

Question 1

Let $f(x) = \sqrt{x+4}$ and $g(x) = \frac{1}{x-2}$. Define the composition $h(x) = f(g(x))$.

Find the domain of $h(x)$, and express your answer in interval notation.

Question 1 - Solution

We are given:

$$f(x) = \sqrt{x+4}, \quad g(x) = \frac{1}{x-2}, \quad h(x) = f(g(x)) = \sqrt{\frac{1}{x-2} + 4}$$

To determine the domain of $h(x)$, we consider the restrictions from both the inner and outer functions.

Step 1: Domain restriction from $g(x)$

The function $g(x) = \frac{1}{x-2}$ is undefined when the denominator is zero. This occurs at $x = 2$, so:

$$x \neq 2$$

Step 2: Domain restriction from the square root

We need the expression inside the square root to be greater than or equal to zero:

$$\frac{1}{x-2} + 4 \geq 0 \Rightarrow \frac{4x-7}{x-2} \geq 0$$

Step 3: Critical points and sign analysis

$$4x - 7 = 0 \Rightarrow x = \frac{7}{4}, \quad x - 2 = 0 \Rightarrow x = 2$$

From the sign chart:

Positive on $(-\infty, \frac{7}{4}]$, Negative on $(\frac{7}{4}, 2)$, Positive again on $(2, \infty)$.

Final Domain:

$$\boxed{(-\infty, \frac{7}{4}] \cup (2, \infty)}$$

Question 2

A piecewise function is defined as follows:

$$f(x) = \begin{cases} x^2 - 4x + 3 & \text{if } x < 2 \\ \sqrt{x+2} & \text{if } x \geq 2 \end{cases}$$

- (a) Determine whether $f(x)$ is continuous at $x = 2$.
- (b) Determine whether $f(x)$ is differentiable at $x = 2$.

Question 2 - Solution

Part (a): Continuity at $x = 2$ We must check:

- $\lim_{x \rightarrow 2^-} f(x)$
- $\lim_{x \rightarrow 2^+} f(x)$
- $f(2)$

For $x < 2$, use $f(x) = x^2 - 4x + 3$:

$$\lim_{x \rightarrow 2^-} f(x) = 2^2 - 4(2) + 3 = 4 - 8 + 3 = -1$$

For $x \geq 2$, use $f(x) = \sqrt{x+2}$:

$$\lim_{x \rightarrow 2^+} f(x) = \sqrt{2+2} = \sqrt{4} = 2$$

Since:

$$\lim_{x \rightarrow 2^-} f(x) = -1, \quad \lim_{x \rightarrow 2^+} f(x) = 2$$

the limit from left and right are not equal.

Conclusion: $f(x)$ is **not continuous** at $x = 2$

Part (b): Differentiability at $x = 2$

A function must be continuous at a point to be differentiable there. Since $f(x)$ is not continuous at $x = 2$, it cannot be differentiable there.

Conclusion: $f(x)$ is **not differentiable** at $x = 2$

Question 3

Let the function f be defined by

$$f(x) = \cos(\sin x)$$

- (a) Determine the period of $f(x)$.
- (b) Prove that $f(x)$ is an even function.
- (c) Find all values of $x \in [0, 2\pi]$ such that $f(x) = 1$.

Question 3 - Solution

(a) Although $\sin x$ has period 2π , the function

$$f(x) = \cos(\sin x)$$

has a smaller period because cosine is even.

Using

$$\sin(x + \pi) = -\sin x$$

and

$$\cos(-u) = \cos(u),$$

we get

$$f(x + \pi) = \cos(\sin(x + \pi)) = \cos(-\sin x) = \cos(\sin x) = f(x).$$

Therefore, $f(x)$ repeats after π . The reason the period changes from 2π to π is that shifting by π changes $\sin x$ to $-\sin x$, but cosine gives the same value for opposite inputs.

Thus, the period is

$$\boxed{\pi}.$$

(b) To prove that $f(x)$ is even, compute $f(-x)$:

$$f(-x) = \cos(\sin(-x)) = \cos(-\sin x) = \cos(\sin x) = f(x).$$

Therefore,

$$\boxed{f(x) \text{ is even.}}$$

(c) Solve

$$\cos(\sin x) = 1.$$

Cosine equals 1 when its input is $2k\pi$. Thus,

$$\sin x = 2k\pi.$$

But since $-1 \leq \sin x \leq 1$, the only possible value is

$$\sin x = 0.$$

On $[0, 2\pi]$, this occurs at

$$x = 0, \pi, 2\pi.$$

Therefore,

$$\boxed{x = 0, \pi, 2\pi}.$$

Question 4

$$\text{Let } f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x \neq 3 \\ c, & x = 3 \end{cases}$$

- (a) Simplify $f(x)$ for $x \neq 3$.
- (b) Find the value of c that makes $f(x)$ continuous at $x = 3$.
- (c) Is $f(x)$ differentiable at $x = 3$? Justify your answer.

Question 4 - Solution

(a) Simplifying for $x \neq 3$:

$$f(x) = \frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3, \quad x \neq 3$$

So the simplified function is:

$$f(x) = x + 3 \quad \text{for } x \neq 3$$

(b) **Continuity at $x = 3$:**

To ensure continuity at $x = 3$, we must have:

$$\lim_{x \rightarrow 3} f(x) = f(3) = c$$

From (a), since $f(x) = x + 3$ for $x \neq 3$, then:

$$\lim_{x \rightarrow 3} f(x) = 3 + 3 = 6 \Rightarrow c = \boxed{6}$$

(c) **Differentiability at $x = 3$:**

We now check if $f(x)$ is differentiable at $x = 3$. The simplified form $f(x) = x + 3$ is linear and differentiable for all $x \neq 3$.

If we define $f(3) = 6$, then the function becomes:

$$f(x) = \begin{cases} x + 3 & x \neq 3 \\ 6 & x = 3 \end{cases}$$

This function is: Continuous at $x = 3$, Equal to $x + 3$ everywhere except possibly at 3, $f(x)$ approaches $x + 3$ smoothly at 3

So,

$$f'(3) = \lim_{h \rightarrow 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \rightarrow 0} \frac{(3 + h + 3) - 6}{h} = \lim_{h \rightarrow 0} \frac{6 + h - 6}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1$$

Conclusion: $f(x)$ is differentiable at $x = 3$, and $f'(3) = \boxed{1}$

Question 5

A function f is defined as:

$$f(x) = \begin{cases} \frac{\ln(1+ax)}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

- (a) Find the value of a for which $f(x)$ is continuous at $x = 0$.
- (b) For that value of a , determine whether f is differentiable at $x = 0$.

Question 5 - Solution

(a) For f to be continuous at $x = 0$, we need

$$\lim_{x \rightarrow 0} f(x) = f(0) = 1.$$

For $x \neq 0$,

$$f(x) = \frac{\ln(1 + ax)}{x}.$$

So compute

$$\lim_{x \rightarrow 0} \frac{\ln(1 + ax)}{x}.$$

As $x \rightarrow 0$, the numerator and denominator both approach 0, so we use L'Hospital's Rule:

$$\lim_{x \rightarrow 0} \frac{\ln(1 + ax)}{x} = \lim_{x \rightarrow 0} \frac{\frac{a}{1+ax}}{1} = a.$$

For continuity, this limit must equal $f(0) = 1$. Therefore,

$$a = 1.$$

Thus,

$$\boxed{a = 1}.$$

(b) Now let $a = 1$. Then

$$f(x) = \frac{\ln(1 + x)}{x}, \quad x \neq 0,$$

and $f(0) = 1$.

To check differentiability at $x = 0$, use the definition:

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}.$$

Substitute $f(h) = \frac{\ln(1 + h)}{h}$ and $f(0) = 1$:

$$f'(0) = \lim_{h \rightarrow 0} \frac{\frac{\ln(1+h)}{h} - 1}{h} = \lim_{h \rightarrow 0} \frac{\ln(1 + h) - h}{h^2}.$$

This is also a 0/0 form, so apply L'Hospital's Rule:

$$f'(0) = \lim_{h \rightarrow 0} \frac{\frac{1}{1+h} - 1}{2h} = \lim_{h \rightarrow 0} \frac{-h}{2h(1 + h)} = \lim_{h \rightarrow 0} \frac{-1}{2(1 + h)} = -\frac{1}{2}.$$

Since this limit exists, f is differentiable at $x = 0$, and

$$\boxed{f'(0) = -\frac{1}{2}}.$$

Question 6

Let $f(x) = |x - 2| + |x + 1|$.

- (a) Express $f(x)$ as a piecewise-defined function.
- (b) Sketch the graph of $f(x)$.
- (c) Determine all values of x where $f(x)$ is not differentiable. Justify your answer.

Question 6 - Solution

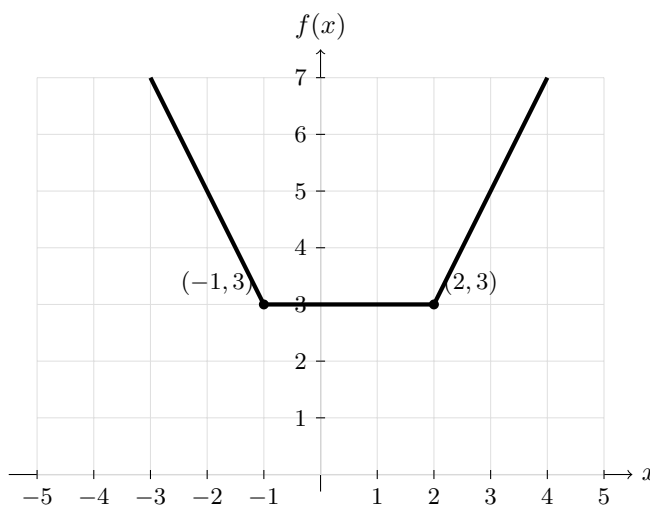
(a) The absolute values change sign at $x = -1$ and $x = 2$.

$$f(x) = \begin{cases} -(x-2) - (x+1), & x < -1, \\ -(x-2) + (x+1), & -1 \leq x < 2, \\ (x-2) + (x+1), & x \geq 2. \end{cases}$$

After simplifying,

$$f(x) = \begin{cases} -2x + 1, & x < -1, \\ 3, & -1 \leq x < 2, \\ 2x - 1, & x \geq 2. \end{cases}$$

(b) The graph decreases until $x = -1$, stays constant from $x = -1$ to $x = 2$, and then increases.



(c) The only possible non-differentiable points are where the absolute value expressions change sign:

$$x = -1 \quad \text{and} \quad x = 2.$$

From the piecewise function, the slopes are

$$\begin{cases} -2, & x < -1, \\ 0, & -1 < x < 2, \\ 2, & x > 2. \end{cases}$$

At $x = -1$, the slope changes from -2 to 0 , so f is not differentiable there.

At $x = 2$, the slope changes from 0 to 2 , so f is not differentiable there.

Therefore,

$$\boxed{x = -1, 2}$$

Question 7

Consider the function

$$f(x) = \frac{x^2 + 2x - 3}{x^2 - 4}.$$

- (a) Determine all vertical and horizontal asymptotes of $f(x)$.
- (b) Identify any holes in the graph of $f(x)$, and specify their coordinates.
- (c) State the domain of $f(x)$.

Question 7 - Solution

We are given

$$f(x) = \frac{x^2 + 2x - 3}{x^2 - 4}.$$

Factor the numerator and denominator:

$$f(x) = \frac{(x+3)(x-1)}{(x-2)(x+2)}.$$

(a) Vertical asymptotes occur where the denominator is zero and no factor cancels. Since

$$x^2 - 4 = (x-2)(x+2),$$

the denominator is zero at

$$x = 2 \quad \text{and} \quad x = -2.$$

There are no common factors with the numerator, so both are vertical asymptotes:

$$\boxed{x = -2, x = 2}.$$

Since the numerator and denominator have the same degree, the horizontal asymptote is the ratio of leading coefficients:

$$y = \frac{1}{1} = 1.$$

Thus,

$$\boxed{y = 1}.$$

(b) Holes occur when a factor cancels from the numerator and denominator. Since

$$\frac{(x+3)(x-1)}{(x-2)(x+2)}$$

has no common factors, there are

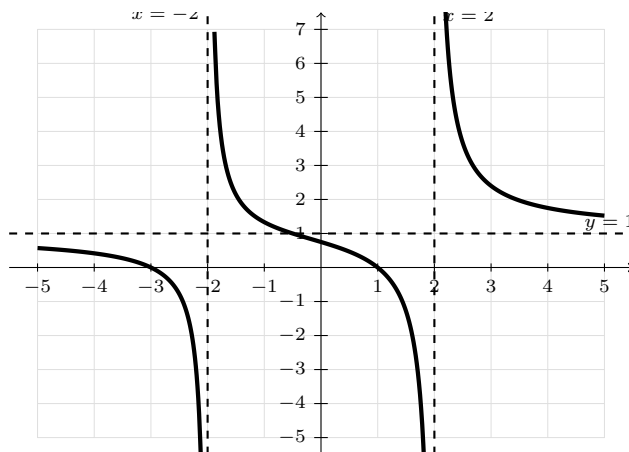
$$\boxed{\text{no holes}}.$$

(c) The domain excludes the values that make the denominator zero:

$$x \neq -2, \quad x \neq 2.$$

Therefore,

$$\boxed{(-\infty, -2) \cup (-2, 2) \cup (2, \infty)}.$$



Question 8

Let

$$f(x) = \frac{1}{x}, \quad g(x) = \sqrt{x-2}.$$

- (a) Find the composite function $(f \circ g)(x)$ and state its domain.
- (b) Determine whether $(f \circ g)(x)$ is defined at $x = 2$. Justify your answer.
- (c) Evaluate $(f \circ g)(6)$.

Question 8 - Solution**(a) Composite function and domain**

The composite function is

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x-2}) = \frac{1}{\sqrt{x-2}}.$$

To determine the domain:

- The square root requires $x - 2 \geq 0 \Rightarrow x \geq 2$.
- The denominator cannot be zero, so $\sqrt{x-2} \neq 0 \Rightarrow x \neq 2$.

Thus, the domain is

$$\boxed{(2, \infty)}.$$

(b) Endpoint analysis

At $x = 2$,

$$(f \circ g)(2) = \frac{1}{\sqrt{2-2}} = \frac{1}{0},$$

which is undefined. Therefore, $x = 2$ is excluded from the domain.

(c) Evaluation

Evaluate the composite at $x = 6$:

$$(f \circ g)(6) = \frac{1}{\sqrt{6-2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}.$$

Answer:

$$\boxed{\frac{1}{2}}$$

Question 9

Let $f(x) = \begin{cases} 2x + 5 & \text{if } x < 1 \\ x^2 + 2 & \text{if } x \geq 1 \end{cases}$

- (a) Is $f(x)$ continuous at $x = 1$? Justify analytically.
- (b) Is $f(x)$ differentiable at $x = 1$? If not, explain why.

Question 9 - Solution**(a) Continuity at $x = 1$:**

Check:

$$\lim_{x \rightarrow 1^-} f(x) = 2(1) + 5 = 7$$

$$\lim_{x \rightarrow 1^+} f(x) = 1^2 + 2 = 3$$

Since:

$$\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x),$$

The two-sided limit does not exist, so:

$f(x)$ is not continuous at $x = 1$

(b) Differentiability at $x = 1$:

A function cannot be differentiable at a point where it is not continuous.

So:

$f(x)$ is not differentiable at $x = 1$

Explanation: The left and right-hand limits do not match at $x = 1$, so there is a jump discontinuity.

Question 10

Let the function f be defined by:

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x \neq 2 \\ c, & x = 2 \end{cases}$$

- (a) Find the limit $\lim_{x \rightarrow 2} f(x)$.
- (b) Determine the value of c that makes $f(x)$ continuous at $x = 2$.
- (c) Is $f(x)$ differentiable at $x = 2$ for the value of c you found? Justify your answer.

Question 10 - Solution

(a) Compute $\lim_{x \rightarrow 2} f(x)$:

Factor the numerator:

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2}, \quad x \neq 2$$

Cancel the common factor:

$$f(x) = x + 2 \quad \text{for } x \neq 2$$

So:

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (x + 2) = 4$$

Answer: $\boxed{4}$

(b) To make f continuous at $x = 2$, set:

$$f(2) = \lim_{x \rightarrow 2} f(x) = 4 \Rightarrow c = \boxed{4}$$

(c) **Differentiability at $x = 2$:**

With $c = 4$, we now have:

$$f(x) = \begin{cases} x + 2, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

The function is: Continuous at $x = 2$, Equal to a linear function near $x = 2$

Compute derivative using definition:

$$f'(2) = \lim_{h \rightarrow 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \rightarrow 0} \frac{(2 + h + 2) - 4}{h} = \frac{4 + h - 4}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1$$

Conclusion: $f(x)$ is differentiable at $x = 2$, with $f'(2) = \boxed{1}$

Question 1

Let the function $f(x) = \frac{2x + 5}{x - 3}$.

- (a) Show that f is one-to-one.
- (b) Find the inverse function $f^{-1}(x)$.
- (c) State the domain and range of both $f(x)$ and $f^{-1}(x)$.

Question 1 - Solution

(a) **Show that f is one-to-one:**

A function is one-to-one if it passes the Horizontal Line Test, or algebraically if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Suppose:

$$\frac{2x_1 + 5}{x_1 - 3} = \frac{2x_2 + 5}{x_2 - 3}$$

Cross-multiplying:

$$(2x_1 + 5)(x_2 - 3) = (2x_2 + 5)(x_1 - 3)$$

Expanding both sides:

$$2x_1x_2 - 6x_1 + 5x_2 - 15 = 2x_1x_2 - 6x_2 + 5x_1 - 15$$

Subtract $2x_1x_2$ and -15 from both sides:

$$-6x_1 + 5x_2 = -6x_2 + 5x_1 \Rightarrow 5x_2 + 6x_2 = 5x_1 + 6x_1 \Rightarrow 11x_2 = 11x_1 \Rightarrow x_1 = x_2$$

Thus, f is one-to-one.

(b) **Find the inverse function:**

Let $y = \frac{2x + 5}{x - 3}$, and solve for x :

$$y(x - 3) = 2x + 5 \Rightarrow yx - 3y = 2x + 5 \Rightarrow yx - 2x = 3y + 5 \Rightarrow x(y - 2) = 3y + 5 \Rightarrow x = \frac{3y + 5}{y - 2}$$

Now swap x and y :

$$f^{-1}(x) = \frac{3x + 5}{x - 2}$$

(c) **Domain and Range:**

For $f(x) = \frac{2x + 5}{x - 3}$: - Denominator cannot be 0 $x \neq 3$ - So domain of f is: $\boxed{(-\infty, 3) \cup (3, \infty)}$

Range: Find y values that cannot occur. From inverse function: $f^{-1}(x) = \frac{3x + 5}{x - 2} \Rightarrow x \neq 2$

So range of f is $\boxed{(-\infty, 2) \cup (2, \infty)}$

Domain and Range:

Domain of $f(x) : (-\infty, 3) \cup (3, \infty)$

Range of $f(x) : (-\infty, 2) \cup (2, \infty)$

Domain of $f^{-1}(x) : (-\infty, 2) \cup (2, \infty)$

Range of $f^{-1}(x) : (-\infty, 3) \cup (3, \infty)$

Question 2

Let $f(x) = \ln(x - 1)$, where $x > 1$.

- (a) Find the inverse function $f^{-1}(x)$.
- (b) State the domain and range of $f(x)$ and $f^{-1}(x)$.
- (c) Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Question 2 - Solution

(a) Find the inverse of $f(x) = \ln(x - 1)$:

Let $y = \ln(x - 1)$

Exponentiate both sides:

$$e^y = x - 1 \Rightarrow x = e^y + 1$$

Now swap x and y to get the inverse:

$$f^{-1}(x) = e^x + 1$$

Answer: $f^{-1}(x) = e^x + 1$

(b) **Domain and Range:**

For $f(x) = \ln(x - 1)$:

Domain: $x > 1 \Rightarrow (1, \infty)$, Range of $\ln(x - 1)$ is all real numbers: $(-\infty, \infty)$

So for the inverse:

Domain of $f^{-1}(x)$: $(-\infty, \infty)$, Range of $f^{-1}(x) = e^x + 1$: since $e^x > 0$, we get $f^{-1}(x) > 1 \Rightarrow (1, \infty)$

Summary:

Domain of $f(x) = (1, \infty)$

Range of $f(x) = (-\infty, \infty)$

Domain of $f^{-1}(x) = (-\infty, \infty)$

Range of $f^{-1}(x) = (1, \infty)$

(c) **Verify compositions:**

$$f(f^{-1}(x)) = \ln((e^x + 1) - 1) = \ln(e^x) = x$$

$$f^{-1}(f(x)) = e^{\ln(x-1)} + 1 = x - 1 + 1 = x$$

Both identities verified.

Question 3

Let $f(x) = \frac{x}{x+1}$, where $x \neq -1$.

- (a) Find the inverse function $f^{-1}(x)$.
- (b) State the domain and range of both $f(x)$ and $f^{-1}(x)$.
- (c) Verify the identity $f(f^{-1}(x)) = x$.

Question 3 - Solution

(a) Find the inverse of $f(x) = \frac{x}{x+1}$:

Let $y = \frac{x}{x+1}$. Solve for x :

Multiply both sides by $x+1$:

$$y(x+1) = x \Rightarrow yx + y = x \Rightarrow yx - x = -y \Rightarrow x(y-1) = -y \Rightarrow x = \frac{-y}{y-1}$$

Now switch x and y :

$$f^{-1}(x) = \frac{-x}{x-1}$$

Answer: $f^{-1}(x) = \frac{-x}{x-1}$

(b) Domain and Range:

For $f(x) = \frac{x}{x+1}$: Domain: $x \neq -1$, As $x \rightarrow \infty$, $f(x) \rightarrow 1$, and as $x \rightarrow -\infty$, $f(x) \rightarrow 1$, Solve $\frac{x}{x+1} = 1 \Rightarrow x = x+1 \Rightarrow 0 = 1 \rightarrow$ contradiction
So $f(x) \neq 1$

Domain: $(-\infty, -1) \cup (-1, \infty)$ **Range:** $(-\infty, 1) \cup (1, \infty)$

Now for $f^{-1}(x) = \frac{-x}{x-1}$:

Domain excludes $x = 1$, since denominator is zero Range excludes $y = -1$, since solving $\frac{-x}{x-1} = -1 \Rightarrow x = \frac{1}{2}$, which contradicts range of original function

Domain of $f^{-1}(x) = (-\infty, 1) \cup (1, \infty)$ **Range of** $f^{-1}(x) = (-\infty, -1) \cup (-1, \infty)$

(c) Verify:

$$f(f^{-1}(x)) = f\left(\frac{-x}{x-1}\right) = \frac{\frac{-x}{x-1}}{\frac{-x}{x-1} + 1} = \frac{\frac{-x}{x-1}}{\frac{-x + (x-1)}{x-1}} = \frac{\frac{-x}{x-1}}{\frac{-1}{x-1}} = x$$

Verified: $f(f^{-1}(x)) = x$

Question 4

Suppose $f(x) = \frac{1}{x^2 + 1}$.

- (a) Determine whether $f(x)$ has an inverse function. Justify.
- (b) If not, restrict the domain of f so that it becomes one-to-one and then find the inverse on that restricted domain.
- (c) State the domain and range of the inverse function you found.

Question 4 - Solution

(a) Does $f(x) = \frac{1}{x^2 + 1}$ have an inverse?

No. Its natural domain is $(-\infty, \infty)$, since $x^2 + 1$ is never zero. To have an inverse function, f must be **one-to-one**: different inputs must give different outputs.

Horizontal line test: Imagine drawing horizontal lines across the graph. Each horizontal line represents a fixed output y . If even one such line crosses the graph more than once, then different inputs give the same output, so the function is not one-to-one.

Testing points: We can show this without drawing the graph by choosing two different inputs and calculating their outputs. Try $x = 1$ and $x = -1$, since squaring either gives 1:

$$f(1) = \frac{1}{1^2 + 1} = \frac{1}{2}, \quad f(-1) = \frac{1}{(-1)^2 + 1} = \frac{1}{2}.$$

Thus, the horizontal line $y = \frac{1}{2}$ meets the graph at both $(1, \frac{1}{2})$ and $(-1, \frac{1}{2})$. An inverse would have to send $\frac{1}{2}$ back to both 1 and -1 , but a function can give only one output for each input.

So, f fails the horizontal line test and has no inverse function on $(-\infty, \infty)$. One pair of different inputs with the same output is enough to prove this; testing a few points with different outputs would not prove that a function is one-to-one everywhere.

(b) **Restrict domain to make f one-to-one:**

Choose only nonnegative inputs. As x increases on $[0, \infty)$, the denominator $x^2 + 1$ increases, so $f(x)$ strictly decreases. Therefore, no output repeats, and the restricted function is one-to-one. We use:

$$x \in [0, \infty)$$

Let $y = \frac{1}{x^2 + 1}$, with $x \geq 0$.
Solve for x :

$$\begin{aligned} y(x^2 + 1) &= 1 \\ x^2 + 1 &= \frac{1}{y} \\ x^2 &= \frac{1}{y} - 1 = \frac{1 - y}{y} \\ x &= \sqrt{\frac{1 - y}{y}}. \end{aligned}$$

We take only the nonnegative square root because our restricted domain requires $x \geq 0$. Now switch x and y to express the inverse using x as its input:

$$f^{-1}(x) = \sqrt{\frac{1 - x}{x}}, \quad \text{for } 0 < x \leq 1$$

(c) **Domain and range of the inverse function:**

The restricted function has domain $[0, \infty)$. Its largest output is $f(0) = 1$. As x increases, its outputs approach 0 but never equal 0, so its range is $(0, 1]$.

An inverse reverses inputs and outputs. Therefore, the original range becomes the inverse's domain, and the original domain becomes the inverse's range:

$$\text{Domain of } f^{-1}(x) = (0, 1], \quad \text{Range of } f^{-1}(x) = [0, \infty)$$

Answer:

$$f^{-1}(x) = \sqrt{\frac{1 - x}{x}}, \quad x \in (0, 1]$$

Question 5

Let $f(x) = \frac{3x - 4}{5x + 2}$, where $x \neq -\frac{2}{5}$.

- (a) Show that f is a one-to-one function.
- (b) Find the inverse function $f^{-1}(x)$.
- (c) Determine the domain and range of $f(x)$ and $f^{-1}(x)$.

Question 5 - Solution

(a) **Show that f is one-to-one:**

Suppose $f(x_1) = f(x_2)$. Then:

$$\frac{3x_1 - 4}{5x_1 + 2} = \frac{3x_2 - 4}{5x_2 + 2}$$

Cross-multiplying:

$$(3x_1 - 4)(5x_2 + 2) = (3x_2 - 4)(5x_1 + 2)$$

Expanding both sides:

$$15x_1x_2 + 6x_1 - 20x_2 - 8 = 15x_1x_2 + 6x_2 - 20x_1 - 8$$

Cancel $15x_1x_2$ and -8 :

$$6x_1 - 20x_2 = 6x_2 - 20x_1 \Rightarrow 6x_1 + 20x_1 = 6x_2 + 20x_2 \Rightarrow 26x_1 = 26x_2 \Rightarrow x_1 = x_2$$

Therefore, f is one-to-one.

(b) **Find the inverse function:**

Let $y = \frac{3x-4}{5x+2}$

Multiply both sides:

$$y(5x + 2) = 3x - 4 \Rightarrow 5xy + 2y = 3x - 4 \Rightarrow 5xy - 3x = -2y - 4 \Rightarrow x(5y - 3) = -2y - 4 \Rightarrow x = \frac{-2y - 4}{5y - 3}$$

Now swap x and y :

$$f^{-1}(x) = \frac{-2x - 4}{5x - 3}$$

(c) **Domain and Range:**

The only restriction on the domain of f is $5x + 2 \neq 0 \Rightarrow x \neq -\frac{2}{5}$

The inverse is undefined when the denominator is 0:

$$5x - 3 \neq 0 \Rightarrow x \neq \frac{3}{5}$$

So:

$$\text{Domain of } f(x) : \boxed{\left(-\infty, -\frac{2}{5}\right) \cup \left(-\frac{2}{5}, \infty\right)}$$

$$\text{Range of } f(x) : \boxed{\left(-\infty, \frac{3}{5}\right) \cup \left(\frac{3}{5}, \infty\right)}$$

$$\text{Domain of } f^{-1}(x) : \boxed{\left(-\infty, \frac{3}{5}\right) \cup \left(\frac{3}{5}, \infty\right)}$$

$$\text{Range of } f^{-1}(x) : \boxed{\left(-\infty, -\frac{2}{5}\right) \cup \left(-\frac{2}{5}, \infty\right)}$$

Question 6

Let $f(x) = \sqrt{2x - 1}$ with domain $x \geq \frac{1}{2}$.

- (a) Show that f is one-to-one and has an inverse.
- (b) Find the inverse function $f^{-1}(x)$.
- (c) Determine the domain and range of $f^{-1}(x)$.
- (d) Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Question 6 - Solution**(a) One-to-One Check:**

Since $f(x) = \sqrt{2x-1}$ is a square root function with a linear expression inside and domain $x \geq \frac{1}{2}$, it is strictly increasing. Therefore, it passes the horizontal line test and is one-to-one.

(b) Find the inverse:

Let $y = \sqrt{2x-1}$. Solve for x :

$$y^2 = 2x - 1 \Rightarrow 2x = y^2 + 1 \Rightarrow x = \frac{y^2 + 1}{2}$$

Now switch x and y :

$$f^{-1}(x) = \frac{x^2 + 1}{2}$$

Answer: $\boxed{f^{-1}(x) = \frac{x^2 + 1}{2}}$

(c) Domain and Range of Inverse:

Since original domain is $x \geq \frac{1}{2}$, the range is:

$$f(x) = \sqrt{2x-1} \geq 0 \Rightarrow \text{Range of } f = [0, \infty)$$

So: - Domain of $f^{-1}(x) = [0, \infty)$ - Range of $f^{-1}(x) = [\frac{1}{2}, \infty)$

(d) Verifying composition:

$$f(f^{-1}(x)) = f\left(\frac{x^2 + 1}{2}\right) = \sqrt{2 \cdot \frac{x^2 + 1}{2} - 1} = \sqrt{x^2 + 1 - 1} = \sqrt{x^2} = x \quad (\text{since } x \geq 0)$$

$$f^{-1}(f(x)) = f^{-1}(\sqrt{2x-1}) = \frac{(\sqrt{2x-1})^2 + 1}{2} = \frac{2x - 1 + 1}{2} = x$$

Both identities verified.

Question 7

Let $f(x) = \frac{6}{x+2} - 1$, with domain $x \neq -2$.

- (a) Show that f is a one-to-one function.
- (b) Find the inverse function $f^{-1}(x)$.
- (c) Determine the domain and range of both $f(x)$ and $f^{-1}(x)$.

Question 7 - Solution

(a) Show $f(x) = \frac{6}{x+2} - 1$ is one-to-one:

Let $f(x_1) = f(x_2)$. Then:

$$\frac{6}{x_1+2} - 1 = \frac{6}{x_2+2} - 1 \Rightarrow \frac{6}{x_1+2} = \frac{6}{x_2+2} \Rightarrow x_1+2 = x_2+2 \Rightarrow x_1 = x_2$$

Thus, f is one-to-one.

(b) Find the inverse function:

Let $y = \frac{6}{x+2} - 1$

Solve for x :

$$y+1 = \frac{6}{x+2} \Rightarrow x+2 = \frac{6}{y+1} \Rightarrow x = \frac{6}{y+1} - 2$$

Now switch x and y :

$$f^{-1}(x) = \frac{6}{x+1} - 2$$

Answer: $f^{-1}(x) = \frac{6}{x+1} - 2$

(c) Domain and Range:

Original function $f(x) = \frac{6}{x+2} - 1$

- Undefined when $x+2=0 \Rightarrow x=-2$ - Domain of f : $(-\infty, -2) \cup (-2, \infty)$

Asymptote at $x=-2$, and horizontal asymptote $y=-1$ - Range of f : $(-\infty, -1) \cup (-1, \infty)$

Inverse: - Domain of f^{-1} : same as range of f : $(-\infty, -1) \cup (-1, \infty)$ - Range of f^{-1} : same as domain of f : $(-\infty, -2) \cup (-2, \infty)$

Question 8

Let $f(x) = \frac{x+3}{x-1}$, where $x \neq 1$.

- (a) Show that f is a one-to-one function.
- (b) Find the inverse function $f^{-1}(x)$.
- (c) State the domain and range of both $f(x)$ and $f^{-1}(x)$.
- (d) Verify that $f(f^{-1}(x)) = x$.

Question 8 - Solution

(a) Show that f is one-to-one:Let $f(x_1) = f(x_2)$:

$$\frac{x_1 + 3}{x_1 - 1} = \frac{x_2 + 3}{x_2 - 1} \Rightarrow (x_1 + 3)(x_2 - 1) = (x_2 + 3)(x_1 - 1)$$

Expanding both sides:

$$x_1x_2 - x_1 + 3x_2 - 3 = x_1x_2 - x_2 + 3x_1 - 3$$

Subtract x_1x_2 and -3 from both sides:

$$-x_1 + 3x_2 = -x_2 + 3x_1 \Rightarrow 3x_2 + x_2 = 3x_1 + x_1 \Rightarrow 4x_2 = 4x_1 \Rightarrow x_1 = x_2$$

Therefore, f is one-to-one.

(b) Find the inverse:

$$\text{Let } y = \frac{x + 3}{x - 1}$$

Multiply both sides:

$$y(x - 1) = x + 3 \Rightarrow yx - y = x + 3 \Rightarrow yx - x = y + 3 \Rightarrow x(y - 1) = y + 3 \Rightarrow x = \frac{y + 3}{y - 1}$$

Switch x and y :

$$f^{-1}(x) = \frac{x + 3}{x - 1}$$

So:

$$\boxed{f^{-1}(x) = \frac{x + 3}{x - 1}}$$

Interestingly, $f^{-1}(x) = f(x)$. The function is its own inverse.

(c) Domain and Range:

From $f(x) = \frac{x + 3}{x - 1}$, we see that: - Domain: $x \neq 1 \Rightarrow \boxed{(-\infty, 1) \cup (1, \infty)}$ - As $x \rightarrow \infty$, $f(x) \rightarrow 1$, and $f(x) \neq 1$ - So range: $\boxed{(-\infty, 1) \cup (1, \infty)}$

The same applies to $f^{-1}(x)$, so: - Domain of $f^{-1}(x) : (-\infty, 1) \cup (1, \infty)$ - Range of $f^{-1}(x) : (-\infty, 1) \cup (1, \infty)$

(d) Verify:

$$f(f^{-1}(x)) = f\left(\frac{x + 3}{x - 1}\right) = \frac{\left(\frac{x + 3}{x - 1}\right) + 3}{\left(\frac{x + 3}{x - 1}\right) - 1} = \frac{\frac{x + 3 + 3(x - 1)}{x - 1}}{\frac{x + 3 - (x - 1)}{x - 1}} = \frac{\frac{x + 3 + 3x - 3}{x - 1}}{\frac{x + 3 - x + 1}{x - 1}} = \frac{\frac{4x}{x - 1}}{\frac{4}{x - 1}} = x$$

Verified: $f(f^{-1}(x)) = x$

Question 9

Let $f(x) = \ln(x - 2)$, with domain $x > 2$.

- (a) Find the inverse function $f^{-1}(x)$.
- (b) Determine the domain and range of f and f^{-1} .
- (c) Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Question 9 - Solution**(a) Find the inverse:**Let $y = \ln(x - 2)$ Solve for x :

$$x - 2 = e^y \Rightarrow x = e^y + 2$$

Now switch x and y :

$$f^{-1}(x) = e^x + 2$$

Answer: $f^{-1}(x) = e^x + 2$ **(b) Domain and Range:**For $f(x) = \ln(x - 2)$: - Domain: $x > 2 \Rightarrow (2, \infty)$ - Range: $\ln(x - 2) \rightarrow -\infty$ as $x \rightarrow 2^+$, and $\rightarrow \infty$ as $x \rightarrow \infty$ Range: $(-\infty, \infty)$ For $f^{-1}(x) = e^x + 2$: - Domain: $(-\infty, \infty)$ - Range: $e^x + 2 > 2 \Rightarrow (2, \infty)$ **(c) Verify compositions:**

$$f(f^{-1}(x)) = f(e^x + 2) = \ln((e^x + 2) - 2) = \ln(e^x) = x$$

$$f^{-1}(f(x)) = f^{-1}(\ln(x - 2)) = e^{\ln(x - 2)} + 2 = x - 2 + 2 = x$$

Verified: $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

Question 10

Let

$$f(x) = \frac{\sqrt{x+4}}{x-1}$$

- (a) Determine the domain of $f(x)$.
- (b) Is f one-to-one on its domain? Justify your answer.
- (c) If possible, find $f^{-1}(x)$. If not, restrict the domain appropriately and find the inverse on that domain.

Question 10 - Solution

Domain. The square root and denominator require

$$D = [-4, 1) \cup (1, \infty).$$

One-to-one. For $x > -4$, $x \neq 1$,

$$f'(x) = \frac{-(x+9)}{2\sqrt{x+4}(x-1)^2} < 0.$$

The left branch decreases from 0 to $-\infty$ and has range $(-\infty, 0]$.

The right branch decreases from $+\infty$ to 0 and has range $(0, \infty)$.

These ranges are disjoint, so f is one-to-one on its entire domain and its range is $\mathbb{R}$.

Inverse. Write $y = \sqrt{x+4}/(x-1)$. Squaring gives

$$y^2x^2 - (2y^2 + 1)x + y^2 - 4 = 0.$$

For $y \neq 0$, the quadratic formula gives

$$x = \frac{2y^2 + 1 \pm \sqrt{1 + 20y^2}}{2y^2}.$$

The original, unsquared equation requires $x < 1$ when $y < 0$ and $x > 1$ when $y > 0$.

Also $y = 0$ gives $x = -4$. Thus

$$f^{-1}(y) = \begin{cases} \frac{2y^2 + 1 - \sqrt{1 + 20y^2}}{2y^2}, & y < 0, \\ -4, & y = 0, \\ \frac{2y^2 + 1 + \sqrt{1 + 20y^2}}{2y^2}, & y > 0. \end{cases}$$

The sign choice restores the original equation after squaring. The inverse has domain $\mathbb{R}$ and range D .

Question 1

Given the function $f(x) = 3\sin\left(2x - \frac{\pi}{3}\right) + 1$, answer the following:

- (a) Determine the amplitude, period, phase shift, and vertical shift.
- (b) Find the maximum and minimum values of $f(x)$.
- (c) Sketch one full period of the graph, clearly labeling key points.
- (d) Determine the x -values where $f(x) = 1$ in the interval $[0, 2\pi]$.

Question 1 - Solution

(a) Identify parameters from $f(x) = 3 \sin(2x - \frac{\pi}{3}) + 1$:• Amplitude: $\boxed{3}$ • Period: $\frac{2\pi}{2} = \boxed{\pi}$

• Phase shift:

$$2x - \frac{\pi}{3} = 0 \Rightarrow x = \frac{\pi}{6} \Rightarrow \boxed{\frac{\pi}{6} \text{ to the right}}$$

• Vertical shift: $\boxed{+1}$

(b) Maximum and Minimum:

Since the sine function oscillates between -1 and 1 :

$$\text{Min of } f(x) = 3(-1) + 1 = -2, \quad \text{Max of } f(x) = 3(1) + 1 = 4$$

$$\boxed{\text{Max} = 4}, \quad \boxed{\text{Min} = -2}$$

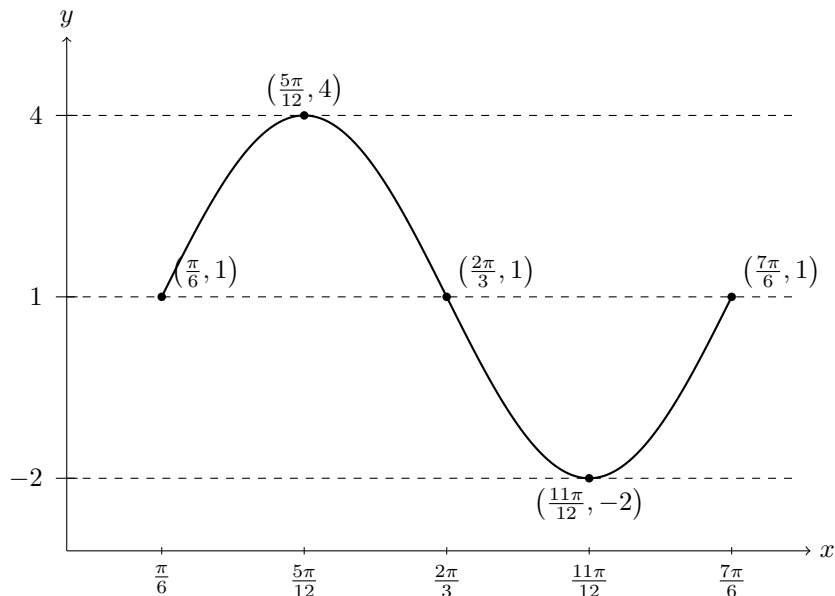
(c) Sketch one full period:

One full period has length π , starting at the phase shift:

$$x = \frac{\pi}{6} \quad \text{to} \quad x = \frac{7\pi}{6}$$

Key points over one period:

x	$f(x)$
$\frac{\pi}{6}$	1 (midline)
$\frac{5\pi}{12}$	4 (maximum)
$\frac{2\pi}{3}$	1 (midline)
$\frac{11\pi}{12}$	-2 (minimum)
$\frac{7\pi}{6}$	1 (midline)



(d) Solve $f(x) = 1$ on $[0, 2\pi]$:

$$3 \sin\left(2x - \frac{\pi}{3}\right) + 1 = 1 \Rightarrow \sin\left(2x - \frac{\pi}{3}\right) = 0$$

$$2x - \frac{\pi}{3} = n\pi \Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{6}$$

Values in $[0, 2\pi]$:

$x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

Question 2

Let

$$g(x) = 2 \tan\left(x - \frac{\pi}{4}\right) + 1$$

- (a) State the period, phase shift, vertical shift, and asymptotes of $g(x)$.
- (b) Sketch one period of the graph, labeling asymptotes and intercepts.
- (c) Find the value(s) of $x \in [0, 2\pi]$ such that $g(x) = 3$.

Question 2 - Solution

(a) **Analyze** $g(x) = 2 \tan\left(x - \frac{\pi}{4}\right) + 1$

- Period: π
- Phase shift: $\frac{\pi}{4}$ to the right
- Vertical shift: $+1$

Asymptotes occur when:

$$x - \frac{\pi}{4} = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z} \Rightarrow x = \frac{3\pi}{4} + k\pi$$

So one period lies on:

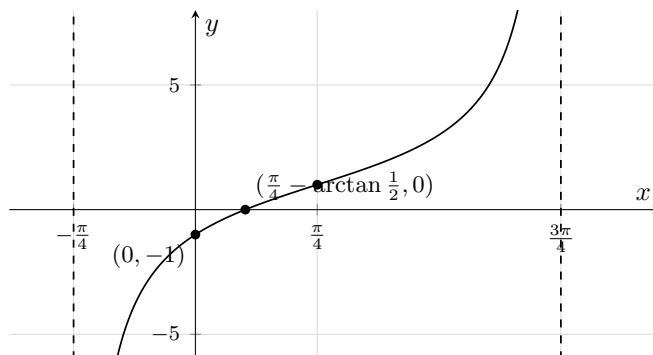
$$\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$$

- All asymptotes: $x = 3\pi/4 + k\pi, \quad k \in \mathbb{Z}$.

(b) **Sketch one period**The intercepts in this period are $(0, -1)$ and $(\pi/4 - \arctan(1/2), 0)$.

Key features:

- Left asymptote: $x = -\frac{\pi}{4}$
- Right asymptote: $x = \frac{3\pi}{4}$
- Midpoint (inflection): $x = \frac{\pi}{4}, \quad g\left(\frac{\pi}{4}\right) = 1$

(c) **Solve** $g(x) = 3$:

$$2 \tan\left(x - \frac{\pi}{4}\right) + 1 = 3 \Rightarrow \tan\left(x - \frac{\pi}{4}\right) = 1$$

$$x - \frac{\pi}{4} = \frac{\pi}{4} + n\pi \Rightarrow x = \frac{\pi}{2} + n\pi$$

Values in $[0, 2\pi]$:

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Question 3

Consider the function

$$f(x) = \cos(2x) - \sin(2x)$$

- (a) Write $f(x)$ as a single trigonometric function of the form $A \cos(2x + \phi)$.
- (b) Determine the amplitude and phase shift of the resulting function.
- (c) Find the maximum and minimum values of $f(x)$.
- (d) Determine all $x \in [0, 2\pi]$ such that $f(x) = 0$.

Question 3 - Solution

(a) Write $f(x)$ as a single cosine function:

We use the identity:

$$A \cos(2x + \phi) = A \cos \phi \cos(2x) - A \sin \phi \sin(2x)$$

Match coefficients with:

$$\cos(2x) - \sin(2x)$$

This gives the system:

$$A \cos \phi = 1, \quad A \sin \phi = 1$$

Square and add:

$$A^2(\cos^2 \phi + \sin^2 \phi) = 1^2 + 1^2 = 2 \Rightarrow A = \sqrt{2}$$

Now:

$$\cos \phi = \frac{1}{\sqrt{2}}, \quad \sin \phi = \frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{\pi}{4}$$

So:

$$f(x) = \sqrt{2} \cos\left(2x + \frac{\pi}{4}\right)$$

(b) Amplitude and Phase Shift:

From the rewritten form:

- Amplitude: $\boxed{\sqrt{2}}$
- Phase shift: Solve $2x + \frac{\pi}{4} = 0 \Rightarrow x = -\frac{\pi}{8}$

So the graph is shifted $\boxed{\frac{\pi}{8}}$ units to the left.

(c) Maximum and Minimum Values:

Since cosine ranges between -1 and 1 :

$$\text{Maximum} = \sqrt{2}, \quad \text{Minimum} = -\sqrt{2}$$

(d) Solve $f(x) = 0$ on $[0, 2\pi]$:

$$\cos(2x) - \sin(2x) = 0 \Rightarrow \cos(2x) = \sin(2x) \Rightarrow \tan(2x) = 1$$

$$2x = \frac{\pi}{4} + n\pi \Rightarrow x = \frac{\pi}{8} + \frac{n\pi}{2}$$

Find values in $[0, 2\pi]$:

$$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$$

Answer:

$$\boxed{x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}}$$

Question 4

Let $f(x) = \sin(x)\cos(x) + \cos^2(x)$

- (a) Express $f(x)$ in terms of a single trigonometric function or a simpler expression.
- (b) Find the maximum and minimum values of $f(x)$ on the interval $[0, 2\pi]$.
- (c) Determine all values of $x \in [0, 2\pi]$ where $f(x) = 1$.

Question 4 - Solution

Simplify. Double-angle identities give

$$f(x) = \frac{1}{2} + \frac{\sqrt{2}}{2} \sin(2x + \pi/4).$$

The sine attains both -1 and 1 on the given interval, so

$$f_{\min} = \frac{1 - \sqrt{2}}{2}, \quad f_{\max} = \frac{1 + \sqrt{2}}{2}.$$

Solve without losing roots. Subtracting 1 from the original expression gives

$$f(x) - 1 = \sin x \cos x - \sin^2 x = \sin x (\cos x - \sin x).$$

Thus $\sin x = 0$ or $\cos x = \sin x$. Restricting both families to $[0, 2\pi]$ gives

$$x = 0, \pi/4, \pi, 5\pi/4, 2\pi.$$

Each listed value makes one factor zero, so each satisfies the original equation.

Question 5

Let $f(x) = \frac{\sin(x)}{1+\cos(x)}$

- (a) Simplify the expression $f(x)$ using trigonometric identities.
- (b) Determine all $x \in (0, 2\pi)$ for which $f(x) = 1$.
- (c) Determine the domain of $f(x)$.
- (d) Determine whether $f(x)$ is an even function, odd function, or neither.

Question 5 - Solution

Simplify with the domain retained. Half-angle identities give

$$\boxed{f(x) = \tan(x/2)}, \quad x \neq (2k+1)\pi, \quad k \in \mathbb{Z}.$$

Indeed $\sin x = 2 \sin(x/2) \cos(x/2)$ and $1 + \cos x = 2 \cos^2(x/2)$.

The alternative $\csc x - \cot x$ is valid only when $\sin x \neq 0$ and loses valid inputs such as $x = 0$.

Solve. $\tan(x/2) = 1$ implies $x = \pi/2 + 2k\pi$, so on $(0, 2\pi)$,

$$\boxed{x = \pi/2}.$$

Domain and parity. The original denominator vanishes precisely at odd multiples of π , hence

$$\boxed{D = \mathbb{R} \setminus \{(2k+1)\pi : k \in \mathbb{Z}\}}.$$

This domain is symmetric about zero, and $f(-x) = -f(x)$ on it. Therefore $\boxed{f \text{ is odd}}$.

Question 6

Let $f(x) = \sin(x) + \sin(3x)$

- (a) Use a trigonometric identity to write $f(x)$ as a product.
- (b) Determine all values of $x \in [0, 2\pi]$ where $f(x) = 0$.
- (c) Sketch the graph of $f(x)$ over one full period.

Question 6 - Solution

(a) Use sum-to-product identities:

We apply:

$$\sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)$$

Let $A = 3x$, $B = x$:

$$f(x) = \sin(x) + \sin(3x) = 2 \sin \left(\frac{3x+x}{2} \right) \cos \left(\frac{3x-x}{2} \right) = 2 \sin(2x) \cos(x)$$

$$f(x) = 2 \sin(2x) \cos(x)$$

(b) Solve $f(x) = 0$ on $[0, 2\pi]$:

$$2 \sin(2x) \cos(x) = 0 \Rightarrow \sin(2x) = 0 \quad \text{or} \quad \cos(x) = 0$$

Solve $\sin(2x) = 0$:

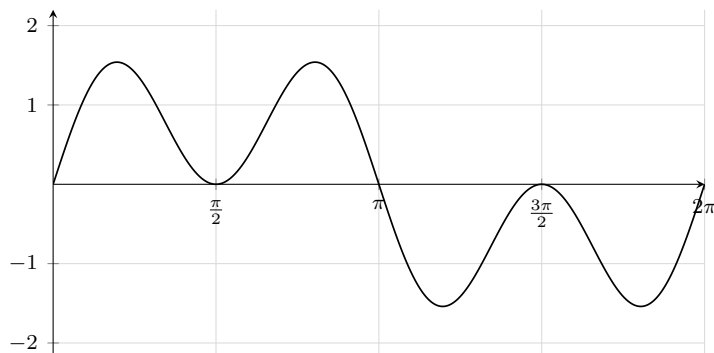
$$2x = n\pi \Rightarrow x = \frac{n\pi}{2} \Rightarrow x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

Solve $\cos(x) = 0$:

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Union of solutions:

$$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

(c) Sketch the graph of $f(x) = \sin(x) + \sin(3x)$ The graph is periodic with the least common multiple of the periods of $\sin(x)$ and $\sin(3x)$.Period of $\sin(x)$: 2π Period of $\sin(3x)$: $\frac{2\pi}{3}$ LCM of 2π and $\frac{2\pi}{3}$ is 2π So sketch one full period over $[0, 2\pi]$. Use the product form $f(x) = 2 \sin(2x) \cos(x)$ to help identify:Zeros: Where either $\sin(2x) = 0$ or $\cos(x) = 0$ Amplitude varies due to the product of two trig functionsImportant points: $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$, and midpoints between for peaks/troughs

Question 7

Let

$$f(x) = \frac{1 + \cos(2x)}{\sin(2x)}$$

- (a) Simplify the expression as much as possible.
- (b) Find all $x \in (0, \pi)$ for which $f(x) = 2$.
- (c) Determine the domain of $f(x)$ over $[0, 2\pi]$.

Question 7 - Solution

Simplify. Where the original function is defined,

$$f(x) = \frac{2 \cos^2 x}{2 \sin x \cos x} = \cot x.$$

The cancellation does not restore points where $\sin(2x) = 0$.

Solve. On $(0, \pi)$, $\cot x = 2$ gives

$$x = \arctan(1/2) \approx 0.4636.$$

This point is in the original domain.

Domain. All multiples of $\pi/2$ must be excluded. On $[0, 2\pi]$ the domain is

$$(0, \pi/2) \cup (\pi/2, \pi) \cup (\pi, 3\pi/2) \cup (3\pi/2, 2\pi).$$

Question 8

Let

$$f(x) = \frac{\cos(x)}{1 - \sin(x)}$$

- (a) Simplify the expression $f(x)$ using trigonometric identities.
- (b) Determine all values of $x \in (0, 2\pi)$ for which $f(x) = 1$.
- (c) Determine whether $f(x)$ is even, odd, or neither.

Question 8 - Solution

Simplification and domain. The original function is defined for $x \neq \pi/2 + 2k\pi$. Where $\cos x \neq 0$, rationalization gives

$$f(x) = \frac{1 + \sin x}{\cos x} = \sec x + \tan x.$$

At $x = 3\pi/2 + 2k\pi$, the original function has value 0, although this alternative expression is undefined.

Solve on the requested interval. If $f(x) = 1$, then $\cos x + \sin x = 1$, with $\sin x \neq 1$.

Using $\cos x = 1 - \sin x$ in $\cos^2 x + \sin^2 x = 1$ gives

$$2 \sin x (\sin x - 1) = 0.$$

The case $\sin x = 1$ is excluded. The other case requires $\sin x = 0$ and $\cos x = 1$, hence $x = 2k\pi$.

Neither endpoint belongs to $(0, 2\pi)$, so

There are no solutions in $(0, 2\pi)$.

Parity. The domain is not symmetric about zero: $-\pi/2$ is allowed but $\pi/2$ is not.

f is neither even nor odd.

Question 9

Let

$$f(x) = \tan(x) - \sec(x)$$

- (a) Show that $f(x) = \frac{\sin(x)-1}{\cos(x)}$
- (b) Determine the values of $x \in (0, 2\pi)$ for which $f(x) = 0$
- (c) Analyze the behavior of $f(x)$ near $x = \frac{\pi}{2}$

Question 9 - Solution

(a) Simplify $f(x) = \tan(x) - \sec(x)$:

We write both terms in terms of sine and cosine:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \sec(x) = \frac{1}{\cos(x)}$$

Then:

$$f(x) = \frac{\sin(x)}{\cos(x)} - \frac{1}{\cos(x)} = \frac{\sin(x) - 1}{\cos(x)}$$

$$f(x) = \frac{\sin(x) - 1}{\cos(x)}$$

(b) Solve $f(x) = 0 \Rightarrow \frac{\sin(x)-1}{\cos(x)} = 0$

This occurs when the numerator is 0 (and denominator is not 0):

$$\sin(x) - 1 = 0 \Rightarrow \sin(x) = 1 \Rightarrow x = \frac{\pi}{2}$$

But at $x = \frac{\pi}{2}$, $\cos(x) = 0$, so $f(x)$ is undefined.Therefore, there is **no solution** to $f(x) = 0$ on $(0, 2\pi)$

No solution in $(0, 2\pi)$

(c) Behavior near $x = \frac{\pi}{2}$ As $x \rightarrow \frac{\pi}{2}^-$, we have: - $\sin(x) \rightarrow 1$ - $\cos(x) \rightarrow 0^+$ - So: $f(x) \rightarrow \frac{1-1}{0^+} = 0$ As $x \rightarrow \frac{\pi}{2}^+$, we have: - $\sin(x) \rightarrow 1$ - $\cos(x) \rightarrow 0^-$ - So: $f(x) \rightarrow \frac{1-1}{0^-} = 0$ But if we take x very close to $\frac{\pi}{2}$, with small deviation, say:

$$f(x) = \frac{\sin(x) - 1}{\cos(x)} \Rightarrow \text{Numerator} \approx 0, \text{ Denominator} \rightarrow 0$$

Apply L'Hôpital's Rule:

$$\lim_{x \rightarrow \frac{\pi}{2}} f(x) = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos(x)}{-\sin(x)} = \frac{0}{-1} = 0$$

So:

$$\lim_{x \rightarrow \frac{\pi}{2}} f(x) = 0$$

But since the function is undefined at $x = \frac{\pi}{2}$, the limit exists but the function is discontinuous there.

Question 10

Define the function:

$$f(x) = \frac{1 - \cos(2x)}{\sin(x)}$$

- (a) Simplify $f(x)$ using appropriate identities.
- (b) Determine the values of $x \in (0, \pi)$ for which $f(x) = 2 \sin(x)$.
- (c) State the domain of $f(x)$ on $[0, 2\pi]$.

Question 10 - Solution**(a) Simplify using identities:**

Using the identity

$$1 - \cos(2x) = 2 \sin^2(x),$$

we obtain

$$f(x) = \frac{2 \sin^2(x)}{\sin(x)}.$$

For $\sin(x) \neq 0$, this simplifies to

$$f(x) = 2 \sin(x).$$

$$f(x) = 2 \sin(x) \quad \text{for } \sin(x) \neq 0$$

(b) Solve $f(x) = 2 \sin(x)$

From part (a), the equation

$$f(x) = 2 \sin(x)$$

is an identity for all x where $f(x)$ is defined.

However, the original function is undefined when

$$\sin(x) = 0 \Rightarrow x = 0, \pi.$$

Since the interval is restricted to $(0, \pi)$, neither excluded value lies in the interval. Therefore, all values in $(0, \pi)$ satisfy the equation.

$$\text{Solution set: } (0, \pi) \quad (\text{excluded: none within the interval})$$

(c) Domain of $f(x) = \frac{1 - \cos(2x)}{\sin(x)}$ on $[0, 2\pi]$

The function is undefined when the denominator is zero:

$$\sin(x) = 0 \Rightarrow x = 0, \pi, 2\pi.$$

Removing these values from the interval $[0, 2\pi]$, the domain is

$$(0, \pi) \cup (\pi, 2\pi).$$

Excluded values: $x = 0, \pi, 2\pi$

Question 1

Solve the equation:

$$2 \sin^2(x) - 3 \sin(x) + 1 = 0$$

for all $x \in [0, 2\pi]$.

Question 1 - Solution

We factor the equation:

$$2 \sin^2(x) - 3 \sin(x) + 1 = 0.$$

Looking for factors of $2 \cdot 1 = 2$ that add to -3 , we write:

$$(2 \sin(x) - 1)(\sin(x) - 1) = 0.$$

Setting each factor equal to zero:

$$2 \sin(x) - 1 = 0 \quad \text{or} \quad \sin(x) - 1 = 0.$$

$$\sin(x) = \frac{1}{2} \quad \text{or} \quad \sin(x) = 1.$$

Now solve for $x \in [0, 2\pi]$:

$$\sin(x) = 1 \Rightarrow x = \frac{\pi}{2}.$$

$$\sin(x) = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}.$$

Therefore, the solutions are:

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}.$$

Question 2

Solve the equation

$$\cos(2x) = 2 \cos^2(x) - 1$$

for all $x \in [0, 2\pi]$, and verify the identity used.

Question 2 - Solution**Step 1: Recognize the identity.**

The right-hand side is a known double-angle identity:

$$\cos(2x) = 2 \cos^2(x) - 1$$

So this equation is always true:

$$\cos(2x) = \cos(2x)$$

Thus, it holds for all x .

But the problem is to solve:

$$\cos(2x) = 2 \cos^2(x) - 1$$

Since it holds for all x , this is an identity. So the equation is true for all $x \in [0, 2\pi]$

$$\boxed{x \in [0, 2\pi]}$$

Verification of identity:

Start with RHS:

$$2 \cos^2(x) - 1 = \cos(2x)$$

This is a standard double-angle identity for cosine:

$$\cos(2x) = 2 \cos^2(x) - 1 \Rightarrow \text{Identity verified.}$$

Question 3

Solve the equation

$$\sin(x) \cos(x) = \frac{1}{2}$$

for all $x \in [0, 2\pi]$.

Question 3 - Solution

We begin by using a known identity:

$$\sin(2x) = 2 \sin(x) \cos(x) \Rightarrow \sin(x) \cos(x) = \frac{1}{2} \sin(2x)$$

Substitute into the equation:

$$\frac{1}{2} \sin(2x) = \frac{1}{2} \Rightarrow \sin(2x) = 1$$

Now solve:

$$\sin(2x) = 1 \Rightarrow 2x = \frac{\pi}{2} + 2n\pi \quad \text{for integer } n \Rightarrow x = \frac{\pi}{4} + n\pi$$

Find solutions in $[0, 2\pi]$:

$$n = 0 \Rightarrow x = \frac{\pi}{4} \quad n = 1 \Rightarrow x = \frac{5\pi}{4} \quad n = 2 \Rightarrow x = \frac{9\pi}{4} > 2\pi \rightarrow \text{discard}$$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

Question 4

Solve the equation

$$\cos(2x) + \sin(x) = 0$$

for all $x \in [0, 2\pi]$.

Question 4 - Solution**Step 1: Use a double-angle identity for $\cos(2x)$:**

We use:

$$\cos(2x) = 1 - 2\sin^2(x)$$

Substitute into the equation:

$$1 - 2\sin^2(x) + \sin(x) = 0 \Rightarrow -2\sin^2(x) + \sin(x) + 1 = 0$$

Multiply through by -1 to simplify:

$$2\sin^2(x) - \sin(x) - 1 = 0$$

Step 2: Solve the quadratic equation in $\sin(x)$:Let $u = \sin(x)$:

$$2u^2 - u - 1 = 0 \Rightarrow u = \frac{1 \pm \sqrt{1^2 + 4(2)(1)}}{2(2)} = \frac{1 \pm \sqrt{9}}{4} = \frac{1 \pm 3}{4}$$

So:

$$u = 1 \quad \text{or} \quad u = -\frac{1}{2}$$

Now solve $\sin(x) = 1 \Rightarrow x = \frac{\pi}{2}$
And $\sin(x) = -\frac{1}{2} \Rightarrow x = \frac{7\pi}{6}, \frac{11\pi}{6}$ **Final answer:**

$x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$
--

Question 5

Solve the equation

$$\sin(x) + \sqrt{3}\cos(x) = 0$$

for all $x \in [0, 2\pi]$.

Question 5 - Solution

We start by solving:

$$\sin(x) + \sqrt{3}\cos(x) = 0 \Rightarrow \sin(x) = -\sqrt{3}\cos(x)$$

Divide both sides by $\cos(x)$, assuming $\cos(x) \neq 0$:

$$\tan(x) = -\sqrt{3}$$

Now solve:

$$\tan(x) = -\sqrt{3} \Rightarrow x = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

But we want values in $[0, 2\pi]$

So find angles with tangent $= -\sqrt{3}$

The reference angle is $\frac{\pi}{3}$, and tangent is negative in quadrants II and IV

$$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}, \quad x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

$$\boxed{x = \frac{2\pi}{3}, \frac{5\pi}{3}}$$

Also check if $\cos(x) = 0$

If $\cos(x) = 0 \Rightarrow \sin(x) = 0$ also must hold (from original equation)

But: - If $\cos(x) = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$ - At $x = \frac{\pi}{2}$: $\sin(x) = 1$, LHS $= 1 + 0 \neq 0$ - At $x = \frac{3\pi}{2}$: $\sin(x) = -1$, LHS $= -1 + 0 \neq 0$

So those are not valid. Final solution:

$$\boxed{x = \frac{2\pi}{3}, \frac{5\pi}{3}}$$

Question 6

Solve the equation

$$2 \cos^2(x) - 1 = \sin(x)$$

for all $x \in [0, 2\pi]$.

Question 6 - Solution

We begin by using the Pythagorean identity:

$$\cos^2(x) = 1 - \sin^2(x)$$

Substitute into the equation:

$$2(1 - \sin^2(x)) - 1 = \sin(x) \Rightarrow 2 - 2\sin^2(x) - 1 = \sin(x) \Rightarrow -2\sin^2(x) - \sin(x) + 1 = 0$$

Multiply through by -1 :

$$2\sin^2(x) + \sin(x) - 1 = 0$$

Let $u = \sin(x)$. Then:

$$2u^2 + u - 1 = 0 \Rightarrow u = \frac{-1 \pm \sqrt{1^2 - 4(2)(-1)}}{2(2)} = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4}$$

$$u = \frac{1}{2}, \quad u = -1$$

So:

$$\sin(x) = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\sin(x) = -1 \Rightarrow x = \frac{3\pi}{2}$$

Final solution:

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

Question 7

Solve the equation

$$\sin(2x) = \sqrt{3} \cos(x)$$

for all $x \in [0, 2\pi]$.

Question 7 - Solution

Use the identity:

$$\sin(2x) = 2 \sin(x) \cos(x)$$

Substitute into the equation:

$$2 \sin(x) \cos(x) = \sqrt{3} \cos(x)$$

Subtract $\sqrt{3} \cos(x)$ from both sides:

$$2 \sin(x) \cos(x) - \sqrt{3} \cos(x) = 0 \Rightarrow \cos(x) (2 \sin(x) - \sqrt{3}) = 0$$

Now solve each factor:

****Case 1:**** $\cos(x) = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

****Case 2:**** $2 \sin(x) - \sqrt{3} = 0 \Rightarrow \sin(x) = \frac{\sqrt{3}}{2}$

$$\sin(x) = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}$$

Final Solution:

$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}$
--

Question 8

Solve the equation

$$\sin^2(x) - \cos(x) = 0$$

for all $x \in [0, 2\pi]$.

Question 8 - Solution

We start with the identity:

$$\sin^2(x) = 1 - \cos^2(x)$$

Substitute into the equation:

$$1 - \cos^2(x) - \cos(x) = 0 \Rightarrow -\cos^2(x) - \cos(x) + 1 = 0 \Rightarrow \cos^2(x) + \cos(x) - 1 = 0$$

Let $u = \cos(x)$, then:

$$u^2 + u - 1 = 0 \Rightarrow u = \frac{-1 \pm \sqrt{1^2 + 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$$

So:

$$\cos(x) = \frac{-1 + \sqrt{5}}{2} \approx 0.618, \quad \cos(x) = \frac{-1 - \sqrt{5}}{2} \approx -1.618$$

Only $\frac{-1 + \sqrt{5}}{2}$ is in the range $[-1, 1]$

Now solve:

$$\cos(x) = \frac{-1 + \sqrt{5}}{2} \Rightarrow x = \cos^{-1}\left(\frac{-1 + \sqrt{5}}{2}\right)$$

Let's call $\theta = \cos^{-1}\left(\frac{-1 + \sqrt{5}}{2}\right)$

Since cosine is positive in quadrants I and IV, and $\cos(x) > 0$, we get:

$$x = \theta, \quad x = 2\pi - \theta$$

So:

$$x = \cos^{-1}\left(\frac{-1 + \sqrt{5}}{2}\right), \quad x = 2\pi - \cos^{-1}\left(\frac{-1 + \sqrt{5}}{2}\right)$$

These are the exact answers in terms of inverse cosine.

Question 9

Solve the equation

$$\tan^2(x) - 3 \tan(x) + 2 = 0$$

for all $x \in [0, 2\pi)$.

Question 9 - Solution

Treat this as a quadratic in $\tan(x)$. Let $u = \tan(x)$:

$$u^2 - 3u + 2 = 0 \Rightarrow u = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(2)}}{2(1)} = \frac{3 \pm \sqrt{9 - 8}}{2} = \frac{3 \pm 1}{2} \Rightarrow u = 1, 2$$

So:

$$\tan(x) = 1 \quad \text{or} \quad \tan(x) = 2$$

Solve $\tan(x) = 1$:

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

Solve $\tan(x) = 2$:

We must use the inverse tangent:

$$x = \tan^{-1}(2) \approx 1.107 \text{ radians}$$

Tangent is positive in quadrants I and III, so:

$$x = \tan^{-1}(2), \pi + \tan^{-1}(2)$$

Approximate values:

$$x \approx 1.107, 4.249$$

Final Answer:

$x = \frac{\pi}{4}, \frac{5\pi}{4}, \tan^{-1}(2), \pi + \tan^{-1}(2)$

Question 10

Solve the equation

$$\sec(x) = 2$$

for all $x \in [0, 2\pi]$.

Question 10 - Solution

Step 1: Recall the definition of secant:

$$\sec(x) = \frac{1}{\cos(x)} \Rightarrow \frac{1}{\cos(x)} = 2 \Rightarrow \cos(x) = \frac{1}{2}$$

Step 2: Solve $\cos(x) = \frac{1}{2}$ in $[0, 2\pi]$:

The cosine is $\frac{1}{2}$ at:

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\boxed{x = \frac{\pi}{3}, \frac{5\pi}{3}}$$

Question 1

Solve the equation

$$\sin(x) = 0.65$$

for all $x \in [0, 2\pi]$, and give your answers rounded to two decimal places.

Question 1 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\sin(1x) = 0.65, \quad u = 1x, \quad 0 \leq u \leq 2\pi.$$

Let $\alpha = \arcsin(0.65)$. All solutions are

$$u = \alpha + 2k\pi \quad \text{or} \quad u = \pi - \alpha + 2k\pi, \quad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1.
Evaluating the inverse function at full precision and rounding only the final values gives

$$x \approx 0.71, 2.43.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

Question 2

Solve the equation

$$\cos(x) = -0.34$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

Question 2 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\cos(1x) = -0.34, \quad u = 1x, \quad 0 \leq u \leq 2\pi.$$

Let $\alpha = \arccos(-0.34)$. All solutions are

$$u = \pm\alpha + 2k\pi, \quad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$x \approx 1.92, 4.37.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

Question 3

Solve the equation

$$\tan(x) = 1.8$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

Question 3 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\tan(1x) = 1.8, \quad u = 1x, \quad 0 \leq u \leq 2\pi.$$

Let $\alpha = \arctan(1.8)$. All solutions are

$$u = \alpha + k\pi, \quad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$x \approx 1.06, 4.21.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

Question 4

Solve the equation

$$\sin(x) = -0.77$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

Question 4 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\sin(1x) = -0.77, \quad u = 1x, \quad 0 \leq u \leq 2\pi.$$

Let $\alpha = \arcsin(-0.77)$. All solutions are

$$u = \alpha + 2k\pi \quad \text{or} \quad u = \pi - \alpha + 2k\pi, \quad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$x \approx 4.02, 5.40.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

Question 5

Solve the equation

$$\cos(x) = 0.29$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

Question 5 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\cos(1x) = 0.29, \quad u = 1x, \quad 0 \leq u \leq 2\pi.$$

Let $\alpha = \arccos(0.29)$. All solutions are

$$u = \pm\alpha + 2k\pi, \quad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1.
Evaluating the inverse function at full precision and rounding only the final values gives

$$x \approx 1.28, 5.01.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

Question 6

Solve the equation

$$\tan(x) = -0.52$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

Question 6 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\tan(1x) = -0.52, \quad u = 1x, \quad 0 \leq u \leq 2\pi.$$

Let $\alpha = \arctan(-0.52)$. All solutions are

$$u = \alpha + k\pi, \quad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$x \approx 2.66, 5.80.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

Question 7

Solve the equation

$$\sin(x) = -0.42$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

Question 7 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\sin(1x) = -0.42, \quad u = 1x, \quad 0 \leq u \leq 2\pi.$$

Let $\alpha = \arcsin(-0.42)$. All solutions are

$$u = \alpha + 2k\pi \quad \text{or} \quad u = \pi - \alpha + 2k\pi, \quad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$x \approx 3.58, 5.85.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

Question 8

Solve the equation

$$\cos(x) = -0.91$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

Question 8 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\cos(1x) = -0.91, \quad u = 1x, \quad 0 \leq u \leq 2\pi.$$

Let $\alpha = \arccos(-0.91)$. All solutions are

$$u = \pm\alpha + 2k\pi, \quad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$x \approx 2.71, 3.57.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

Question 9

Solve the equation

$$\tan(x) = -1.25$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

Question 9 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\tan(1x) = -1.25, \quad u = 1x, \quad 0 \leq u \leq 2\pi.$$

Let $\alpha = \arctan(-1.25)$. All solutions are

$$u = \alpha + k\pi, \quad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$x \approx 2.25, 5.39.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

Question 10

Solve the equation

$$\sin(x) = 0.83$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

Question 10 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\sin(1x) = 0.83, \quad u = 1x, \quad 0 \leq u \leq 2\pi.$$

Let $\alpha = \arcsin(0.83)$. All solutions are

$$u = \alpha + 2k\pi \quad \text{or} \quad u = \pi - \alpha + 2k\pi, \quad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1.
Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 0.98, 2.16}.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 1

Solve the equation

$$2 \cos(x) + 1 = 0$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places if necessary.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 1 - Solution

Step 1: Isolate the cosine term

$$2 \cos(x) + 1 = 0 \Rightarrow \cos(x) = -\frac{1}{2}$$

Step 2: Solve the equation using inverse cosine

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \Rightarrow \text{Reference angle} = \frac{\pi}{3}$$

Since cosine is negative, the solutions are in Quadrants II and III:

- Quadrant II: $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ - Quadrant III: $x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 2

Solve the equation

$$\sin(2x) = 0.6$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 2 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\sin(2x) = 0.6, \quad u = 2x, \quad 0 \leq u \leq 4\pi.$$

Let $\alpha = \arcsin(0.6)$. All solutions are

$$u = \alpha + 2k\pi \quad \text{or} \quad u = \pi - \alpha + 2k\pi, \quad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 4\pi]$ and divide by 2. Evaluating the inverse function at full precision and rounding only the final values gives

$$x \approx 0.32, 1.25, 3.46, 4.39.$$

These are all 4 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 3

Solve the equation

$$\tan(2x) = -1.7$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 3 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\tan(2x) = -1.7, \quad u = 2x, \quad 0 \leq u \leq 4\pi.$$

Let $\alpha = \arctan(-1.7)$. All solutions are

$$u = \alpha + k\pi, \quad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 4\pi]$ and divide by 2. Evaluating the inverse function at full precision and rounding only the final values gives

$$x \approx 1.05, 2.62, 4.19, 5.76.$$

These are all 4 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 4

Solve the equation

$$\cos(3x) = 0.2$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 4 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\cos(3x) = 0.2, \quad u = 3x, \quad 0 \leq u \leq 6\pi.$$

Let $\alpha = \arccos(0.2)$. All solutions are

$$u = \pm\alpha + 2k\pi, \quad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 6\pi]$ and divide by 3. Evaluating the inverse function at full precision and rounding only the final values gives

$$x \approx 0.46, 1.64, 2.55, 3.73, 4.65, 5.83.$$

These are all 6 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 5

Solve the equation

$$2\sin(x) - 1 = 0.3$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 5 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\sin(1x) = 0.65, \quad u = 1x, \quad 0 \leq u \leq 2\pi.$$

Let $\alpha = \arcsin(0.65)$. All solutions are

$$u = \alpha + 2k\pi \quad \text{or} \quad u = \pi - \alpha + 2k\pi, \quad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$x \approx 0.71, 2.43.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 6

Solve the equation

$$1 + 3 \cos(x) = 0.4$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 6 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\cos(1x) = -0.2, \quad u = 1x, \quad 0 \leq u \leq 2\pi.$$

Let $\alpha = \arccos(-0.2)$. All solutions are

$$u = \pm\alpha + 2k\pi, \quad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$x \approx 1.77, 4.51.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 7

Solve the equation

$$4 \sin(2x) = 3$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 7 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\sin(2x) = 0.75, \quad u = 2x, \quad 0 \leq u \leq 4\pi.$$

Let $\alpha = \arcsin(0.75)$. All solutions are

$$u = \alpha + 2k\pi \quad \text{or} \quad u = \pi - \alpha + 2k\pi, \quad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 4\pi]$ and divide by 2. Evaluating the inverse function at full precision and rounding only the final values gives

$$x \approx 0.42, 1.15, 3.57, 4.29.$$

These are all 4 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 8

Solve the equation

$$2 \cos(2x) - 1 = 0.6$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 8 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\cos(2x) = 0.8, \quad u = 2x, \quad 0 \leq u \leq 4\pi.$$

Let $\alpha = \arccos(0.8)$. All solutions are

$$u = \pm\alpha + 2k\pi, \quad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 4\pi]$ and divide by 2. Evaluating the inverse function at full precision and rounding only the final values gives

$$x \approx 0.32, 2.82, 3.46, 5.96.$$

These are all 4 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 9

Solve the equation

$$\tan(x) + 2 = 1.5$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 9 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\tan(1x) = -0.5, \quad u = 1x, \quad 0 \leq u \leq 2\pi.$$

Let $\alpha = \arctan(-0.5)$. All solutions are

$$u = \alpha + k\pi, \quad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$x \approx 2.68, 5.82.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 10

Solve the equation

$$\sin(x) + \cos(x) = 1.2$$

for all $x \in [0, 2\pi]$, rounding your answers to two decimal places.

Calculus I Section 1-5: Solving Trig Equations with Calculators II

Question 10 - Solution

Use

$$\sin x + \cos x = \sqrt{2} \sin(x + \pi/4).$$

Let $\alpha = \arcsin(1.2/\sqrt{2})$. The two solution families are

$$x = \alpha - \pi/4 + 2k\pi, \quad x = 3\pi/4 - \alpha + 2k\pi.$$

Only $k = 0$ gives values in $[0, 2\pi]$, hence

$$\boxed{x \approx 0.23, 1.34}.$$

Both satisfy the unsquared equation; adding π reverses the sign and does not give another solution.

Question 1

The population of a bacterial culture is modeled by the function

$$P(t) = 500e^{0.3t}$$

where t is measured in hours.

- (a) What is the population after 5 hours?
- (b) After how many hours will the population reach 3000?
- (c) By what factor does the population grow every 4 hours?

Round your answers to two decimal places.

Question 1 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$P(5) = 500e^{1.5}$$

2240.84 bacteria

(b)

$$500e^{0.3t} = 3000 \implies t = \ln(6)/0.3$$

5.97 hours

(c)

$$P(t+4)/P(t) = e^{1.2}$$

3.32 growth factor

Question 2

A radioactive substance decays according to the formula

$$A(t) = A_0 e^{-kt}$$

where $A_0 = 100$ grams and the half-life of the substance is 10 hours.

- (a) Find the decay constant k .
- (b) How much of the substance remains after 24 hours?
- (c) After how many hours will only 20 grams remain?

Round all answers to two decimal places.

Question 2 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$e^{-10k} = 1/2 \implies k = \ln(2)/10$$

$$\boxed{0.07 \text{ hr}^{-1}}$$

The exact value of k above is used in later parts; its decimal value is 0.069315.

(b)

$$A(24) = 100e^{-24\ln(2)/10}$$

$$\boxed{18.95 \text{ grams}}$$

(c)

$$100e^{-kt} = 20 \implies t = 10 \ln(5)/\ln(2)$$

$$\boxed{23.22 \text{ hours}}$$

Question 3

A cup of coffee is initially at 190°F and is left to cool in a room that is maintained at 70°F. After 15 minutes, the coffee has cooled to 130°F.

Assume the temperature of the coffee follows Newton's Law of Cooling:

$$T(t) = T_{\text{room}} + (T_0 - T_{\text{room}})e^{-kt}$$

- (a) Find the value of the constant k .
- (b) What will the temperature be after 30 minutes?
- (c) After how many minutes will the coffee reach 100°F?

Round all answers to two decimal places.

Question 3 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$130 = 70 + 120e^{-15k} \implies k = \ln(2)/15$$

$$\boxed{0.05 \text{ min}^{-1}}$$

The exact value of k above is used in later parts; its decimal value is 0.046210.

(b)

$$T(30) = 70 + 120e^{-2 \ln 2}$$

$$\boxed{100.00 \text{ }^{\circ}\text{F}}$$

(c)

$$100 = 70 + 120e^{-kt} \implies t = \ln(4)/k$$

$$\boxed{30.00 \text{ minutes}}$$

Question 4

The value of a car decreases exponentially over time according to the function

$$V(t) = 25000e^{-0.18t}$$

where $V(t)$ is the value in dollars and t is the number of years since purchase.

- (a) What is the value of the car after 3 years?
- (b) How long will it take for the car to be worth half its original value?
- (c) After how many years will the car be worth \$5,000?

Question 4 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$V(3) = 25000e^{-0.54}$$

$$\boxed{14568.71 \text{ dollars}}$$

(b)

$$e^{-0.18t} = 1/2 \implies t = \ln(2)/0.18$$

$$\boxed{3.85 \text{ years}}$$

(c)

$$25000e^{-0.18t} = 5000 \implies t = \ln(5)/0.18$$

$$\boxed{8.94 \text{ years}}$$

Question 5

A scientist is growing a bacteria culture in a lab. The population of bacteria at time t hours is given by the function:

$$P(t) = 1200e^{0.55t}$$

- (a) What is the population after 4 hours?
- (b) After how many hours will the population reach 10,000?
- (c) By what percentage does the population increase each hour?

Round all answers to two decimal places.

Question 5 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$P(4) = 1200e^{2.2}$$

$$\boxed{10830.02 \text{ bacteria}}$$

(b)

$$t = \ln(10000/1200)/0.55$$

$$\boxed{3.86 \text{ hours}}$$

(c)

$$100(P(t+1)/P(t) - 1) = 100(e^{0.55} - 1)$$

$$\boxed{73.33 \text{ \%}}$$

Question 6

A forest fire is spreading in a circular pattern and the area (in square kilometers) covered by the fire after t hours is modeled by the function:

$$A(t) = 2.5e^{0.4t}$$

- (a) What is the area affected after 6 hours?
- (b) After how many hours will the fire have covered 50 square kilometers?
- (c) How fast is the area growing (in km^2/hr) after 6 hours?

Round all answers to two decimal places.

Question 6 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$A(6) = 2.5e^{2.4}$$

$$\boxed{27.56 \text{ km}^2}$$

(b)

$$t = \ln(50/2.5)/0.4$$

$$\boxed{7.49 \text{ hours}}$$

(c)

$$A'(t) = e^{0.4t}, \quad A'(6) = e^{2.4}$$

$$\boxed{11.02 \text{ km}^2/\text{hr}}$$

Question 7

The number of viewers watching a viral video on a social media platform can be modeled by the function

$$V(t) = 1500 \cdot e^{0.7t}$$

where $V(t)$ is the number of viewers and t is the time in days since the video was posted.

- (a) How many people have viewed the video after 2 days?
- (b) After how many days will the video have 50,000 viewers?
- (c) What is the average rate of change of viewers between day 1 and day 3?

Round all answers to two decimal places.

Question 7 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$V(2) = 1500e^{1.4}$$

$$\boxed{6082.80 \text{ viewers}}$$

(b)

$$t = \ln(50000/1500)/0.7$$

$$\boxed{5.01 \text{ days}}$$

(c)

$$\frac{V(3) - V(1)}{3 - 1} = 750(e^{2.1} - e^{0.7})$$

$$\boxed{4614.31 \text{ viewers/day}}$$

Question 8

A certain medication is administered intravenously and metabolized by the body over time. The amount of the drug in the bloodstream (in milligrams) is modeled by the function:

$$C(t) = 80e^{-0.35t}$$

where t is the number of hours after the drug is administered.

- (a) What is the amount of the drug in the bloodstream after 2 hours?
- (b) After how many hours will only 10 mg remain in the bloodstream?
- (c) What percentage of the drug is lost every hour?

Round all answers to two decimal places.

Question 8 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$C(2) = 80e^{-0.7}$$

39.73 mg

(b)

$$t = \ln(80/10)/0.35$$

5.94 hours

(c)

$$100(1 - C(t+1)/C(t)) = 100(1 - e^{-0.35})$$

29.53 %

Question 9

A savings account earns interest compounded continuously at an annual rate of 4.5

$$A(t) = 3000e^{0.045t}$$

- (a) How much money will be in the account after 6 years?
- (b) How long will it take for the account to reach \$5,000?
- (c) How much interest is earned between years 4 and 8?

Round all answers to two decimal places.

Question 9 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$A(6) = 3000e^{0.27}$$

3929.89 dollars

(b)

$$t = \ln(5000/3000)/0.045$$

11.35 years

(c)

$$A(8) - A(4) = 3000(e^{0.36} - e^{0.18})$$

708.34 dollars

Question 10

A certain species of insect is introduced to a new habitat. Its population is modeled by the exponential function:

$$P(t) = 500 \cdot e^{0.6t}$$

where $P(t)$ is the population size after t weeks.

- (a) What will the population be after 5 weeks?
- (b) After how many weeks will the population exceed 10,000?
- (c) How fast is the population growing at $t = 5$ weeks?

Round all answers to two decimal places.

Question 10 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$P(5) = 500e^3$$

10042.77 insects

(b)

$$t_* = \ln(10000/500)/0.6$$

4.99 weeks

(c)

$$P'(t) = 300e^{0.6t}, \quad P'(5) = 300e^3$$

6025.66 insects/week

The population equals 10000 at $t = t_*$ and exceeds it precisely when $t > t_*$. The rounded threshold is not an exact inequality boundary.

Question 1

Solve the equation for x , showing all steps:

$$\log_3(x + 5) + \log_3(x - 2) = 2$$

- (a) Solve the logarithmic equation algebraically.
- (b) State any restrictions on the domain.
- (c) Check for extraneous solutions.

Question 1 - Solution

(a) Combine the logs using log rules:

$$\log_3(x+5) + \log_3(x-2) = \log_3[(x+5)(x-2)] = 2 \Rightarrow \log_3(x^2 + 3x - 10) = 2$$

Rewrite in exponential form:

$$x^2 + 3x - 10 = 3^2 = 9 \Rightarrow x^2 + 3x - 10 = 9 \Rightarrow x^2 + 3x - 19 = 0$$

Solve with quadratic formula:

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-19)}}{2(1)} = \frac{-3 \pm \sqrt{9 + 76}}{2} = \frac{-3 \pm \sqrt{85}}{2}$$

(b) **Domain restrictions:**

Since $\log_3(x+5)$ and $\log_3(x-2)$ are defined only if:

$$x+5 > 0 \Rightarrow x > -5, \quad x-2 > 0 \Rightarrow x > 2 \Rightarrow \text{Overall domain: } x > 2$$

(c) **Check solutions:**

$$\sqrt{85} \approx 9.22 \Rightarrow x_1 = \frac{-3 + 9.22}{2} \approx 3.11, \quad x_2 = \frac{-3 - 9.22}{2} \approx -6.11$$

Only $x_1 \approx 3.11$ satisfies $x > 2$. So:

$$\boxed{x = \frac{-3 + \sqrt{85}}{2}} \quad (\text{valid solution})$$

Question 2

A scientist is studying the pH level of a solution, which is defined by the formula:

$$\text{pH} = -\log_{10}[H^+]$$

where $[H^+]$ is the concentration of hydrogen ions in moles per liter.

- (a) If a solution has a hydrogen ion concentration of $[H^+] = 3.2 \times 10^{-5}$, find its pH.
- (b) If another solution has a pH of 4.6, what is its hydrogen ion concentration?
- (c) Which solution is more acidic, and by how many times is it more acidic?

Question 2 - Solution

(a) **Compute pH:**

$$\begin{aligned}\text{pH} &= -\log_{10}(3.2 \times 10^{-5}) = -[\log_{10}(3.2) + \log_{10}(10^{-5})] \\ &= -[\log_{10}(3.2) - 5] \approx -[0.5051 - 5] = -(-4.4949) = \boxed{4.49}\end{aligned}$$

(b) **Compute hydrogen ion concentration from pH:**

$$\text{pH} = -\log_{10}[H^+] \Rightarrow \log_{10}[H^+] = -4.6 \Rightarrow [H^+] = 10^{-4.6} \approx \boxed{2.51 \times 10^{-5}}$$

(c) **Compare acidity:**

Recall: lower pH = more acidic.

The first solution has pH ≈ 4.49 , second has pH 4.6, so first solution is more acidic.

To find how many times more acidic:

$$\frac{[H^+]_{\text{first}}}{[H^+]_{\text{second}}} = \frac{3.2 \times 10^{-5}}{2.51 \times 10^{-5}} \approx \boxed{1.27 \text{ times more acidic}}$$

Question 3

Consider the function:

$$f(x) = \log_2(4x^2 - 12x)$$

- (a) Determine the domain of the function.
- (b) Solve the equation $f(x) = 3$.
- (c) State any restrictions or extraneous solutions.

Question 3 - Solution**(a) Domain:**

We must have:

$$4x^2 - 12x > 0 \Rightarrow 4x(x - 3) > 0$$

Using a sign chart or test intervals:

- Critical points: $x = 0$, $x = 3$ - Positive intervals: $x < 0$ or $x > 3$

$$\text{Domain: } \boxed{(-\infty, 0) \cup (3, \infty)}$$

(b) Solve $f(x) = 3$:

$$\log_2(4x^2 - 12x) = 3 \Rightarrow 4x^2 - 12x = 2^3 = 8 \Rightarrow 4x^2 - 12x - 8 = 0$$

Divide by 4:

$$x^2 - 3x - 2 = 0 \Rightarrow x = \frac{3 \pm \sqrt{(-3)^2 + 4 \cdot 2}}{2} = \frac{3 \pm \sqrt{9 + 8}}{2} = \frac{3 \pm \sqrt{17}}{2}$$

$$\boxed{x = \frac{3 \pm \sqrt{17}}{2}}$$

(c) Check for extraneous solutions:

Approximate roots:

$$\sqrt{17} \approx 4.12 \Rightarrow x \approx \frac{3 \pm 4.12}{2} \Rightarrow x_1 \approx 3.56, \quad x_2 \approx -0.56$$

From domain: $x < 0$ or $x > 3$, so both are valid.

$$\boxed{\text{No extraneous solutions}}$$

Question 4

Let $f(x) = \ln(x^2 - 6x + 10)$.

- (a) Determine the domain of $f(x)$.
- (b) Find the exact value of $f(3)$.
- (c) Solve the equation $\ln(x^2 - 6x + 10) = 0$.

Question 4 - Solution**(a) Domain:**

We require the argument of the logarithm to be positive:

$$x^2 - 6x + 10 > 0$$

Consider the quadratic:

$$x^2 - 6x + 10 = (x - 3)^2 + 1$$

Since $(x - 3)^2 \geq 0$, we know:

$$(x - 3)^2 + 1 \geq 1 > 0 \quad \text{for all real } x$$

Domain: $(-\infty, \infty)$

(b) Evaluate $f(3)$:

$$f(3) = \ln(3^2 - 6 \cdot 3 + 10) = \ln(9 - 18 + 10) = \ln(1) = \boxed{0}$$

(c) Solve $\ln(x^2 - 6x + 10) = 0$:

$$\ln(x^2 - 6x + 10) = 0 \Rightarrow x^2 - 6x + 10 = e^0 = 1 \Rightarrow x^2 - 6x + 9 = 0 \Rightarrow (x - 3)^2 = 0 \Rightarrow \boxed{x = 3}$$

Question 5

Let $f(x) = \log_5(x - 1) + \log_5(x + 3)$.

- (a) Determine the domain of $f(x)$.
- (b) Simplify the expression using logarithmic properties.
- (c) Solve the equation $\log_5(x - 1) + \log_5(x + 3) = 2$.

Question 5 - Solution**(a) Domain:**

We need both logarithmic expressions to be defined:

$$x - 1 > 0 \Rightarrow x > 1, \quad x + 3 > 0 \Rightarrow x > -3$$

The more restrictive condition is $x > 1$, so:

Domain: $(1, \infty)$

(b) Use the product rule for logarithms:

$$f(x) = \log_5[(x-1)(x+3)] = \log_5(x^2 + 2x - 3)$$

(c) Solve the equation:

$$\log_5(x^2 + 2x - 3) = 2 \Rightarrow x^2 + 2x - 3 = 5^2 = 25 \Rightarrow x^2 + 2x - 28 = 0$$

Use the quadratic formula:

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{(2)^2 + 4 \cdot 28}}{2} = \frac{-2 \pm \sqrt{4 + 112}}{2} = \frac{-2 \pm \sqrt{116}}{2} \\ &= \frac{-2 \pm 2\sqrt{29}}{2} = -1 \pm \sqrt{29} \end{aligned}$$

Check for extraneous solutions:

$$-1 + \sqrt{29} \approx -1 + 5.385 \approx 4.385 \text{ (valid)}, \quad -1 - \sqrt{29} < 0 \text{ (invalid)}$$

Only the positive root satisfies $x > 1$, so:

$x = -1 + \sqrt{29}$

Question 6

The intensity I of an earthquake is measured on the Richter scale using the formula:

$$R = \log_{10} \left(\frac{I}{I_0} \right)$$

where R is the Richter magnitude and $I_0 = 10^{-4}$ is the reference intensity.

- (a) If an earthquake has intensity $I = 10^{-1.5}$, what is its magnitude R ?
- (b) If an earthquake has a Richter magnitude of $R = 6.2$, what is its intensity I ?
- (c) How many times more intense is an earthquake of magnitude 7.2 compared to one of magnitude 5.2?

Question 6 - Solution

(a) Compute the magnitude R :

$$R = \log_{10} \left(\frac{10^{-1.5}}{10^{-4}} \right) = \log_{10}(10^{2.5}) = \boxed{2.5}$$

(b) Find the intensity for $R = 6.2$:

$$R = \log_{10} \left(\frac{I}{10^{-4}} \right) \Rightarrow 10^R = \frac{I}{10^{-4}} \Rightarrow I = 10^R \cdot 10^{-4}$$

$$I = 10^{6.2-4} = 10^{2.2} \approx \boxed{158.49}$$

(c) Compare intensities of magnitudes **7.2** and **5.2**:

Let I_1 and I_2 be the intensities corresponding to magnitudes 7.2 and 5.2, respectively.

$$\log_{10} \left(\frac{I_1}{I_0} \right) = 7.2 \Rightarrow I_1 = 10^{7.2} \cdot I_0, \quad \log_{10} \left(\frac{I_2}{I_0} \right) = 5.2 \Rightarrow I_2 = 10^{5.2} \cdot I_0$$

$$\frac{I_1}{I_2} = \frac{10^{7.2} \cdot I_0}{10^{5.2} \cdot I_0} = 10^{7.2-5.2} = 10^2 = \boxed{100}$$

So the magnitude 7.2 earthquake is **100 times more intense** than the 5.2 earthquake.

Question 7

Solve the following logarithmic equation:

$$2\log_4(x+1) - \log_4(x-3) = 1$$

- (a) Solve the equation algebraically.
- (b) State any domain restrictions and eliminate extraneous solutions.

Question 7 - Solution**(a) Combine the logarithms:**

Use the power rule:

$$\log_4(x+1)^2 - \log_4(x-3) = \log_4\left(\frac{(x+1)^2}{x-3}\right)$$

So the equation becomes:

$$\log_4\left(\frac{(x+1)^2}{x-3}\right) = 1 \Rightarrow \frac{(x+1)^2}{x-3} = 4^1 = 4$$

Multiply both sides by $x-3$:

$$(x+1)^2 = 4(x-3) \Rightarrow x^2 + 2x + 1 = 4x - 12 \Rightarrow x^2 - 2x + 13 = 0$$

Solve with the quadratic formula:

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(13)}}{2} = \frac{2 \pm \sqrt{4 - 52}}{2} = \frac{2 \pm \sqrt{-48}}{2} = \frac{2 \pm 4i\sqrt{3}}{2} = 1 \pm 2i\sqrt{3}$$

(b) Domain restrictions:

$$x+1 > 0 \Rightarrow x > -1, \quad x-3 > 0 \Rightarrow x > 3$$

So domain is: $x > 3$

The resulting quadratic has no real roots, so the original real logarithmic equation has no solution.

No real solution.

Question 8

Let $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$. Suppose that

$$f(2) = 1 \quad \text{and} \quad f(5) \approx 2.32193.$$

- (a) Estimate $f(10)$ using logarithm properties.
- (b) Estimate $f(5/2)$ using logarithm properties.
- (c) Explain the properties used.

Question 8 - Solution

The product and quotient laws hold for positive arguments:

$$f(10) = f(2) + f(5) \approx 1 + 2.32193 = \boxed{3.32193},$$

$$f(5/2) = f(5) - f(2) \approx 2.32193 - 1 = \boxed{1.32193}.$$

The datum $\log_b 2 = 1$ fixes $b = 2$, so the supplied approximation to $\log_b 5$ is consistent with the same base.

Question 9

A bacteria culture grows according to the function

$$N(t) = N_0 e^{kt}$$

where N_0 is the initial population, k is a constant, and t is the time in hours.

Suppose a culture starts with 500 bacteria and grows to 2000 bacteria in 6 hours.

- (a) Find the growth rate k .
- (b) Use your result to estimate the population after 10 hours.
- (c) After how many hours will the population reach 10,000?

Question 9 - Solution

Growth rate. $2000 = 500e^{6k}$ gives

$$k = \frac{\ln 4}{6} \approx 0.231049 \text{ hr}^{-1}.$$

Population after ten hours.

$$N(10) = 500 \cdot 4^{10/6} \approx 5039.68,$$

so the estimated whole-number population is $\boxed{5040}$.

Time to reach 10000.

$$500e^{kt} = 10000 \quad \implies \quad t = \frac{6 \ln 20}{\ln 4} \approx 12.97 \text{ hours}.$$

Substitution into the exact model verifies both target populations.

Question 10

Solve the logarithmic equation:

$$\ln(x + 2) + \ln(x - 1) = \ln(3x)$$

- (a) Solve the equation algebraically.
- (b) State the domain and eliminate any extraneous solutions.

Question 10 - Solution**(a) Combine the logarithmic expressions:**

Use the product rule:

$$\ln[(x+2)(x-1)] = \ln(3x) \Rightarrow (x+2)(x-1) = 3x \Rightarrow x^2 + x - 2 = 3x \Rightarrow x^2 - 2x - 2 = 0$$

Use the quadratic formula:

$$x = \frac{2 \pm \sqrt{(-2)^2 + 4(1)(2)}}{2} = \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = \boxed{1 \pm \sqrt{3}}$$

(b) Domain restrictions:

We need all logarithms to have positive arguments:

$$x+2 > 0 \Rightarrow x > -2, \quad x-1 > 0 \Rightarrow x > 1, \quad 3x > 0 \Rightarrow x > 0$$

Combining all: $\boxed{x > 1}$

Now test each solution: - $x = 1 + \sqrt{3} \approx 2.73 > 1 \rightarrow$ valid - $x = 1 - \sqrt{3} \approx -0.73 < 1 \rightarrow$ invalid (violates domain)

Final answer: $\boxed{x = 1 + \sqrt{3}}$

Question 1

Solve the following equation for x :

$$3^{2x-1} = 7^{x+2}$$

Instructions: Solve algebraically using logarithms and express your answer in exact form. Approximate the result to 3 decimal places.

Question 1 - Solution

Taking natural logarithms (both sides are positive) gives

$$(2x - 1) \ln 3 = (x + 2) \ln 7.$$

Collect the terms in x and divide by their nonzero coefficient:

$$x = \frac{\ln 3 + 2 \ln 7}{2 \ln 3 - \ln 7} \approx 19.857.$$

The logarithm is one-to-one, so solving this equivalent linear equation introduces no extraneous solutions. The exact expression verifies the original equation.

Question 2

Solve the following equation:

$$e^{2x} - 7e^x + 10 = 0$$

- (a) Solve algebraically for all real solutions.
- (b) Check each solution for validity.

Question 2 - Solution

(a) **Let** $y = e^x$. Then the equation becomes:

$$y^2 - 7y + 10 = 0 \Rightarrow (y - 5)(y - 2) = 0 \Rightarrow y = 5 \quad \text{or} \quad y = 2$$

Now solve for x :

$$e^x = 5 \Rightarrow x = \ln(5)$$

$$e^x = 2 \Rightarrow x = \ln(2)$$

(b) **Check domain/validity:**

Both $\ln(2)$ and $\ln(5)$ are defined for positive inputs, and since both values came from valid solutions for $y = e^x > 0$, no extraneous solutions.

Final Answer:

$x = \ln(2) \quad \text{or} \quad x = \ln(5)$

Question 3

Solve the following equation:

$$\log_3(x^2 - 4x) = 1 + \log_3(x - 2)$$

- (a) Solve the equation algebraically.
- (b) Identify any extraneous solutions.

Question 3 - Solution

(a) Combine the logarithms:

Move the constant:

$$\log_3(x^2 - 4x) - \log_3(x - 2) = 1$$

Apply the quotient rule:

$$\log_3\left(\frac{x^2 - 4x}{x - 2}\right) = 1$$

Factor the numerator:

$$x^2 - 4x = x(x - 4) \Rightarrow \frac{x(x - 4)}{x - 2}$$

So:

$$\log_3\left(\frac{x(x - 4)}{x - 2}\right) = 1 \Rightarrow \frac{x(x - 4)}{x - 2} = 3^1 = 3$$

Multiply both sides:

$$x(x - 4) = 3(x - 2) \Rightarrow x^2 - 4x = 3x - 6 \Rightarrow x^2 - 7x + 6 = 0 \Rightarrow (x - 6)(x - 1) = 0 \Rightarrow x = 6, 1$$

(b) Check for extraneous solutions:

Logarithms are only defined for positive arguments.

Check domain: $x^2 - 4x > 0 \Rightarrow x(x - 4) > 0 \Rightarrow x < 0$ or $x > 4$, $x - 2 > 0 \Rightarrow x > 2$

Only values that satisfy both: $x > 4$

So: $x = 6 \rightarrow$ valid, $x = 1 \rightarrow$ not valid (fails both domain conditions)

Final Answer:

$$\boxed{x = 6}$$

Question 4

Solve for x in the equation:

$$2^{3x+1} = 5^{x-2}$$

Give your answer in exact form using logarithms, and approximate the result to three decimal places.

Question 4 - Solution

Taking natural logarithms (both sides are positive) gives

$$(3x + 1) \ln 2 = (x - 2) \ln 5.$$

Collect the terms in x and divide by their nonzero coefficient:

$$x = \frac{-\ln 2 - 2 \ln 5}{3 \ln 2 - \ln 5} \approx -8.323.$$

The logarithm is one-to-one, so solving this equivalent linear equation introduces no extraneous solutions. The exact expression verifies the original equation.

Question 5

Solve the equation:

$$\ln(x - 1) + \ln(x + 1) = \ln(6)$$

- (a) Solve for x algebraically.
- (b) Identify and exclude any extraneous solutions.

Question 5 - Solution

(a) Combine the logarithmic expressions using properties:

$$\ln(x-1) + \ln(x+1) = \ln[(x-1)(x+1)] = \ln(x^2-1)$$

So the equation becomes:

$$\ln(x^2-1) = \ln(6) \Rightarrow x^2-1 = 6 \Rightarrow x^2 = 7 \Rightarrow x = \pm\sqrt{7}$$

(b) **Domain Check:**

We must have:

$$x-1 > 0 \Rightarrow x > 1 \quad \text{and} \quad x+1 > 0 \Rightarrow x > -1 \Rightarrow \text{Combined: } x > 1$$

So $x = \sqrt{7} \approx 2.6458$ is valid, but $x = -\sqrt{7} \approx -2.6458$ is not.

$$\text{Final Answer: } \boxed{x = \sqrt{7}}$$

Question 6

Solve the equation:

$$4e^{2x} - 12e^x + 5 = 0$$

- (a) Solve algebraically for all real solutions.
- (b) Provide both exact and approximate answers.

Question 6 - Solution

(a) **Let** $y = e^x$. Then the equation becomes:

$$4y^2 - 12y + 5 = 0$$

Use the quadratic formula:

$$y = \frac{12 \pm \sqrt{(-12)^2 - 4(4)(5)}}{2(4)} = \frac{12 \pm \sqrt{144 - 80}}{8} = \frac{12 \pm \sqrt{64}}{8} = \frac{12 \pm 8}{8}$$

So:

$$y = \frac{20}{8} = 2.5 \quad \text{or} \quad y = \frac{4}{8} = 0.5$$

Now solve for x :

$$e^x = 2.5 \Rightarrow x = \ln(2.5), \quad e^x = 0.5 \Rightarrow x = \ln(0.5)$$

(b) Final Answers:

Exact:

$$\boxed{x = \ln(2.5) \quad \text{or} \quad x = \ln(0.5)}$$

Approximate:

$$x \approx \ln(2.5) \approx 0.9163, \quad x \approx \ln(0.5) \approx -0.6931$$

Question 7

Solve for all real solutions:

$$\log_2(x + 3) + \log_2(x - 1) = 3$$

Instructions: Solve algebraically and check for extraneous solutions.

Question 7 - Solution

We are given:

$$\log_2(x+3) + \log_2(x-1) = 3$$

Step 1: Combine the logs using the product rule:

$$\log_2[(x+3)(x-1)] = 3$$

Step 2: Convert from logarithmic to exponential form:

$$(x+3)(x-1) = 2^3 = 8 \Rightarrow x^2 + 2x - 3 = 8 \Rightarrow x^2 + 2x - 11 = 0$$

Step 3: Solve the quadratic:

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-11)}}{2(1)} = \frac{-2 \pm \sqrt{4+44}}{2} = \frac{-2 \pm \sqrt{48}}{2} = \frac{-2 \pm 4\sqrt{3}}{2} = \boxed{-1 \pm 2\sqrt{3}}$$

Step 4: Domain check:

We need:

$$x+3 > 0 \Rightarrow x > -3, \quad x-1 > 0 \Rightarrow x > 1 \Rightarrow \text{So } x > 1$$

Now test the two roots: $x = -1+2\sqrt{3} \approx -1+3.464 \approx 2.464 \rightarrow \text{valid}$, $x = -1-2\sqrt{3} \approx -1-3.464 \approx -4.464 \rightarrow \text{invalid}$

Final Answer:

$$\boxed{x = -1 + 2\sqrt{3}}$$

Question 8

Solve the equation:

$$5^{x+1} = 2^{3x-2}$$

Instructions: Solve the equation algebraically using logarithms. Provide both an exact expression and an approximate value rounded to 3 decimal places.

Question 8 - Solution

Taking natural logarithms (both sides are positive) gives

$$(x + 1) \ln 5 = (3x - 2) \ln 2.$$

Collect the terms in x and divide by their nonzero coefficient:

$$x = \frac{-\ln 5 - 2 \ln 2}{\ln 5 - 3 \ln 2} \approx 6.374.$$

The logarithm is one-to-one, so solving this equivalent linear equation introduces no extraneous solutions. The exact expression verifies the original equation.

Question 9

Solve for all real solutions:

$$\log(x^2 + 1) = 1 + \log(x - 2)$$

Instructions: Solve algebraically and check for extraneous solutions.

Question 9 - Solution

We are given:

$$\log(x^2 + 1) = 1 + \log(x - 2)$$

Step 1: Use properties of logarithms.

Subtract $\log(x - 2)$ from both sides:

$$\log(x^2 + 1) - \log(x - 2) = 1$$

Apply the quotient rule:

$$\log\left(\frac{x^2 + 1}{x - 2}\right) = 1$$

Step 2: Convert to exponential form.

Recall that $\log_b(A) = C \Rightarrow A = b^C$. Since no base is written, it is base 10:

$$\frac{x^2 + 1}{x - 2} = 10^1 = 10$$

$$x^2 + 1 = 10(x - 2) \Rightarrow x^2 + 1 = 10x - 20 \Rightarrow x^2 - 10x + 21 = 0 \Rightarrow (x - 3)(x - 7) = 0 \Rightarrow x = 3 \text{ or } x = 7$$

Step 3: Domain check.

We need: $x - 2 > 0 \Rightarrow x > 2$

Both $x = 3$ and $x = 7$ satisfy the domain.

Final Answer:

$x = 3 \quad \text{or} \quad x = 7$

Question 10

Solve the equation:

$$2^{x+1} + 2^x = 48$$

Instructions: Solve the equation algebraically. Express your answer in exact form and give an approximation rounded to three decimal places.

Question 10 - Solution

We are given:

$$2^{x+1} + 2^x = 48$$

Step 1: Use the property $2^{x+1} = 2 \cdot 2^x$:

$$2 \cdot 2^x + 2^x = 48 \Rightarrow 3 \cdot 2^x = 48 \Rightarrow 2^x = \frac{48}{3} = 16$$

Step 2: Solve for x :

$$2^x = 16 = 2^4 \Rightarrow \boxed{x = 4}$$

Exact answer: $\boxed{x = 4}$

Approximate answer: $\boxed{x = 4.000}$

Question 1

Sketch the graph of the function below. Identify the domain, range, intercepts, asymptotes, and any transformations applied to a basic parent function.

$$f(x) = -\frac{1}{x-2} + 3$$

Instructions: Analyze the function using transformations of the reciprocal function $f(x) = \frac{1}{x}$.

Question 1 - Solution

We are given:

$$f(x) = -\frac{1}{x-2} + 3$$

Parent Function: $\frac{1}{x}$

Transformations:

- Horizontal shift right by 2
- Reflection over x-axis (due to the negative sign)
- Vertical shift up by 3

Domain: $(-\infty, 2) \cup (2, \infty)$

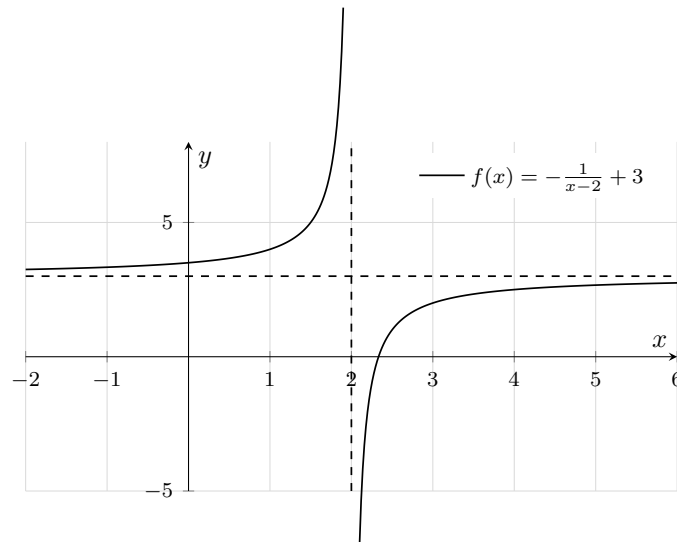
Range: $(-\infty, 3) \cup (3, \infty)$

Vertical Asymptote: $x = 2$

Horizontal Asymptote: $y = 3$

Y-intercept: $f(0) = \frac{7}{2}$

X-intercept: $f(x) = 0 \Rightarrow x = \frac{7}{3}$



Question 2

Sketch the graph of the function:

$$f(x) = |x + 1| - 2$$

Instructions: Identify the vertex, axis of symmetry, domain, range, and x- and y-intercepts. Then sketch the graph.

Question 2 - Solution

We are given:

$$f(x) = |x + 1| - 2$$

Step 1: Transform the parent function $f(x) = |x|$

- Shift left by 1: $x \rightarrow x + 1$
- Shift down by 2: $f(x) \rightarrow f(x) - 2$

Vertex: The vertex of $f(x) = |x|$ is at $(0, 0)$, so for this function it's at $(-1, -2)$

Axis of Symmetry: $x = -1$

Domain: $(-\infty, \infty)$

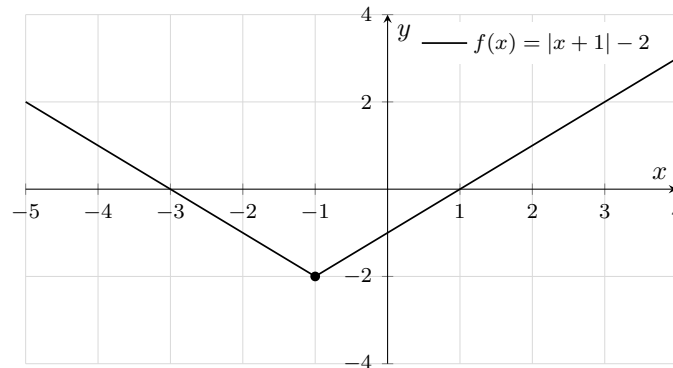
Range: $[-2, \infty)$

X-intercepts:

$$|x + 1| - 2 = 0 \Rightarrow |x + 1| = 2 \Rightarrow x + 1 = \pm 2 \Rightarrow x = 1, -3$$

Y-intercept:

$$f(0) = |0 + 1| - 2 = 1 - 2 = -1$$



Question 3

Sketch the graph of the piecewise function:

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0 \end{cases}$$

Instructions: Clearly identify continuity, domain, range, intercepts, and sketch the graph.

Question 3 - Solution

We are given a piecewise function:

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0 \end{cases}$$

Step 1: Domain

Both pieces are defined on their respective intervals, so:

$$\text{Domain: } (-\infty, \infty)$$

Step 2: Range

For $x < 0$: $f(x) = x^2 + 1 \Rightarrow f(x) > 1$, For $x \geq 0$: $f(x) = 2x + 1 \Rightarrow f(x) \geq 1$

So the function reaches its minimum at $f(0) = 1$, and increases in both directions:

$$\text{Range: } [1, \infty)$$

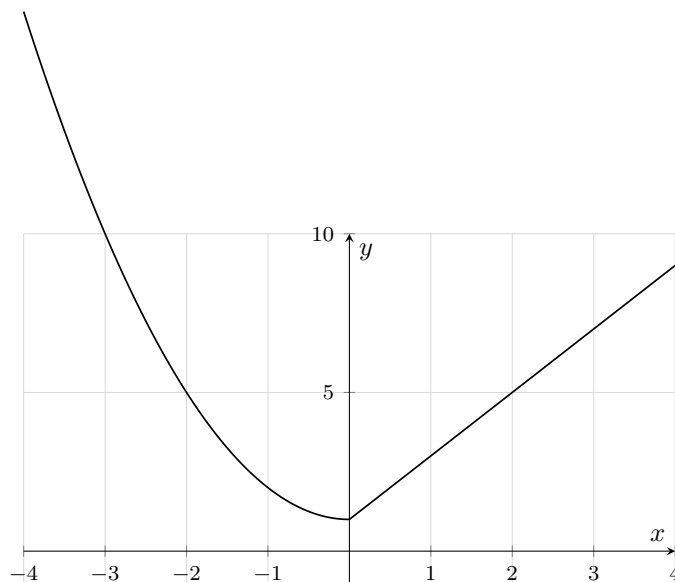
Step 3: Continuity at $x = 0$

$$\lim_{x \rightarrow 0^-} f(x) = 0^2 + 1 = 1, \quad \lim_{x \rightarrow 0^+} f(x) = 2(0) + 1 = 1 \Rightarrow \text{Function is continuous at } x = 0$$

Step 4: Intercepts

Y-intercept: $f(0) = 1$ **X-intercept:** Solve $f(x) = 0$: , $x^2 + 1 = 0 \Rightarrow$ no real solution , $2x + 1 = 0 \Rightarrow x = -\frac{1}{2} \notin [0, \infty) \Rightarrow$ no x-intercept

Conclusion: Domain: $(-\infty, \infty)$, Range: $[1, \infty)$, No x-intercepts, y-intercept at $(0, 1)$, Continuous everywhere



$$f(x) = \begin{cases} x^2 + 1 & x < 0 \\ 2x + 1 & x \geq 0 \end{cases}$$

Question 4

Sketch the graph of the function:

$$f(x) = \sqrt{x+2} - 1$$

Instructions: Identify the domain, range, intercepts, and transformations applied to the parent square root function. Then provide a sketch.

Question 4 - Solution

We are given:

$$f(x) = \sqrt{x+2} - 1$$

Step 1: Parent function: $f(x) = \sqrt{x}$

Transformations:

- Horizontal shift left by 2: $x \rightarrow x + 2$
- Vertical shift down by 1

Step 2: Domain:

$$x + 2 \geq 0 \Rightarrow x \geq -2 \Rightarrow \boxed{\text{Domain: } [-2, \infty)}$$

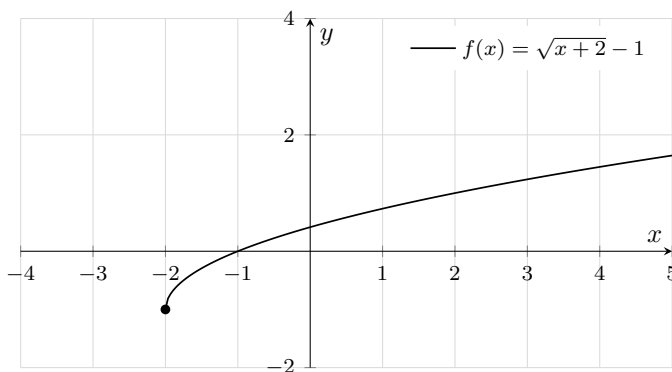
Step 3: Range: Since square root outputs ≥ 0 , and we subtract 1:

$$\boxed{\text{Range: } [-1, \infty)}$$

Step 4: Intercepts

- **Y-intercept:** $f(0) = \sqrt{2} - 1 \approx 0.414$
- **X-intercept:** Set $f(x) = 0 \Rightarrow \sqrt{x+2} = 1 \Rightarrow x+2 = 1 \Rightarrow x = -1$

Key Point: The graph starts at $(-2, -1)$



Question 5

Sketch the graph of the function:

$$f(x) = -|x - 3| + 2$$

Instructions: Identify the vertex, axis of symmetry, domain, range, and intercepts. Then sketch the graph of the function using transformations.

Question 5 - Solution

We are given:

$$f(x) = -|x - 3| + 2$$

Step 1: Parent function: $f(x) = |x|$

Transformations:

- Shift right by 3: $x \rightarrow x - 3$
- Reflect over x-axis: negative sign
- Shift up by 2

Step 2: Vertex:

Vertex at $(3, 2)$

Step 3: Axis of symmetry:

$$x = 3$$

Step 4: Domain and Range

Domain: $(-\infty, \infty)$

Range: $(-\infty, 2]$

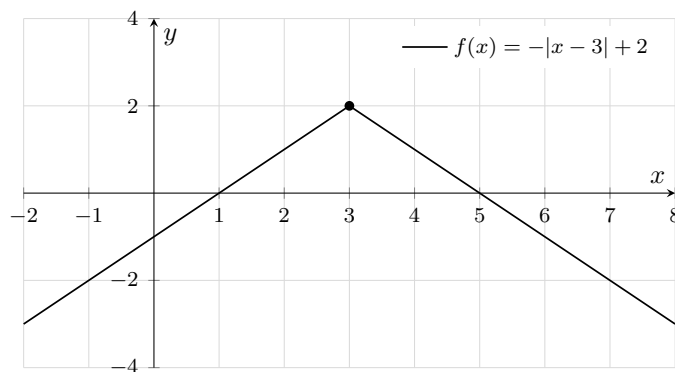
Step 5: Intercepts

- **Y-intercept:** Set $x = 0 \Rightarrow f(0) = -|0 - 3| + 2 = -3 + 2 = -1$ - **X-intercepts:** Solve $f(x) = 0$

$$-|x - 3| + 2 = 0 \Rightarrow |x - 3| = 2 \Rightarrow x - 3 = \pm 2 \Rightarrow x = 1, 5$$

Final Answer Summary:

- Vertex: $(3, 2)$
- Axis of symmetry: $x = 3$
- Domain: $(-\infty, \infty)$
- Range: $(-\infty, 2]$
- X-intercepts: $x = 1, 5$
- Y-intercept: $y = -1$



Question 6

Sketch the graph of the rational function:

$$f(x) = \frac{x^2 - 1}{x^2 - 4}$$

Instructions: Identify all intercepts, vertical and horizontal asymptotes, domain, and any removable discontinuities. Then sketch the graph based on your analysis.

Question 6 - Solution

We are given:

$$f(x) = \frac{x^2 - 1}{x^2 - 4} = \frac{(x-1)(x+1)}{(x-2)(x+2)}$$

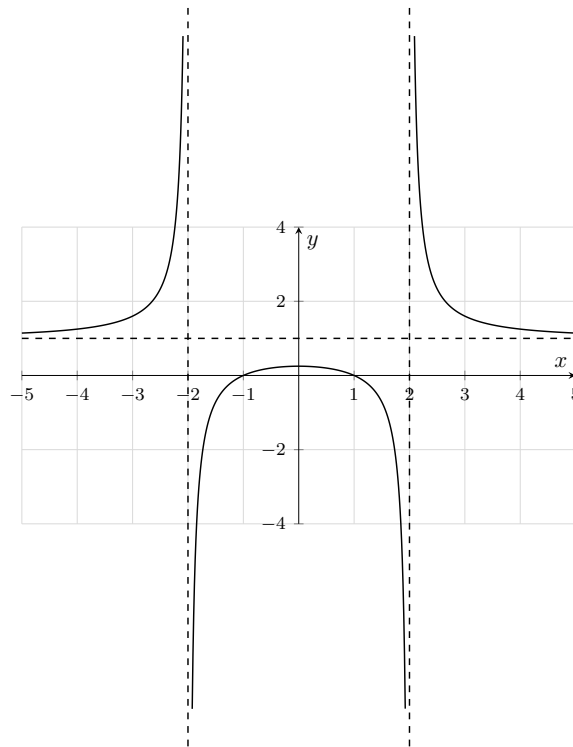
Domain: $x \neq \pm 2 \Rightarrow (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Vertical Asymptotes: $x = -2$ and $x = 2$

Horizontal Asymptote: Degrees of numerator and denominator are equal: $y = \frac{1}{1} = y = 1$

X-intercepts: Set numerator = 0: $x = \pm 1$

Y-intercept: $f(0) = \frac{-1}{-4} = \frac{1}{4}$



Question 7

Sketch the graph of the function:

$$f(x) = \frac{1}{x^2 + 1}$$

Instructions: Identify the domain, range, intercepts, end behavior, and any symmetry of the function. Then sketch the graph.

Question 7 - Solution

We are given:

$$f(x) = \frac{1}{x^2 + 1}$$

Step 1: Domain

Since $x^2 + 1 > 0$ for all real x , the domain is:

$$(-\infty, \infty)$$

Step 2: Y-intercept

Set $x = 0$:

$$f(0) = \frac{1}{0^2 + 1} = 1$$

Step 3: X-intercepts

Set $f(x) = 0$:

$$\frac{1}{x^2 + 1} = 0 \Rightarrow \text{no solution}$$

So, **no x-intercepts**.

Step 4: Range

Since the denominator is always ≥ 1 , the maximum occurs at $x = 0$, and the function decreases on both sides. Therefore:

$$\text{Range: } (0, 1]$$

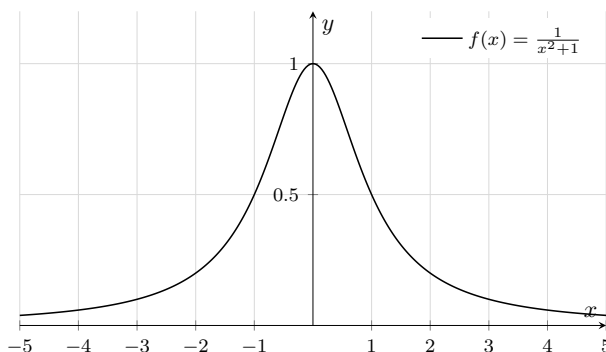
Step 5: Symmetry

$$f(-x) = \frac{1}{(-x)^2 + 1} = \frac{1}{x^2 + 1} = f(x)$$

So the function is **even**, symmetric about the y-axis.

Step 6: End Behavior

As $x \rightarrow \pm\infty$, $f(x) \rightarrow 0$



Question 8

Sketch the graph of the piecewise function:

$$f(x) = \begin{cases} x + 2 & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x \leq 2 \\ 4 - x & \text{if } x > 2 \end{cases}$$

Instructions: Clearly indicate open and closed points on the graph. Identify points of discontinuity, if any.

Question 8 - Solution

We are given a piecewise function:

$$f(x) = \begin{cases} x + 2 & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x \leq 2 \\ 4 - x & \text{if } x > 2 \end{cases}$$

Step 1: Evaluate endpoints to check continuity.

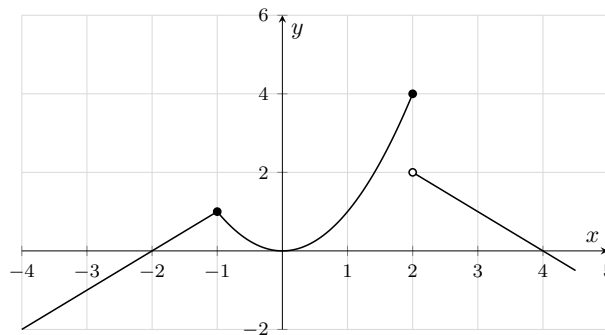
From the left of $x = -1$, $\lim_{x \rightarrow -1^-} f(x) = -1 + 2 = 1$

At $x = -1$, $f(-1) = (-1)^2 = 1 \Rightarrow$ continuous

From the right of $x = 2$, $\lim_{x \rightarrow 2^+} f(x) = 4 - 2 = 2$

At $x = 2$, $f(2) = 2^2 = 4 \Rightarrow$ discontinuous at $x = 2$

Discontinuity: Jump discontinuity at $x = 2$



Question 9

Sketch the graph of the function:

$$f(x) = \ln(x^2 + 1)$$

Instructions: Identify the domain, intercepts, end behavior, and any symmetry. Then sketch the graph.

Question 9 - Solution

We are given:

$$f(x) = \ln(x^2 + 1)$$

Domain: Since $x^2 + 1 > 0$ for all real x ,

Domain: $(-\infty, \infty)$

Y-intercept:

$$f(0) = \ln(0^2 + 1) = \ln(1) = 0$$

X-intercepts: Set $f(x) = 0$:

$$\ln(x^2 + 1) = 0 \Rightarrow x^2 + 1 = 1 \Rightarrow x^2 = 0 \Rightarrow x = 0$$

So the only intercept is at the origin.

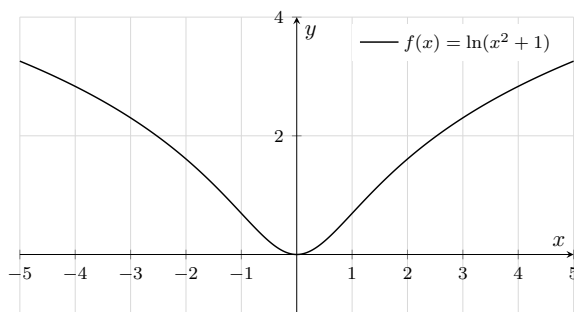
Symmetry:

$$f(-x) = \ln((-x)^2 + 1) = \ln(x^2 + 1) = f(x) \Rightarrow \text{even function}$$

End behavior: As $x \rightarrow \pm\infty$, $x^2 + 1 \rightarrow \infty \Rightarrow f(x) \rightarrow \infty$

Range: Minimum value is at $x = 0$: $\ln(1) = 0$, and it increases without bound.

Range: $[0, \infty)$



Question 10

Sketch the graph of the function:

$$f(x) = \frac{x}{\sqrt{x^2 + 1}}$$

Instructions: Identify the domain, range, intercepts, asymptotic behavior, and any symmetry. Then sketch the graph clearly.

Question 10 - Solution

We are given:

$$f(x) = \frac{x}{\sqrt{x^2 + 1}}$$

Step 1: Domain

The denominator is always positive and defined for all real x :

$$\text{Domain: } (-\infty, \infty)$$

Step 2: Intercepts

Y-intercept:

$$f(0) = \frac{0}{\sqrt{0^2 + 1}} = 0$$

X-intercept:

$$f(x) = 0 \Rightarrow \frac{x}{\sqrt{x^2 + 1}} = 0 \Rightarrow x = 0$$

Step 3: Symmetry

Check if the function is odd:

$$f(-x) = \frac{-x}{\sqrt{(-x)^2 + 1}} = \frac{-x}{\sqrt{x^2 + 1}} = -f(x)$$

So the function is **odd**, symmetric about the origin.

Step 4: End Behavior

As $x \rightarrow \pm\infty$:

$$f(x) = \frac{x}{\sqrt{x^2 + 1}} \rightarrow \pm 1$$

So horizontal asymptotes:

$$\boxed{y = 1} \quad \text{as } x \rightarrow \infty \quad \text{and} \quad \boxed{y = -1} \quad \text{as } x \rightarrow -\infty$$

Step 5: Range

$$\text{Range: } (-1, 1)$$

The function approaches 1 and -1 but never reaches them.

