Question 6

Evaluate the limit:

$$\lim_{x \rightarrow -\infty} \left(\frac{3x^3 + 2x^2 - x + 5}{\sqrt{x^6 + x^2 + 1}} \right)$$

Question 6 - Solution

We are given:

$$\lim_{x \rightarrow -\infty} \left(\frac{3x^3 + 2x^2 - x + 5}{\sqrt{x^6 + x^2 + 1}} \right)$$

Step 1: Factor out highest powers

Numerator is a cubic polynomial: factor x^3 :

$$3x^3 + 2x^2 - x + 5 = x^3 \left(3 + \frac{2}{x} - \frac{1}{x^2} + \frac{5}{x^3} \right)$$

Denominator is a square root of a degree-6 polynomial:

$$\sqrt{x^6 + x^2 + 1} = |x^3| \sqrt{1 + \frac{1}{x^4} + \frac{1}{x^6}}$$

Note: Since $x \rightarrow -\infty$, we have $|x^3| = -x^3$

So,

$$\sqrt{x^6 + x^2 + 1} = -x^3 \sqrt{1 + \frac{1}{x^4} + \frac{1}{x^6}}$$

Step 2: Substitute into the expression

$$\frac{x^3 \left(3 + \frac{2}{x} - \frac{1}{x^2} + \frac{5}{x^3} \right)}{-x^3 \sqrt{1 + \frac{1}{x^4} + \frac{1}{x^6}}}$$

Cancel x^3 :

$$\frac{3 + \frac{2}{x} - \frac{1}{x^2} + \frac{5}{x^3}}{-\sqrt{1 + \frac{1}{x^4} + \frac{1}{x^6}}}$$

Step 3: Take the limit as $x \rightarrow -\infty$

All terms with $\frac{1}{x^n} \rightarrow 0$. So:

$$\lim_{x \rightarrow -\infty} \frac{3 + \frac{2}{x} - \frac{1}{x^2} + \frac{5}{x^3}}{-\sqrt{1 + \frac{1}{x^4} + \frac{1}{x^6}}} = \frac{3}{-1} = -3$$

Final Answer:

$$\boxed{-3}$$

Question 7

Evaluate the limit:

$$\lim_{x \rightarrow \infty} x \left(\sqrt{x^2 + 5x} - x \right).$$

Question 7 - Solution

We evaluate

$$\lim_{x \rightarrow \infty} x \left(\sqrt{x^2 + 5x} - x \right).$$

At first glance, as $x \rightarrow \infty$,

$$\sqrt{x^2 + 5x} \rightarrow \infty \quad \text{and} \quad x \rightarrow \infty,$$

so the expression inside the parentheses has the indeterminate form $\infty - \infty$.

Step 1: Rationalize the Expression

Multiply the expression inside the parentheses by its conjugate:

$$\sqrt{x^2 + 5x} - x = \frac{(\sqrt{x^2 + 5x} - x)(\sqrt{x^2 + 5x} + x)}{\sqrt{x^2 + 5x} + x}.$$

Using the identity $(a - b)(a + b) = a^2 - b^2$, we obtain

$$= \frac{(x^2 + 5x) - x^2}{\sqrt{x^2 + 5x} + x} = \frac{5x}{\sqrt{x^2 + 5x} + x}.$$

Thus the original limit becomes

$$\lim_{x \rightarrow \infty} x \cdot \frac{5x}{\sqrt{x^2 + 5x} + x}.$$

Step 2: Simplify

Rewrite the expression as

$$\lim_{x \rightarrow \infty} \frac{5x^2}{\sqrt{x^2 + 5x} + x}.$$

Factor x out of the square root:

$$\sqrt{x^2 + 5x} = x\sqrt{1 + \frac{5}{x}} \quad (x > 0).$$

Substitute:

$$\frac{5x^2}{x \left(\sqrt{1 + \frac{5}{x}} + 1 \right)} = \frac{5x}{\sqrt{1 + \frac{5}{x}} + 1}.$$

Step 3: Take the Limit

As $x \rightarrow \infty$, $\frac{5}{x} \rightarrow 0$, so

$$\sqrt{1 + \frac{5}{x}} \rightarrow 1.$$

Therefore,

$$\lim_{x \rightarrow \infty} \frac{5x}{\sqrt{1 + \frac{5}{x}} + 1} = \lim_{x \rightarrow \infty} \frac{5x}{2} = \infty.$$

Final Answer

The limit diverges to ∞ .

Question 8

Evaluate the limit:

$$\lim_{x \rightarrow \infty} x^2 \left(\sqrt{x^2 + 4x} - x \right).$$

Question 8 - Solution

We evaluate

$$\lim_{x \rightarrow \infty} x^2 \left(\sqrt{x^2 + 4x} - x \right).$$

As $x \rightarrow \infty$, both $\sqrt{x^2 + 4x}$ and x approach infinity, so the expression inside the parentheses has the indeterminate form $\infty - \infty$. Multiplying by x^2 significantly changes the behavior of the limit, so careful analysis is required.

Step 1: Rationalize the Expression

We first rationalize the difference:

$$\sqrt{x^2 + 4x} - x = \frac{(\sqrt{x^2 + 4x} - x)(\sqrt{x^2 + 4x} + x)}{\sqrt{x^2 + 4x} + x}.$$

Using the identity $(a - b)(a + b) = a^2 - b^2$, this becomes

$$= \frac{(x^2 + 4x) - x^2}{\sqrt{x^2 + 4x} + x} = \frac{4x}{\sqrt{x^2 + 4x} + x}.$$

Substitute into the original expression:

$$\lim_{x \rightarrow \infty} x^2 \cdot \frac{4x}{\sqrt{x^2 + 4x} + x} = \lim_{x \rightarrow \infty} \frac{4x^3}{\sqrt{x^2 + 4x} + x}.$$

Step 2: Simplify

Factor x out of the square root:

$$\sqrt{x^2 + 4x} = x\sqrt{1 + \frac{4}{x}} \quad (x > 0).$$

Then

$$\frac{4x^3}{x\left(\sqrt{1 + \frac{4}{x}} + 1\right)} = \frac{4x^2}{\sqrt{1 + \frac{4}{x}} + 1}.$$

Step 3: Take the Limit

As $x \rightarrow \infty$, $\frac{4}{x} \rightarrow 0$, so

$$\sqrt{1 + \frac{4}{x}} \rightarrow 1.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{4x^2}{\sqrt{1 + \frac{4}{x}} + 1} = \lim_{x \rightarrow \infty} \frac{4x^2}{2} = \infty.$$

Final Answer

The limit diverges to ∞ .

Question 9

Evaluate the limit:

$$\lim_{x \rightarrow -\infty} \frac{x^3 + 4x^2 - 7}{2x^3 - 5x + 1}$$

Question 9 - Solution

We are given the rational function:

$$\lim_{x \rightarrow -\infty} \frac{x^3 + 4x^2 - 7}{2x^3 - 5x + 1}$$

—

Step 1: Analyze degrees of numerator and denominator

- The highest degree in both numerator and denominator is 3. - For limits at infinity, when the degrees match, the limit is the ratio of the leading coefficients.

So:

$$\lim_{x \rightarrow -\infty} \frac{x^3 + 4x^2 - 7}{2x^3 - 5x + 1} = \frac{1}{2}$$

—

Step 2: Justify with algebraic manipulation (optional)

Divide numerator and denominator by x^3 :

$$= \frac{1 + \frac{4}{x} - \frac{7}{x^3}}{2 - \frac{5}{x^2} + \frac{1}{x^3}}$$

As $x \rightarrow -\infty$, the fractions go to 0:

$$\frac{1 + 0 - 0}{2 - 0 + 0} = \frac{1}{2}$$

—

Final Answer:

$$\boxed{\frac{1}{2}}$$

Question 10

Evaluate the limit:

$$\lim_{x \rightarrow \infty} x \left(\sqrt{x^2 + 3x} - x \right).$$

Question 10 - Solution

We evaluate

$$\lim_{x \rightarrow \infty} x \left(\sqrt{x^2 + 3x} - x \right).$$

As $x \rightarrow \infty$, both $\sqrt{x^2 + 3x}$ and x grow without bound, so the expression inside the parentheses has the indeterminate form $\infty - \infty$. Multiplying by x changes the rate at which the expression grows, so careful analysis is required.

Step 1: Rationalize the Expression

We first rationalize the difference:

$$\sqrt{x^2 + 3x} - x = \frac{(\sqrt{x^2 + 3x} - x)(\sqrt{x^2 + 3x} + x)}{\sqrt{x^2 + 3x} + x}.$$

Using the identity $(a - b)(a + b) = a^2 - b^2$, this becomes

$$= \frac{(x^2 + 3x) - x^2}{\sqrt{x^2 + 3x} + x} = \frac{3x}{\sqrt{x^2 + 3x} + x}.$$

Substitute this into the original limit:

$$\lim_{x \rightarrow \infty} x \cdot \frac{3x}{\sqrt{x^2 + 3x} + x} = \lim_{x \rightarrow \infty} \frac{3x^2}{\sqrt{x^2 + 3x} + x}.$$

Step 2: Simplify

Factor x out of the square root:

$$\sqrt{x^2 + 3x} = x\sqrt{1 + \frac{3}{x}} \quad (x > 0).$$

Then

$$\frac{3x^2}{x \left(\sqrt{1 + \frac{3}{x}} + 1 \right)} = \frac{3x}{\sqrt{1 + \frac{3}{x}} + 1}.$$

Step 3: Take the Limit

As $x \rightarrow \infty$, $\frac{3}{x} \rightarrow 0$, so

$$\sqrt{1 + \frac{3}{x}} \rightarrow 1.$$

Therefore,

$$\lim_{x \rightarrow \infty} \frac{3x}{\sqrt{1 + \frac{3}{x}} + 1} = \lim_{x \rightarrow \infty} \frac{3x}{2} = \infty.$$

Final Answer

The limit diverges to ∞ .

Question 1

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} kx^2 + 3x + 1, & \text{if } x < 2 \\ x^3 - 3kx + c, & \text{if } x \geq 2 \end{cases}$$

Find the values of constants k and c such that $f(x)$ is continuous and differentiable at $x = 2$.

Question 1 - Solution

We are given:

$$f(x) = \begin{cases} kx^2 + 3x + 1, & \text{if } x < 2 \\ x^3 - 3kx + c, & \text{if } x \geq 2 \end{cases}$$

We want $f(x)$ to be both:

1. Continuous at $x = 2$
2. Differentiable at $x = 2$

Step 1: Continuity at $x = 2$

For continuity, we need:

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

Left-hand limit:

$$\lim_{x \rightarrow 2^-} f(x) = k(2)^2 + 3(2) + 1 = 4k + 6 + 1 = 4k + 7$$

Right-hand limit:

$$\lim_{x \rightarrow 2^+} f(x) = (2)^3 - 3k(2) + c = 8 - 6k + c$$

Set them equal:

$$4k + 7 = 8 - 6k + c \quad (1)$$

Step 2: Differentiability at $x = 2$

Derivatives:

$$\text{Left: } f'(x) = 2kx + 3 \Rightarrow f'(2^-) = 4k + 3$$

$$\text{Right: } f'(x) = 3x^2 - 3k \Rightarrow f'(2^+) = 12 - 3k$$

Set equal:

$$4k + 3 = 12 - 3k \quad (2)$$

Step 3: Solve the system

From (2):

$$4k + 3 = 12 - 3k \Rightarrow 7k = 9 \Rightarrow k = \frac{9}{7}$$

Substitute into (1):

$$4\left(\frac{9}{7}\right) + 7 = 8 - 6\left(\frac{9}{7}\right) + c \Rightarrow \frac{36}{7} + 7 = 8 - \frac{54}{7} + c \Rightarrow \frac{85}{7} = \frac{2}{7} + c \Rightarrow c = \frac{83}{7}$$

Final Answer:

$$\boxed{k = \frac{9}{7}, \quad c = \frac{83}{7}}$$

Question 2

For real constants a and b , let

$$f(x) = \begin{cases} \frac{\sin(ax)}{x}, & x \neq 0, \\ b, & x = 0. \end{cases}$$

Classify the pairs (a, b) for which f is continuous at zero and those for which it has a removable discontinuity there.

State how to remove the discontinuity.

Question 2 - Solution

For $a \neq 0$, write

$$\frac{\sin(ax)}{x} = a \frac{\sin(ax)}{ax} \longrightarrow a.$$

For $a = 0$, the quotient is identically zero away from zero, so the same limit holds.

Thus $\lim_{x \rightarrow 0} f(x) = a$ for every real a .

Since $f(0) = b$, continuity holds exactly when $\boxed{b = a}$.

When $\boxed{b \neq a}$ the limit exists but differs from $f(0)$, so the discontinuity is removable by resetting $f(0) = a$.

A function cannot be both continuous and discontinuous at the same point.

Question 3

Let the function $f(x)$ be defined by:

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x \neq 2 \\ k, & x = 2 \end{cases}$$

Determine the value of k such that $f(x)$ is continuous at $x = 2$.

Question 3 - Solution

We are given:

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x \neq 2 \\ k, & x = 2 \end{cases}$$

To make $f(x)$ continuous at $x = 2$, the following condition must hold:

$$\lim_{x \rightarrow 2} f(x) = f(2)$$

Step 1: Compute the limit as $x \rightarrow 2$

Factor the numerator:

$$x^2 - 4 = (x - 2)(x + 2)$$

For $x \neq 2$, the function simplifies to:

$$f(x) = x + 2$$

So,

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (x + 2) = 4$$

Step 2: Match the function value

Continuity requires:

$$f(2) = k = 4$$

Final Answer:

$$\boxed{k = 4}$$

Question 4

Consider the function

$$f(x) = \begin{cases} \frac{\sin(3x)}{x}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

Find the value of c that makes $f(x)$ continuous at $x = 0$.

Question 4 - Solution

We are given:

$$f(x) = \begin{cases} \frac{\sin(3x)}{x}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

We want $f(x)$ to be continuous at $x = 0$. This means:

$$\lim_{x \rightarrow 0} f(x) = f(0) = c$$

Step 1: Compute the limit as $x \rightarrow 0$

We use the standard limit identity:

$$\lim_{x \rightarrow 0} \frac{\sin(kx)}{x} = k$$

So, in our case:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = 3$$

Step 2: Set the limit equal to the function value

To ensure continuity at $x = 0$, we must have:

$$c = 3$$

Final Answer:

$$\boxed{c = 3}$$

Question 5

Let the function $f(x)$ be defined as

$$f(x) = \begin{cases} kx^2 + 2x, & x < 3 \\ x^2 - 4kx + 9, & x \geq 3 \end{cases}$$

Find the value of k that makes $f(x)$ continuous at $x = 3$.

Question 5 - Solution

We are given:

$$f(x) = \begin{cases} kx^2 + 2x, & x < 3 \\ x^2 - 4kx + 9, & x \geq 3 \end{cases}$$

To ensure continuity at $x = 3$, the left-hand and right-hand limits must agree, and they must equal the function value:

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = f(3)$$

Step 1: Left-hand limit as $x \rightarrow 3^-$

From the first piece:

$$\lim_{x \rightarrow 3^-} f(x) = k(3)^2 + 2(3) = 9k + 6$$

Step 2: Right-hand limit and $f(3)$

From the second piece:

$$f(3) = (3)^2 - 4k(3) + 9 = 9 - 12k + 9 = 18 - 12k$$

Step 3: Set the two expressions equal for continuity

$$9k + 6 = 18 - 12k$$

Solve for k :

$$9k + 12k = 18 - 6 \quad 21k = 12 \quad k = \frac{12}{21} = \frac{4}{7}$$

Final Answer:

$$\boxed{k = \frac{4}{7}}$$

Question 6

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} \frac{\sin(ax)}{x}, & x \neq 0 \\ b, & x = 0 \end{cases}$$

Find the values of constants a and b such that $f(x)$ is continuous at $x = 0$.

Question 6 - Solution

To ensure continuity at $x = 0$, we must have:

$$\lim_{x \rightarrow 0} f(x) = f(0) = b$$

We are given:

$$f(x) = \frac{\sin(ax)}{x}, \quad x \neq 0$$
$$f(0) = b$$

We evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(ax)}{x} = \lim_{x \rightarrow 0} a \cdot \frac{\sin(ax)}{ax} = a \cdot \lim_{u \rightarrow 0} \frac{\sin u}{u} = a \cdot 1 = a$$

So for continuity, we must have:

$$\lim_{x \rightarrow 0} f(x) = a = b$$

Final Answer:

Any $a \in \mathbb{R}$ and $b = a$

Question 7

Let the function $f(x)$ be defined by

$$f(x) = \begin{cases} x^2 - 4, & x < 2 \\ kx + 1, & x \geq 2 \end{cases}$$

Determine the value of k that makes $f(x)$ continuous at $x = 2$.

Question 7 - Solution

We are given a piecewise function:

$$f(x) = \begin{cases} x^2 - 4, & x < 2 \\ kx + 1, & x \geq 2 \end{cases}$$

To ensure continuity at $x = 2$, we require:

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

Evaluate the left-hand limit:

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x^2 - 4) = 4 - 4 = 0$$

Evaluate the right-hand limit:

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (kx + 1) = 2k + 1$$

Set the two limits equal for continuity:

$$2k + 1 = 0 \quad \Rightarrow \quad k = -\frac{1}{2}$$

Final Answer:

$$\boxed{k = -\frac{1}{2}}$$

Question 8

Let

$$f(x) = \begin{cases} \frac{\sin(3x)}{x}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

Find the value of c that makes $f(x)$ continuous at $x = 0$.

Question 8 - Solution

To ensure continuity at $x = 0$, we require:

$$\lim_{x \rightarrow 0} f(x) = f(0) = c$$

Compute the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = \lim_{x \rightarrow 0} \frac{3 \sin(3x)}{3x} = 3 \cdot \lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} = 3 \cdot 1 = 3$$

Thus, to make f continuous at $x = 0$, we must set:

$$\boxed{c = 3}$$

Question 9

Let

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ k, & x = 2 \end{cases}$$

Determine the value of k such that $f(x)$ is continuous at $x = 2$.

Question 9 - Solution

We want f to be continuous at $x = 2$, so we must have:

$$\lim_{x \rightarrow 2} f(x) = f(2) = k$$

First, compute the limit:

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2}$$

For $x \neq 2$, we can cancel the common factor:

$$f(x) = x + 2$$

Therefore:

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (x + 2) = 4$$

To make f continuous at $x = 2$, we set:

$$\boxed{k = 4}$$

Question 10

Let

$$f(x) = \begin{cases} \frac{\sin(3x)}{x}, & x \neq 0 \\ a, & x = 0 \end{cases}$$

- (a) Find the value of a such that $f(x)$ is continuous at $x = 0$.
- (b) Is $f(x)$ differentiable at $x = 0$? Justify your answer.

Question 10 - Solution

(a) For continuity at $x = 0$, we must have

$$\lim_{x \rightarrow 0} f(x) = f(0) = a.$$

We compute the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = \lim_{x \rightarrow 0} 3 \cdot \frac{\sin(3x)}{3x} = 3 \cdot 1 = 3.$$

Thus, to make $f(x)$ continuous at $x = 0$, we must set

$$\boxed{a = 3}.$$

(b) Differentiability at $x = 0$:

We use the definition of the derivative:

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{\sin(3h)}{h} - 3}{h}.$$

Rewrite the expression:

$$\frac{\frac{\sin(3h)}{h} - 3}{h} = \frac{\sin(3h) - 3h}{h^2}.$$

Rescale by $u = 3h$ (so $u^2 = 9h^2$):

$$\frac{\sin(3h) - 3h}{h^2} = 9 \cdot \frac{\sin(3h) - 3h}{(3h)^2}.$$

Let $u = 3h$. Then as $h \rightarrow 0$, $u \rightarrow 0$, and the limit becomes

$$f'(0) = 9 \lim_{u \rightarrow 0} \frac{\sin u - u}{u^2}.$$

Using the standard limit

$$\lim_{u \rightarrow 0} \frac{\sin u - u}{u^2} = 0,$$

we conclude that

$$f'(0) = 0.$$

Conclusion: The limit exists, so $f(x)$ is differentiable at $x = 0$, and

$$\boxed{f'(0) = 0}.$$

Question 1

Let

$$f(x) = 4x - 1$$

Use the ε - δ definition of the limit to prove that:

$$\lim_{x \rightarrow 2} f(x) = 7$$

Question 1 - Solution

We are given:

$$f(x) = 4x - 1, \quad \lim_{x \rightarrow 2} f(x) = 7$$

We will prove this using the ε - δ definition:

Goal: For every $\varepsilon > 0$, we must find $\delta > 0$ such that:

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |f(x) - 7| < \varepsilon$$

Step 1: Write out $|f(x) - 7|$:

$$|f(x) - 7| = |4x - 1 - 7| = |4x - 8| = 4|x - 2|$$

Step 2: Make $4|x - 2| < \varepsilon$:

We can achieve this by choosing:

$$|x - 2| < \frac{\varepsilon}{4}$$

Step 3: Let $\delta = \frac{\varepsilon}{4}$.

Then, for all x such that $0 < |x - 2| < \delta$, we have:

$$|f(x) - 7| = 4|x - 2| < 4\delta = 4 \cdot \frac{\varepsilon}{4} = \varepsilon$$

Conclusion:

Therefore, $\lim_{x \rightarrow 2} f(x) = 7$ by the ε - δ definition.

Question 2

Let

$$f(x) = x^2 + 3x$$

Use the ε - δ definition of a limit to prove that:

$$\lim_{x \rightarrow 1} f(x) = 4$$

Question 2 - Solution

We are given:

$$f(x) = x^2 + 3x, \quad \lim_{x \rightarrow 1} f(x) = 4$$

We want to prove using the ε - δ definition:

Goal: For every $\varepsilon > 0$, we must find $\delta > 0$ such that:

$$0 < |x - 1| < \delta \quad \Rightarrow \quad |f(x) - 4| < \varepsilon$$

Step 1: Write the expression $|f(x) - 4|$:

$$|f(x) - 4| = |x^2 + 3x - 4|$$

Factor the expression:

$$x^2 + 3x - 4 = (x - 1)(x + 4) \Rightarrow |f(x) - 4| = |x - 1||x + 4|$$

We want:

$$|x - 1||x + 4| < \varepsilon$$

Step 2: Estimate $|x + 4|$:

Assume $|x - 1| < 1 \Rightarrow x \in (0, 2) \Rightarrow x + 4 \in (4, 6) \Rightarrow |x + 4| < 6$

Then:

$$|f(x) - 4| = |x - 1||x + 4| < 6|x - 1|$$

To make $|f(x) - 4| < \varepsilon$, it is enough to make:

$$6|x - 1| < \varepsilon \Rightarrow |x - 1| < \frac{\varepsilon}{6}$$

Step 3: Let $\delta = \min \left\{ 1, \frac{\varepsilon}{6} \right\}$

Then if $0 < |x - 1| < \delta$, we have:

$$|f(x) - 4| < \varepsilon$$

Conclusion:

Therefore, $\lim_{x \rightarrow 1} (x^2 + 3x) = 4$ by the ε - δ definition.

Question 3

Let

$$f(x) = 5x - 1$$

Prove, using the ε - δ definition of a limit, that:

$$\lim_{x \rightarrow 2} f(x) = 9$$

Question 3 - Solution

We are given:

$$f(x) = 5x - 1, \quad \lim_{x \rightarrow 2} f(x) = 9$$

We will use the ε - δ definition of a limit:

Goal: For every $\varepsilon > 0$, find $\delta > 0$ such that:

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |f(x) - 9| < \varepsilon$$

Step 1: Express $|f(x) - 9|$:

$$|f(x) - 9| = |5x - 1 - 9| = |5x - 10| = 5|x - 2|$$

We want:

$$5|x - 2| < \varepsilon \quad \Rightarrow \quad |x - 2| < \frac{\varepsilon}{5}$$

Step 2: Choose δ :

Let:

$$\delta = \frac{\varepsilon}{5}$$

Then if $0 < |x - 2| < \delta$, we have:

$$|f(x) - 9| = 5|x - 2| < 5\delta = \varepsilon$$

Conclusion:

Therefore, $\lim_{x \rightarrow 2} (5x - 1) = 9$ by the ε - δ definition.

Question 4

Let

$$f(x) = \frac{1}{x}$$

Prove, using the ε - δ definition of a limit, that:

$$\lim_{x \rightarrow 1} \frac{1}{x} = 1$$

Question 4 - Solution

We are given:

$$f(x) = \frac{1}{x}, \quad \lim_{x \rightarrow 1} f(x) = 1$$

We will prove this using the ε - δ definition of the limit.

Goal: For every $\varepsilon > 0$, find $\delta > 0$ such that:

$$0 < |x - 1| < \delta \quad \Rightarrow \quad \left| \frac{1}{x} - 1 \right| < \varepsilon$$

Step 1: Manipulate the expression

We begin by simplifying:

$$\left| \frac{1}{x} - 1 \right| = \left| \frac{1 - x}{x} \right| = \frac{|x - 1|}{|x|}$$

So we want:

$$\frac{|x - 1|}{|x|} < \varepsilon$$

Step 2: Bound $|x|$ away from zero

Choose $\delta \leq \frac{1}{2}$, so $|x - 1| < \frac{1}{2} \Rightarrow \frac{1}{2} < x < \frac{3}{2}$

This gives us a lower bound: $|x| > \frac{1}{2}$

So:

$$\frac{|x - 1|}{|x|} < \frac{|x - 1|}{1/2} = 2|x - 1|$$

Step 3: Ensure $2|x - 1| < \varepsilon$

To make this less than ε , we require:

$$2|x - 1| < \varepsilon \quad \Rightarrow \quad |x - 1| < \frac{\varepsilon}{2}$$

Step 4: Choose δ

Let:

$$\delta = \min\left(\frac{1}{2}, \frac{\varepsilon}{2}\right)$$

Then whenever $0 < |x - 1| < \delta$, we get:

$$\left| \frac{1}{x} - 1 \right| = \frac{|x - 1|}{|x|} < 2|x - 1| < 2\delta \leq \varepsilon$$

Conclusion:

Thus, by the ε - δ definition of a limit,

$$\boxed{\lim_{x \rightarrow 1} \frac{1}{x} = 1}$$

Question 5

Let

$$f(x) = 5x - 3$$

Prove, using the ε - δ definition of a limit, that:

$$\lim_{x \rightarrow 2} (5x - 3) = 7$$

Question 5 - Solution

We are given:

$$f(x) = 5x - 3$$

We want to prove:

$$\lim_{x \rightarrow 2} (5x - 3) = 7$$

using the formal ε - δ definition of the limit.

Definition: For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |(5x - 3) - 7| < \varepsilon$$

Step 1: Simplify the expression

$$|(5x - 3) - 7| = |5x - 10| = 5|x - 2|$$

So we want:

$$5|x - 2| < \varepsilon \quad \Rightarrow \quad |x - 2| < \frac{\varepsilon}{5}$$

Step 2: Choose δ

Let:

$$\delta = \frac{\varepsilon}{5}$$

Step 3: Conclusion

Then whenever $0 < |x - 2| < \delta$, we have:

$$|(5x - 3) - 7| = 5|x - 2| < 5\delta = \varepsilon$$

Therefore, by the ε - δ definition,

$$\boxed{\lim_{x \rightarrow 2} (5x - 3) = 7}$$

Question 6

Let

$$f(x) = \frac{1}{2}x + 4$$

Prove using the ε - δ definition of a limit that:

$$\lim_{x \rightarrow -2} \left(\frac{1}{2}x + 4 \right) = 3$$

Question 6 - Solution

We are given:

$$f(x) = \frac{1}{2}x + 4$$

and want to prove:

$$\lim_{x \rightarrow -2} \left(\frac{1}{2}x + 4 \right) = 3$$

using the formal ε - δ definition of a limit.

Definition: For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x + 2| < \delta \quad \Rightarrow \quad \left| \left(\frac{1}{2}x + 4 \right) - 3 \right| < \varepsilon$$

Step 1: Simplify the expression

$$\left| \left(\frac{1}{2}x + 4 \right) - 3 \right| = \left| \frac{1}{2}x + 1 \right| = \left| \frac{1}{2}(x + 2) \right| = \frac{1}{2}|x + 2|$$

So we want:

$$\frac{1}{2}|x + 2| < \varepsilon \quad \Rightarrow \quad |x + 2| < 2\varepsilon$$

Step 2: Choose δ

Let:

$$\delta = 2\varepsilon$$

Step 3: Conclusion

Then whenever $0 < |x + 2| < \delta$, we have:

$$\left| \left(\frac{1}{2}x + 4 \right) - 3 \right| = \frac{1}{2}|x + 2| < \frac{1}{2} \cdot \delta = \frac{1}{2}(2\varepsilon) = \varepsilon$$

Therefore, by the ε - δ definition,

$$\boxed{\lim_{x \rightarrow -2} \left(\frac{1}{2}x + 4 \right) = 3}$$

Question 7

Let

$$f(x) = 3x - 1$$

Use the ε - δ definition of a limit to prove that:

$$\lim_{x \rightarrow 2} f(x) = 5$$

Question 7 - Solution

We are given the function:

$$f(x) = 3x - 1$$

and we are asked to prove:

$$\lim_{x \rightarrow 2} (3x - 1) = 5$$

using the ε - δ definition.

Step 1: Recall the definition.

We must show that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |(3x - 1) - 5| < \varepsilon$$

Step 2: Simplify the expression inside the absolute value.

$$|(3x - 1) - 5| = |3x - 6| = 3|x - 2|$$

So we require:

$$3|x - 2| < \varepsilon \quad \Rightarrow \quad |x - 2| < \frac{\varepsilon}{3}$$

Step 3: Choose δ .

Let

$$\delta = \frac{\varepsilon}{3}$$

Step 4: Verification.

If $0 < |x - 2| < \delta = \frac{\varepsilon}{3}$, then:

$$|f(x) - 5| = |3x - 6| = 3|x - 2| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon$$

Conclusion: By the ε - δ definition, we have:

$$\boxed{\lim_{x \rightarrow 2} (3x - 1) = 5}$$

Question 8

Let

$$f(x) = x^2$$

Use the ε - δ definition of a limit to prove that:

$$\lim_{x \rightarrow 3} f(x) = 9$$

Question 8 - Solution

We are given:

$$f(x) = x^2, \quad \lim_{x \rightarrow 3} x^2 = 9$$

We want to prove that:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - 3| < \delta \Rightarrow |x^2 - 9| < \varepsilon$$

Step 1: Manipulate the expression.

$$|x^2 - 9| = |x - 3||x + 3|$$

We want to bound this expression. Assume $|x - 3| < 1$. Then:

$$2 < x < 4 \Rightarrow 5 < x + 3 < 7 \Rightarrow |x + 3| < 7$$

So:

$$|x^2 - 9| = |x - 3||x + 3| < |x - 3| \cdot 7$$

We want:

$$|x - 3| \cdot 7 < \varepsilon \Rightarrow |x - 3| < \frac{\varepsilon}{7}$$

Step 2: Choose δ .

Let:

$$\delta = \min\left(1, \frac{\varepsilon}{7}\right)$$

Step 3: Verify.

If $0 < |x - 3| < \delta$, then:

$$|x^2 - 9| = |x - 3||x + 3| < \delta \cdot 7 \leq \frac{\varepsilon}{7} \cdot 7 = \varepsilon$$

Conclusion: By the ε - δ definition of the limit:

$$\lim_{x \rightarrow 3} x^2 = 9$$

Question 9

Let

$$f(x) = 5x - 1$$

Use the ε - δ definition of a limit to prove that:

$$\lim_{x \rightarrow 2} f(x) = 9$$

Question 9 - Solution

We are given:

$$f(x) = 5x - 1, \quad \lim_{x \rightarrow 2} (5x - 1) = 9$$

We want to show that:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that if } 0 < |x - 2| < \delta, \text{ then } |f(x) - 9| < \varepsilon$$

Step 1: Work with the expression $|f(x) - 9|$

$$|f(x) - 9| = |5x - 1 - 9| = |5x - 10| = 5|x - 2|$$

We want:

$$5|x - 2| < \varepsilon \Rightarrow |x - 2| < \frac{\varepsilon}{5}$$

Step 2: Choose δ

Let:

$$\delta = \frac{\varepsilon}{5}$$

Step 3: Conclusion

Then for all $\varepsilon > 0$, if $0 < |x - 2| < \delta$, we have:

$$|f(x) - 9| = 5|x - 2| < 5 \cdot \delta = \varepsilon$$

Therefore, by the ε - δ definition of the limit:

$$\boxed{\lim_{x \rightarrow 2} (5x - 1) = 9}$$

Question 10

Let

$$f(x) = \frac{x^2 - 1}{x - 1}$$

Use the ε - δ definition of a limit to prove:

$$\lim_{x \rightarrow 1} f(x) = 2$$

Question 10 - Solution

We are given:

$$f(x) = \frac{x^2 - 1}{x - 1}$$

Note that:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x+1 \quad \text{for } x \neq 1$$

So for $x \neq 1$, $f(x) = x+1$, and:

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (x+1) = 2$$

We now prove this using the ε - δ definition.

We want to show:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - 1| < \delta \Rightarrow |f(x) - 2| < \varepsilon$$

Since $f(x) = x+1$ for $x \neq 1$, we compute:

$$|f(x) - 2| = |x+1-2| = |x-1|$$

So:

$$|f(x) - 2| < \varepsilon \quad \text{whenever} \quad |x - 1| < \varepsilon$$

Step: Choose $\delta = \varepsilon$

Then if $0 < |x - 1| < \delta$, we get:

$$|f(x) - 2| = |x - 1| < \varepsilon$$

Conclusion: By the ε - δ definition, we have:

$$\boxed{\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2}$$