Calculus I

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Question 1

Let $f(x) = \sqrt{x+4}$ and $g(x) = \dfrac{1}{x-2}$. Define the composition $h(x) = f(g(x))$.

Find the domain of $h(x)$, and express your answer in interval notation.

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Question 1 - Solution

We are given:

$$f(x) = \sqrt{x+4}, \qquad g(x) = \frac{1}{x-2}, \qquad h(x) = f(g(x)) = \sqrt{\frac{1}{x-2}+4}$$

To determine the domain of $h(x)$, we consider the restrictions from both the inner and outer functions.

Step 1: Domain restriction from $g(x)$
The function $g(x) = \dfrac{1}{x-2}$ is undefined when the denominator is zero. This occurs at $x = 2$, so:

$$x \neq 2$$

Step 2: Domain restriction from the square root
We need the expression inside the square root to be greater than or equal to zero:

$$\frac{1}{x-2} + 4 \geq 0 \Rightarrow \frac{4x-7}{x-2} \geq 0$$

Step 3: Critical points and sign analysis

$$4x - 7 = 0 \Rightarrow x = \frac{7}{4}, \qquad x - 2 = 0 \Rightarrow x = 2$$

From the sign chart:
Positive on $(-\infty, \frac{7}{4}]$, Negative on $(\frac{7}{4}, 2)$, Positive again on $(2, \infty)$.

Final Domain:

$$\boxed{(-\infty,\ \tfrac{7}{4}] \cup (2,\ \infty)}$$

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Question 2

A piecewise function is defined as follows:

$$f(x) = \begin{cases} x^2 - 4x + 3 & \text{if } x < 2 \\ \sqrt{x+2} & \text{if } x \geq 2 \end{cases}$$

(a) Determine whether $f(x)$ is continuous at $x = 2$.

(b) Determine whether $f(x)$ is differentiable at $x = 2$.

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Question 2 - Solution

Part (a): Continuity at $x = 2$ We must check:

- $\lim_{x\to 2^-} f(x)$
- $\lim_{x\to 2^+} f(x)$
- $f(2)$

For $x < 2$, use $f(x) = x^2 - 4x + 3$:

$$\lim_{x\to 2^-} f(x) = 2^2 - 4(2) + 3 = 4 - 8 + 3 = -1$$

For $x \geq 2$, use $f(x) = \sqrt{x+2}$:

$$\lim_{x\to 2^+} f(x) = \sqrt{2+2} = \sqrt{4} = 2$$

Since:

$$\lim_{x\to 2^-} f(x) = -1, \quad \lim_{x\to 2^+} f(x) = 2$$

the limit from left and right are not equal.

Conclusion: $f(x)$ is **not continuous** at $x = 2$

Part (b): Differentiability at $x = 2$

A function must be continuous at a point to be differentiable there. Since $f(x)$ is not continuous at $x = 2$, it cannot be differentiable there.

Conclusion: $f(x)$ is **not differentiable** at $x = 2$

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Question 3

Let the function f be defined by

$$f(x) = \cos(\sin x)$$

(a) Determine the period of $f(x)$.

(b) Prove that $f(x)$ is an even function.

(c) Find all values of $x \in [0, 2\pi]$ such that $f(x) = 1$.

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Question 3 - Solution

(a) Although $\sin x$ has period 2π, the function

$$f(x) = \cos(\sin x)$$

has a smaller period because cosine is even.

Using

$$\sin(x + \pi) = -\sin x$$

and

$$\cos(-u) = \cos(u),$$

we get

$$f(x + \pi) = \cos(\sin(x + \pi)) = \cos(-\sin x) = \cos(\sin x) = f(x).$$

Therefore, $f(x)$ repeats after π. The reason the period changes from 2π to π is that shifting by π changes $\sin x$ to $-\sin x$, but cosine gives the same value for opposite inputs.

Thus, the period is

$$\boxed{\pi}.$$

(b) To prove that $f(x)$ is even, compute $f(-x)$:

$$f(-x) = \cos(\sin(-x)) = \cos(-\sin x) = \cos(\sin x) = f(x).$$

Therefore,

$$\boxed{f(x) \text{ is even.}}$$

(c) Solve

$$\cos(\sin x) = 1.$$

Cosine equals 1 when its input is $2k\pi$. Thus,

$$\sin x = 2k\pi.$$

But since $-1 \leq \sin x \leq 1$, the only possible value is

$$\sin x = 0.$$

On $[0, 2\pi]$, this occurs at

$$x = 0,\ \pi,\ 2\pi.$$

Therefore,

$$\boxed{x = 0,\ \pi,\ 2\pi}.$$

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Question 4

Let $f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & x \neq 3 \\ c, & x = 3 \end{cases}$

(a) Simplify $f(x)$ for $x \neq 3$.

(b) Find the value of c that makes $f(x)$ continuous at $x = 3$.

(c) Is $f(x)$ differentiable at $x = 3$? Justify your answer.

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Question 4 - Solution

(a) Simplifying for $x \neq 3$:

$$f(x) = \frac{x^2 - 9}{x - 3} = \frac{(x-3)(x+3)}{x-3} = x + 3, \quad x \neq 3$$

So the simplified function is:

$$f(x) = x + 3 \quad \text{for } x \neq 3$$

(b) Continuity at $x = 3$:
To ensure continuity at $x = 3$, we must have:

$$\lim_{x \to 3} f(x) = f(3) = c$$

From (a), since $f(x) = x + 3$ for $x \neq 3$, then:

$$\lim_{x \to 3} f(x) = 3 + 3 = 6 \Rightarrow c = \boxed{6}$$

(c) Differentiability at $x = 3$:
We now check if $f(x)$ is differentiable at $x = 3$. The simplified form $f(x) = x+3$ is linear and differentiable for all $x \neq 3$.

If we define $f(3) = 6$, then the function becomes:

$$f(x) = \begin{cases} x + 3 & x \neq 3 \\ 6 & x = 3 \end{cases}$$

This function is: Continuous at $x = 3$, Equal to $x+3$ everywhere except possibly at 3 , $f(x)$ approaches $x + 3$ smoothly at 3

So,

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3+h+3) - 6}{h} = \lim_{h \to 0} \frac{6 + h - 6}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

Conclusion: $f(x)$ is differentiable at $x = 3$, and $f'(3) = \boxed{1}$

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Question 5

A function f is defined as:

$$f(x) = \begin{cases} \dfrac{\ln(1+ax)}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

(a) Find the value of a for which $f(x)$ is continuous at $x = 0$.

(b) For that value of a, determine whether f is differentiable at $x = 0$.

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Question 5 - Solution

(a) For f to be continuous at $x = 0$, we need

$$\lim_{x\to 0} f(x) = f(0) = 1.$$

For $x \neq 0$,

$$f(x) = \frac{\ln(1+ax)}{x}.$$

So compute

$$\lim_{x\to 0} \frac{\ln(1+ax)}{x}.$$

As $x \to 0$, the numerator and denominator both approach 0, so we use L'Hospital's Rule:

$$\lim_{x\to 0} \frac{\ln(1+ax)}{x} = \lim_{x\to 0} \frac{\frac{a}{1+ax}}{1} = a.$$

For continuity, this limit must equal $f(0) = 1$. Therefore,

$$a = 1.$$

Thus,

$$\boxed{a = 1}.$$

(b) Now let $a = 1$. Then

$$f(x) = \frac{\ln(1+x)}{x}, \qquad x \neq 0,$$

and $f(0) = 1$.

To check differentiability at $x = 0$, use the definition:

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}.$$

Substitute $f(h) = \dfrac{\ln(1+h)}{h}$ and $f(0) = 1$:

$$f'(0) = \lim_{h\to 0} \frac{\frac{\ln(1+h)}{h} - 1}{h} = \lim_{h\to 0} \frac{\ln(1+h) - h}{h^2}.$$

This is also a $0/0$ form, so apply L'Hospital's Rule:

$$f'(0) = \lim_{h\to 0} \frac{\frac{1}{1+h} - 1}{2h} = \lim_{h\to 0} \frac{-h}{2h(1+h)} = \lim_{h\to 0} \frac{-1}{2(1+h)} = -\frac{1}{2}.$$

Since this limit exists, f is differentiable at $x = 0$, and

$$\boxed{f'(0) = -\frac{1}{2}}.$$

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Question 6

Let $f(x) = |x - 2| + |x + 1|$.

(a) Express $f(x)$ as a piecewise-defined function.
(b) Sketch the graph of $f(x)$.
(c) Determine all values of x where $f(x)$ is not differentiable. Justify your answer.

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Question 6 - Solution

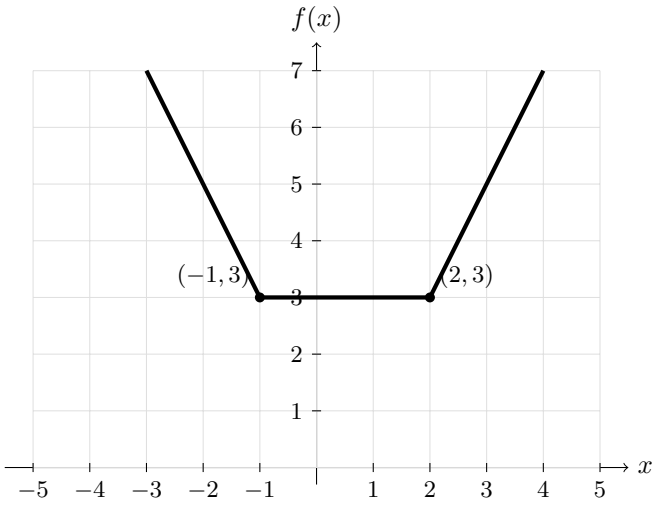
(a) The absolute values change sign at $x = -1$ and $x = 2$.

$$f(x) = \begin{cases} -(x-2) - (x+1), & x < -1, \\ -(x-2) + (x+1), & -1 \le x < 2, \\ (x-2) + (x+1), & x \ge 2. \end{cases}$$

After simplifying,

$$f(x) = \begin{cases} -2x + 1, & x < -1, \\ 3, & -1 \le x < 2, \\ 2x - 1, & x \ge 2. \end{cases}$$

(b) The graph decreases until $x = -1$, stays constant from $x = -1$ to $x = 2$, and then increases.



(c) The only possible non-differentiable points are where the absolute value expressions change sign:

$$x = -1 \qquad \text{and} \qquad x = 2.$$

From the piecewise function, the slopes are

$$\begin{cases} -2, & x < -1, \\ 0, & -1 < x < 2, \\ 2, & x > 2. \end{cases}$$

At $x = -1$, the slope changes from -2 to 0, so f is not differentiable there.
At $x = 2$, the slope changes from 0 to 2, so f is not differentiable there.
Therefore,

$$\boxed{x = -1,\ 2}$$

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Question 7

Consider the function

$$f(x) = \frac{x^2 + 2x - 3}{x^2 - 4}.$$

(a) Determine all vertical and horizontal asymptotes of $f(x)$.

(b) Identify any holes in the graph of $f(x)$, and specify their coordinates.

(c) State the domain of $f(x)$.

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Question 7 - Solution

We are given

$$f(x) = \frac{x^2 + 2x - 3}{x^2 - 4}.$$

Factor the numerator and denominator:

$$f(x) = \frac{(x+3)(x-1)}{(x-2)(x+2)}.$$

(a) Vertical asymptotes occur where the denominator is zero and no factor cancels. Since

$$x^2 - 4 = (x-2)(x+2),$$

the denominator is zero at

$$x = 2 \quad \text{and} \quad x = -2.$$

There are no common factors with the numerator, so both are vertical asymptotes:

$$\boxed{x = -2,\ x = 2}.$$

Since the numerator and denominator have the same degree, the horizontal asymptote is the ratio of leading coefficients:

$$y = \frac{1}{1} = 1.$$

Thus,

$$\boxed{y = 1}.$$

(b) Holes occur when a factor cancels from the numerator and denominator. Since

$$\frac{(x+3)(x-1)}{(x-2)(x+2)}$$

has no common factors, there are

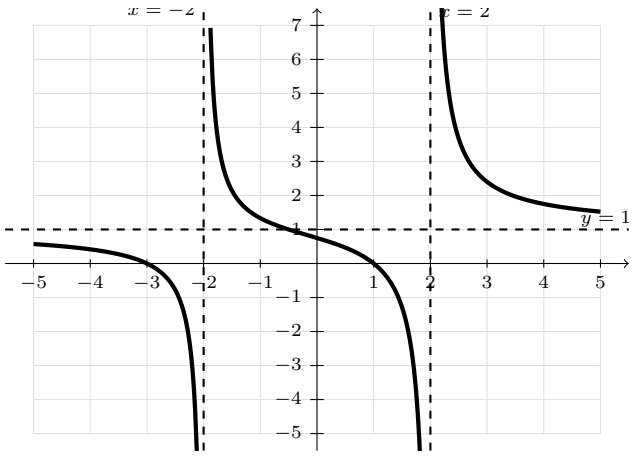
$$\boxed{\text{no holes}}.$$

(c) The domain excludes the values that make the denominator zero:

$$x \neq -2, \quad x \neq 2.$$

Therefore,

$$\boxed{(-\infty, -2) \cup (-2, 2) \cup (2, \infty)}.$$



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Question 8

Let

$$f(x) = \frac{1}{x}, \qquad g(x) = \sqrt{x-2}.$$

(a) Find the composite function $(f \circ g)(x)$ and state its domain.

(b) Determine whether $(f \circ g)(x)$ is defined at $x = 2$. Justify your answer.

(c) Evaluate $(f \circ g)(6)$.

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Question 8 - Solution

(a) Composite function and domain

The composite function is

$$(f \circ g)(x) = f(g(x)) = f\big(\sqrt{x-2}\big) = \frac{1}{\sqrt{x-2}}.$$

To determine the domain:

- The square root requires $x - 2 \geq 0 \Rightarrow x \geq 2$.
- The denominator cannot be zero, so $\sqrt{x-2} \neq 0 \Rightarrow x \neq 2$.

Thus, the domain is

$$\boxed{(2, \infty)}.$$

(b) Endpoint analysis

At $x = 2$,

$$(f \circ g)(2) = \frac{1}{\sqrt{2-2}} = \frac{1}{0},$$

which is undefined. Therefore, $x = 2$ is excluded from the domain.

(c) Evaluation

Evaluate the composite at $x = 6$:

$$(f \circ g)(6) = \frac{1}{\sqrt{6-2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}.$$

Answer:

$$\boxed{\frac{1}{2}}$$

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Question 9

Let $f(x) = \begin{cases} 2x + 5 & \text{if } x < 1 \\ x^2 + 2 & \text{if } x \geq 1 \end{cases}$

(a) Is $f(x)$ continuous at $x = 1$? Justify analytically.

(b) Is $f(x)$ differentiable at $x = 1$? If not, explain why.

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Question 9 - Solution

(a) Continuity at $x = 1$:

Check:

$$\lim_{x \to 1^-} f(x) = 2(1) + 5 = 7$$

$$\lim_{x \to 1^+} f(x) = 1^2 + 2 = 3$$

Since:

$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x),$$

The two-sided limit does not exist, so:

$$\boxed{f(x) \text{ is not continuous at } x = 1}$$

(b) Differentiability at $x = 1$:

A function cannot be differentiable at a point where it is not continuous.

So:

$$\boxed{f(x) \text{ is not differentiable at } x = 1}$$

Explanation: The left and right-hand limits do not match at $x = 1$, so there is a jump discontinuity.

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Question 10

Let the function f be defined by:

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ c, & x = 2 \end{cases}$$

(a) Find the limit $\lim_{x \to 2} f(x)$.

(b) Determine the value of c that makes $f(x)$ continuous at $x = 2$.

(c) Is $f(x)$ differentiable at $x = 2$ for the value of c you found? Justify your answer.

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Question 10 - Solution

(a) Compute $\lim_{x\to 2} f(x)$:

Factor the numerator:

$$f(x) = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2}, \quad x \neq 2$$

Cancel the common factor:

$$f(x) = x+2 \quad \text{for } x \neq 2$$

So:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}(x+2) = 4$$

Answer: $\boxed{4}$

(b) To make f continuous at $x = 2$, set:

$$f(2) = \lim_{x\to 2} f(x) = 4 \Rightarrow c = \boxed{4}$$

(c) Differentiability at $x = 2$:

With $c = 4$, we now have:

$$f(x) = \begin{cases} x+2, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

The function is: Continuous at $x = 2$, Equal to a linear function near $x = 2$

Compute derivative using definition:

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{(2+h+2)-4}{h} = \frac{4+h-4}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

Conclusion: $f(x)$ is differentiable at $x = 2$, with $f'(2) = \boxed{1}$

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Question 1

Let the function $f(x) = \dfrac{2x+5}{x-3}$.

(a) Show that f is one-to-one.

(b) Find the inverse function $f^{-1}(x)$.

(c) State the domain and range of both $f(x)$ and $f^{-1}(x)$.

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Question 1 - Solution

(a) Show that f is one-to-one:

A function is one-to-one if it passes the Horizontal Line Test, or algebraically if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.
Suppose:

$$\frac{2x_1 + 5}{x_1 - 3} = \frac{2x_2 + 5}{x_2 - 3}$$

Cross-multiplying:

$$(2x_1 + 5)(x_2 - 3) = (2x_2 + 5)(x_1 - 3)$$

Expanding both sides:

$$2x_1x_2 - 6x_1 + 5x_2 - 15 = 2x_1x_2 - 6x_2 + 5x_1 - 15$$

Subtract $2x_1x_2$ and -15 from both sides:

$$-6x_1 + 5x_2 = -6x_2 + 5x_1 \Rightarrow 5x_2 + 6x_2 = 5x_1 + 6x_1 \Rightarrow 11x_2 = 11x_1 \Rightarrow x_1 = x_2$$

Thus, f is one-to-one.

(b) Find the inverse function:

Let $y = \dfrac{2x + 5}{x - 3}$, and solve for x:

$$y(x - 3) = 2x + 5 \Rightarrow yx - 3y = 2x + 5 \Rightarrow yx - 2x = 3y + 5 \Rightarrow x(y - 2) = 3y + 5 \Rightarrow x = \frac{3y + 5}{y - 2}$$

Now swap x and y:

$$f^{-1}(x) = \frac{3x + 5}{x - 2}$$

(c) Domain and Range:

For $f(x) = \dfrac{2x + 5}{x - 3}$: - Denominator cannot be 0 $\quad x \neq 3$ - So domain of f is: $\boxed{(-\infty, 3) \cup (3, \infty)}$

Range: Find y values that cannot occur. From inverse function: $f^{-1}(x) = \dfrac{3x + 5}{x - 2} \Rightarrow x \neq 2$

So range of f is $\boxed{(-\infty, 2) \cup (2, \infty)}$

Domain and Range:

$$\text{Domain of } f(x) : (-\infty, 3) \cup (3, \infty)$$
$$\text{Range of } f(x) : (-\infty, 2) \cup (2, \infty)$$
$$\text{Domain of } f^{-1}(x) : (-\infty, 2) \cup (2, \infty)$$
$$\text{Range of } f^{-1}(x) : (-\infty, 3) \cup (3, \infty)$$

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Question 2

Let $f(x) = \ln(x-1)$, where $x > 1$.

(a) Find the inverse function $f^{-1}(x)$.

(b) State the domain and range of $f(x)$ and $f^{-1}(x)$.

(c) Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

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Question 2 - Solution

(a) Find the inverse of $f(x) = \ln(x-1)$:

Let $y = \ln(x-1)$

Exponentiate both sides:

$$e^y = x - 1 \Rightarrow x = e^y + 1$$

Now swap x and y to get the inverse:

$$f^{-1}(x) = e^x + 1$$

Answer: $\boxed{f^{-1}(x) = e^x + 1}$

(b) Domain and Range:

For $f(x) = \ln(x-1)$:

Domain: $x > 1 \Rightarrow \boxed{(1, \infty)}$, Range of $\ln(x-1)$ is all real numbers: $\boxed{(-\infty, \infty)}$

So for the inverse:

Domain of $f^{-1}(x)$: $\boxed{(-\infty, \infty)}$, Range of $f^{-1}(x) = e^x + 1$: since $e^x > 0$, we get $f^{-1}(x) > 1 \Rightarrow \boxed{(1, \infty)}$

Summary:

$$\text{Domain of } f(x) = (1, \infty)$$
$$\text{Range of } f(x) = (-\infty, \infty)$$
$$\text{Domain of } f^{-1}(x) = (-\infty, \infty)$$
$$\text{Range of } f^{-1}(x) = (1, \infty)$$

(c) Verify compositions:

$$f(f^{-1}(x)) = \ln((e^x + 1) - 1) = \ln(e^x) = x$$

$$f^{-1}(f(x)) = e^{\ln(x-1)} + 1 = x - 1 + 1 = x$$

Both identities verified.

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Question 3

Let $f(x) = \dfrac{x}{x+1}$, where $x \neq -1$.

(a) Find the inverse function $f^{-1}(x)$.

(b) State the domain and range of both $f(x)$ and $f^{-1}(x)$.

(c) Verify the identity $f(f^{-1}(x)) = x$.

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Question 3 - Solution

(a) Find the inverse of $f(x) = \dfrac{x}{x+1}$:

Let $y = \dfrac{x}{x+1}$. Solve for x:

Multiply both sides by $x+1$:

$$y(x+1) = x \Rightarrow yx + y = x \Rightarrow yx - x = -y \Rightarrow x(y-1) = -y \Rightarrow x = \frac{-y}{y-1}$$

Now switch x and y:

$$f^{-1}(x) = \frac{-x}{x-1}$$

Answer: $\boxed{f^{-1}(x) = \dfrac{-x}{x-1}}$

(b) Domain and Range:

For $f(x) = \dfrac{x}{x+1}$: Domain: $x \neq -1$, As $x \to \infty$, $f(x) \to 1$, and as $x \to -\infty$, $f(x) \to 1$, Solve $\dfrac{x}{x+1} = 1 \Rightarrow x = x+1 \Rightarrow 0 = 1 \to$ contradiction

So $f(x) \neq 1$

Domain: $\boxed{(-\infty,-1) \cup (-1,\infty)}$ **Range:** $\boxed{(-\infty,1) \cup (1,\infty)}$

Now for $f^{-1}(x) = \dfrac{-x}{x-1}$:

Domain excludes $x = 1$, since denominator is zero Range excludes $y = -1$, since solving $\frac{-x}{x-1} = -1 \Rightarrow x = \frac{1}{2}$, which contradicts range of original function

Domain of $f^{-1}(x) = \boxed{(-\infty,1) \cup (1,\infty)}$ **Range of** $f^{-1}(x) = \boxed{(-\infty,-1) \cup (-1,\infty)}$

(c) Verify:

$$f(f^{-1}(x)) = f\left(\frac{-x}{x-1}\right) = \frac{\frac{-x}{x-1}}{\frac{-x}{x-1}+1} = \frac{\frac{-x}{x-1}}{\frac{-x+(x-1)}{x-1}} = \frac{\frac{-x}{x-1}}{\frac{-1}{x-1}} = x$$

Verified: $\boxed{f(f^{-1}(x)) = x}$

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Question 4

Suppose $f(x) = \dfrac{1}{x^2+1}$.

(a) Determine whether $f(x)$ has an inverse function. Justify.

(b) If not, restrict the domain of f so that it becomes one-to-one and then find the inverse on that restricted domain.

(c) State the domain and range of the inverse function you found.

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Question 4 - Solution

(a) Does $f(x) = \dfrac{1}{x^2+1}$ have an inverse?

No. Its natural domain is $(-\infty, \infty)$, since $x^2 + 1$ is never zero. To have an inverse function, f must be **one-to-one**: different inputs must give different outputs.

Horizontal line test: Imagine drawing horizontal lines across the graph. Each horizontal line represents a fixed output y. If even one such line crosses the graph more than once, then different inputs give the same output, so the function is not one-to-one.

Testing points: We can show this without drawing the graph by choosing two different inputs and calculating their outputs. Try $x = 1$ and $x = -1$, since squaring either gives 1:

$$f(1) = \frac{1}{1^2+1} = \frac{1}{2}, \qquad f(-1) = \frac{1}{(-1)^2+1} = \frac{1}{2}.$$

Thus, the horizontal line $y = \frac{1}{2}$ meets the graph at both $(1, \frac{1}{2})$ and $(-1, \frac{1}{2})$. An inverse would have to send $\frac{1}{2}$ back to both 1 and -1, but a function can give only one output for each input.

So, f fails the horizontal line test and has no inverse function on $(-\infty, \infty)$. One pair of different inputs with the same output is enough to prove this; testing a few points with different outputs would not prove that a function is one-to-one everywhere.

(b) Restrict domain to make f one-to-one:

Choose only nonnegative inputs. As x increases on $[0, \infty)$, the denominator $x^2 + 1$ increases, so $f(x)$ strictly decreases. Therefore, no output repeats, and the restricted function is one-to-one. We use:

$$\boxed{x \in [0, \infty)}$$

Let $y = \frac{1}{x^2+1}$, with $x \geq 0$.
Solve for x:

$$\begin{aligned} y(x^2+1) &= 1 \\ x^2 + 1 &= \frac{1}{y} \\ x^2 &= \frac{1}{y} - 1 = \frac{1-y}{y} \\ x &= \sqrt{\frac{1-y}{y}}. \end{aligned}$$

We take only the nonnegative square root because our restricted domain requires $x \geq 0$. Now switch x and y to express the inverse using x as its input:

$$f^{-1}(x) = \sqrt{\frac{1-x}{x}}, \quad \text{for } 0 < x \leq 1$$

(c) Domain and range of the inverse function:

The restricted function has domain $[0, \infty)$. Its largest output is $f(0) = 1$. As x increases, its outputs approach 0 but never equal 0, so its range is $(0, 1]$.

An inverse reverses inputs and outputs. Therefore, the original range becomes the inverse's domain, and the original domain becomes the inverse's range:

$$\text{Domain of } f^{-1}(x) = (0, 1], \quad \text{Range of } f^{-1}(x) = [0, \infty)$$

Answer:

$$f^{-1}(x) = \sqrt{\frac{1-x}{x}}, \quad x \in (0, 1]$$

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Question 5

Let $f(x) = \dfrac{3x-4}{5x+2}$, where $x \neq -\dfrac{2}{5}$.

(a) Show that f is a one-to-one function.

(b) Find the inverse function $f^{-1}(x)$.

(c) Determine the domain and range of $f(x)$ and $f^{-1}(x)$.

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Question 5 - Solution

(a) Show that f is one-to-one:

Suppose $f(x_1) = f(x_2)$. Then:

$$\frac{3x_1 - 4}{5x_1 + 2} = \frac{3x_2 - 4}{5x_2 + 2}$$

Cross-multiplying:

$$(3x_1 - 4)(5x_2 + 2) = (3x_2 - 4)(5x_1 + 2)$$

Expanding both sides:

$$15x_1x_2 + 6x_1 - 20x_2 - 8 = 15x_1x_2 + 6x_2 - 20x_1 - 8$$

Cancel $15x_1x_2$ and -8:

$$6x_1 - 20x_2 = 6x_2 - 20x_1 \Rightarrow 6x_1 + 20x_1 = 6x_2 + 20x_2 \Rightarrow 26x_1 = 26x_2 \Rightarrow x_1 = x_2$$

Therefore, f is one-to-one.

(b) Find the inverse function:

Let $y = \frac{3x-4}{5x+2}$

Multiply both sides:

$$y(5x + 2) = 3x - 4 \Rightarrow 5xy + 2y = 3x - 4 \Rightarrow 5xy - 3x = -2y - 4 \Rightarrow x(5y - 3) = -2y - 4 \Rightarrow x = \frac{-2y - 4}{5y - 3}$$

Now swap x and y:

$$f^{-1}(x) = \frac{-2x - 4}{5x - 3}$$

(c) Domain and Range:

The only restriction on the domain of f is $5x + 2 \neq 0 \Rightarrow x \neq -\frac{2}{5}$

The inverse is undefined when the denominator is 0:

$$5x - 3 \neq 0 \Rightarrow x \neq \frac{3}{5}$$

So:

$$\text{Domain of } f(x): \boxed{(-\infty, -\tfrac{2}{5}) \cup (-\tfrac{2}{5}, \infty)}$$

$$\text{Range of } f(x): \boxed{(-\infty, \tfrac{3}{5}) \cup (\tfrac{3}{5}, \infty)}$$

$$\text{Domain of } f^{-1}(x): \boxed{(-\infty, \tfrac{3}{5}) \cup (\tfrac{3}{5}, \infty)}$$

$$\text{Range of } f^{-1}(x): \boxed{(-\infty, -\tfrac{2}{5}) \cup (-\tfrac{2}{5}, \infty)}$$

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Question 6

Let $f(x) = \sqrt{2x-1}$ with domain $x \geq \frac{1}{2}$.

(a) Show that f is one-to-one and has an inverse.

(b) Find the inverse function $f^{-1}(x)$.

(c) Determine the domain and range of $f^{-1}(x)$.

(d) Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

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Question 6 - Solution

(a) One-to-One Check:

Since $f(x) = \sqrt{2x-1}$ is a square root function with a linear expression inside and domain $x \geq \frac{1}{2}$, it is strictly increasing. Therefore, it passes the horizontal line test and is one-to-one.

(b) Find the inverse:
Let $y = \sqrt{2x-1}$. Solve for x:

$$y^2 = 2x - 1 \Rightarrow 2x = y^2 + 1 \Rightarrow x = \frac{y^2+1}{2}$$

Now switch x and y:

$$f^{-1}(x) = \frac{x^2+1}{2}$$

Answer: $\boxed{f^{-1}(x) = \frac{x^2+1}{2}}$

(c) Domain and Range of Inverse:
Since original domain is $x \geq \frac{1}{2}$, the range is:

$$f(x) = \sqrt{2x-1} \geq 0 \Rightarrow \text{Range of } f = [0, \infty)$$

So: - Domain of $f^{-1}(x) = [0, \infty)$ - Range of $f^{-1}(x) = \left[\frac{1}{2}, \infty\right)$

(d) Verifying composition:

$$f(f^{-1}(x)) = f\left(\frac{x^2+1}{2}\right) = \sqrt{2 \cdot \frac{x^2+1}{2} - 1} = \sqrt{x^2+1-1} = \sqrt{x^2} = x \quad (\text{since } x \geq 0)$$

$$f^{-1}(f(x)) = f^{-1}\left(\sqrt{2x-1}\right) = \frac{(\sqrt{2x-1})^2+1}{2} = \frac{2x-1+1}{2} = x$$

Both identities verified.

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Question 7

Let $f(x) = \dfrac{6}{x+2} - 1$, with domain $x \neq -2$.

(a) Show that f is a one-to-one function.

(b) Find the inverse function $f^{-1}(x)$.

(c) Determine the domain and range of both $f(x)$ and $f^{-1}(x)$.

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Question 7 - Solution

(a) Show $f(x) = \dfrac{6}{x+2} - 1$ is one-to-one:

Let $f(x_1) = f(x_2)$. Then:

$$\frac{6}{x_1+2} - 1 = \frac{6}{x_2+2} - 1 \Rightarrow \frac{6}{x_1+2} = \frac{6}{x_2+2} \Rightarrow x_1 + 2 = x_2 + 2 \Rightarrow x_1 = x_2$$

Thus, f is one-to-one.

(b) Find the inverse function:

Let $y = \dfrac{6}{x+2} - 1$

Solve for x:

$$y + 1 = \frac{6}{x+2} \Rightarrow x + 2 = \frac{6}{y+1} \Rightarrow x = \frac{6}{y+1} - 2$$

Now switch x and y:

$$f^{-1}(x) = \frac{6}{x+1} - 2$$

Answer: $\boxed{f^{-1}(x) = \dfrac{6}{x+1} - 2}$

(c) Domain and Range:

Original function $f(x) = \dfrac{6}{x+2} - 1$

- Undefined when $x + 2 = 0 \Rightarrow x = -2$ - Domain of f: $\boxed{(-\infty, -2) \cup (-2, \infty)}$

Asymptote at $x = -2$, and horizontal asymptote $y = -1$ - Range of f: $\boxed{(-\infty, -1) \cup (-1, \infty)}$

Inverse: - Domain of f^{-1}: same as range of f: $\boxed{(-\infty, -1) \cup (-1, \infty)}$ - Range of f^{-1}: same as domain of f: $\boxed{(-\infty, -2) \cup (-2, \infty)}$

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Question 8

Let $f(x) = \dfrac{x+3}{x-1}$, where $x \neq 1$.

(a) Show that f is a one-to-one function.

(b) Find the inverse function $f^{-1}(x)$.

(c) State the domain and range of both $f(x)$ and $f^{-1}(x)$.

(d) Verify that $f(f^{-1}(x)) = x$.

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Question 8 - Solution

(a) Show that f is one-to-one:

Let $f(x_1) = f(x_2)$:

$$\frac{x_1+3}{x_1-1} = \frac{x_2+3}{x_2-1} \Rightarrow (x_1+3)(x_2-1) = (x_2+3)(x_1-1)$$

Expanding both sides:

$$x_1x_2 - x_1 + 3x_2 - 3 = x_1x_2 - x_2 + 3x_1 - 3$$

Subtract x_1x_2 and -3 from both sides:

$$-x_1 + 3x_2 = -x_2 + 3x_1 \Rightarrow 3x_2 + x_2 = 3x_1 + x_1 \Rightarrow 4x_2 = 4x_1 \Rightarrow x_1 = x_2$$

Therefore, f is one-to-one.

(b) Find the inverse:

Let $y = \dfrac{x+3}{x-1}$

Multiply both sides:

$$y(x-1) = x+3 \Rightarrow yx - y = x+3 \Rightarrow yx - x = y+3 \Rightarrow x(y-1) = y+3 \Rightarrow x = \frac{y+3}{y-1}$$

Switch x and y:

$$f^{-1}(x) = \frac{x+3}{x-1}$$

So:

$$\boxed{f^{-1}(x) = \frac{x+3}{x-1}}$$

Interestingly, $f^{-1}(x) = f(x)$. The function is its own inverse.

(c) Domain and Range:

From $f(x) = \dfrac{x+3}{x-1}$, we see that: - Domain: $x \neq 1 \Rightarrow \boxed{(-\infty, 1) \cup (1, \infty)}$ - As $x \to \infty$, $f(x) \to 1$, and $f(x) \neq 1$ - So range: $\boxed{(-\infty, 1) \cup (1, \infty)}$

The same applies to $f^{-1}(x)$, so: - Domain of $f^{-1}(x)$: $(-\infty, 1) \cup (1, \infty)$ - Range of $f^{-1}(x)$: $(-\infty, 1) \cup (1, \infty)$

(d) Verify:

$$f(f^{-1}(x)) = f\left(\frac{x+3}{x-1}\right) = \frac{\left(\frac{x+3}{x-1}\right)+3}{\left(\frac{x+3}{x-1}\right)-1} = \frac{\frac{x+3+3(x-1)}{x-1}}{\frac{x+3-(x-1)}{x-1}} = \frac{\frac{x+3+3x-3}{x-1}}{\frac{x+3-x+1}{x-1}} = \frac{\frac{4x}{x-1}}{\frac{4}{x-1}} = x$$

Verified: $f(f^{-1}(x)) = x$

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Question 9

Let $f(x) = \ln(x - 2)$, with domain $x > 2$.

(a) Find the inverse function $f^{-1}(x)$.

(b) Determine the domain and range of f and f^{-1}.

(c) Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

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Question 9 - Solution

(a) Find the inverse:

Let $y = \ln(x-2)$

Solve for x:

$$x - 2 = e^y \Rightarrow x = e^y + 2$$

Now switch x and y:

$$f^{-1}(x) = e^x + 2$$

Answer: $\boxed{f^{-1}(x) = e^x + 2}$

(b) Domain and Range:

For $f(x) = \ln(x-2)$: - Domain: $x > 2 \Rightarrow \boxed{(2, \infty)}$ - Range: $\ln(x-2) \to -\infty$ as $x \to 2^+$, and $\to \infty$ as $x \to \infty$

Range: $\boxed{(-\infty, \infty)}$

For $f^{-1}(x) = e^x + 2$: - Domain: $\boxed{(-\infty, \infty)}$ - Range: $e^x + 2 > 2 \Rightarrow \boxed{(2, \infty)}$

(c) Verify compositions:

$$f(f^{-1}(x)) = f(e^x + 2) = \ln((e^x + 2) - 2) = \ln(e^x) = x$$

$$f^{-1}(f(x)) = f^{-1}(\ln(x-2)) = e^{\ln(x-2)} + 2 = x - 2 + 2 = x$$

Verified: $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

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Question 10

Let

$$f(x) = \frac{\sqrt{x+4}}{x-1}$$

(a) Determine the domain of $f(x)$.

(b) Is f one-to-one on its domain? Justify your answer.

(c) If possible, find $f^{-1}(x)$. If not, restrict the domain appropriately and find the inverse on that domain.

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Question 10 - Solution

Domain. The square root and denominator require

$$\boxed{D = [-4, 1) \cup (1, \infty)}.$$

One-to-one. For $x > -4$, $x \neq 1$,

$$f'(x) = \frac{-(x+9)}{2\sqrt{x+4}(x-1)^2} < 0.$$

The left branch decreases from 0 to $-\infty$ and has range $(-\infty, 0]$.
The right branch decreases from $+\infty$ to 0 and has range $(0, \infty)$.
These ranges are disjoint, so f is one-to-one on its entire domain and its range is $\mathbb{R}$.

Inverse. Write $y = \sqrt{x+4}/(x-1)$. Squaring gives

$$y^2x^2 - (2y^2+1)x + y^2 - 4 = 0.$$

For $y \neq 0$, the quadratic formula gives

$$x = \frac{2y^2 + 1 \pm \sqrt{1+20y^2}}{2y^2}.$$

The original, unsquared equation requires $x < 1$ when $y < 0$ and $x > 1$ when $y > 0$.
Also $y = 0$ gives $x = -4$. Thus

$$\boxed{f^{-1}(y) = \begin{cases} \dfrac{2y^2+1-\sqrt{1+20y^2}}{2y^2}, & y < 0, \\ -4, & y = 0, \\ \dfrac{2y^2+1+\sqrt{1+20y^2}}{2y^2}, & y > 0. \end{cases}}$$

The sign choice restores the original equation after squaring. The inverse has domain $\mathbb{R}$ and range D.

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Question 1

Given the function $f(x) = 3\sin\left(2x - \frac{\pi}{3}\right) + 1$, answer the following:

(a) Determine the amplitude, period, phase shift, and vertical shift.

(b) Find the maximum and minimum values of $f(x)$.

(c) Sketch one full period of the graph, clearly labeling key points.

(d) Determine the x-values where $f(x) = 1$ in the interval $[0, 2\pi]$.

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Question 1 - Solution

(a) Identify parameters from $f(x) = 3\sin\left(2x - \frac{\pi}{3}\right) + 1$:

- Amplitude: $\boxed{3}$
- Period: $\frac{2\pi}{2} = \boxed{\pi}$
- Phase shift:

$$2x - \frac{\pi}{3} = 0 \Rightarrow x = \frac{\pi}{6} \Rightarrow \boxed{\frac{\pi}{6} \text{ to the right}}$$

- Vertical shift: $\boxed{+1}$

(b) Maximum and Minimum:
Since the sine function oscillates between -1 and 1:

$$\text{Min of } f(x) = 3(-1) + 1 = -2, \quad \text{Max of } f(x) = 3(1) + 1 = 4$$

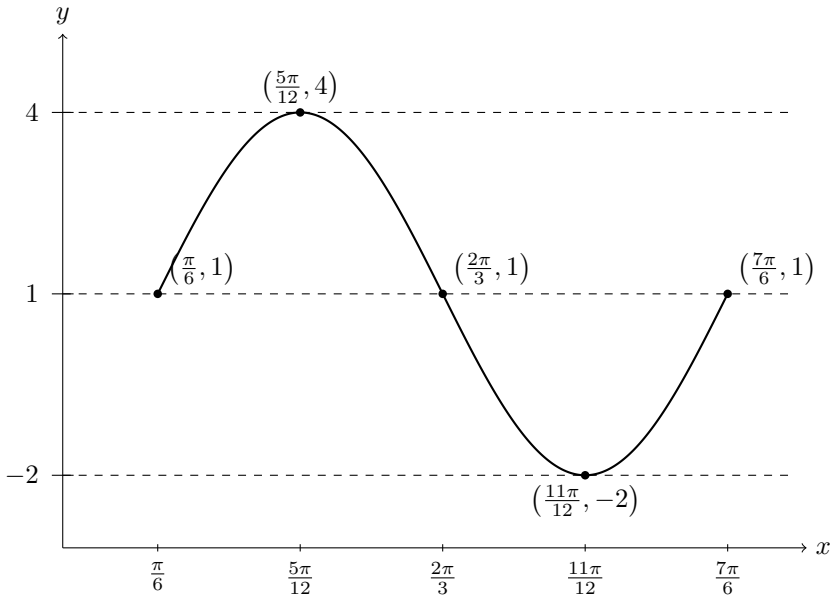
$$\boxed{\text{Max} = 4}, \quad \boxed{\text{Min} = -2}$$

(c) Sketch one full period:
One full period has length π, starting at the phase shift:

$$x = \frac{\pi}{6} \quad \text{to} \quad x = \frac{7\pi}{6}$$

Key points over one period:

x	$f(x)$
$\frac{\pi}{6}$	1 (midline)
$\frac{5\pi}{12}$	4 (maximum)
$\frac{2\pi}{3}$	1 (midline)
$\frac{11\pi}{12}$	-2 (minimum)
$\frac{7\pi}{6}$	1 (midline)



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(d) Solve $f(x) = 1$ on $[0, 2\pi]$:

$$3\sin\left(2x - \frac{\pi}{3}\right) + 1 = 1 \Rightarrow \sin\left(2x - \frac{\pi}{3}\right) = 0$$

$$2x - \frac{\pi}{3} = n\pi \Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{6}$$

Values in $[0, 2\pi]$:

$$\boxed{x = \frac{\pi}{6}, \ \frac{2\pi}{3}, \ \frac{7\pi}{6}, \ \frac{5\pi}{3}}$$

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Question 2

Let

$$g(x) = 2\tan\left(x - \frac{\pi}{4}\right) + 1$$

(a) State the period, phase shift, vertical shift, and asymptotes of $g(x)$.

(b) Sketch one period of the graph, labeling asymptotes and intercepts.

(c) Find the value(s) of $x \in [0, 2\pi]$ such that $g(x) = 3$.

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Question 2 - Solution

(a) Analyze $g(x) = 2\tan\left(x - \frac{\pi}{4}\right) + 1$

- Period: $\boxed{\pi}$
- Phase shift: $\boxed{\frac{\pi}{4} \text{ to the right}}$
- Vertical shift: $\boxed{+1}$

Asymptotes occur when:

$$x - \frac{\pi}{4} = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z} \Rightarrow x = \frac{3\pi}{4} + k\pi$$

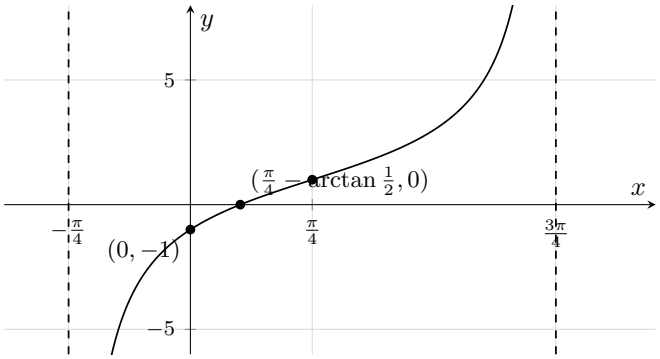
So one period lies on:

$$\left(-\frac{\pi}{4},\ \frac{3\pi}{4}\right)$$

- All asymptotes: $\boxed{x = 3\pi/4 + k\pi,\ k \in \mathbb{Z}}$.

(b) Sketch one period
The intercepts in this period are $(0, -1)$ and $(\pi/4 - \arctan(1/2), 0)$.
Key features:

- Left asymptote: $x = -\frac{\pi}{4}$
- Right asymptote: $x = \frac{3\pi}{4}$
- Midpoint (inflection): $x = \frac{\pi}{4},\ g\left(\frac{\pi}{4}\right) = 1$



(c) Solve $g(x) = 3$**:**

$$2\tan\left(x - \frac{\pi}{4}\right) + 1 = 3 \Rightarrow \tan\left(x - \frac{\pi}{4}\right) = 1$$

$$x - \frac{\pi}{4} = \frac{\pi}{4} + n\pi \Rightarrow x = \frac{\pi}{2} + n\pi$$

Values in $[0, 2\pi]$:

$$\boxed{x = \frac{\pi}{2},\ \frac{3\pi}{2}}$$

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Question 3

Consider the function

$$f(x) = \cos(2x) - \sin(2x)$$

(a) Write $f(x)$ as a single trigonometric function of the form $A\cos(2x + \phi)$.

(b) Determine the amplitude and phase shift of the resulting function.

(c) Find the maximum and minimum values of $f(x)$.

(d) Determine all $x \in [0, 2\pi]$ such that $f(x) = 0$.

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Question 3 - Solution

(a) Write $f(x)$ as a single cosine function:

We use the identity:

$$A\cos(2x+\phi) = A\cos\phi\cos(2x) - A\sin\phi\sin(2x)$$

Match coefficients with:

$$\cos(2x) - \sin(2x)$$

This gives the system:

$$A\cos\phi = 1, \quad A\sin\phi = 1$$

Square and add:

$$A^2(\cos^2\phi + \sin^2\phi) = 1^2 + 1^2 = 2 \Rightarrow A = \sqrt{2}$$

Now:

$$\cos\phi = \frac{1}{\sqrt{2}}, \quad \sin\phi = \frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{\pi}{4}$$

So:

$$f(x) = \sqrt{2}\cos\left(2x + \frac{\pi}{4}\right)$$

(b) Amplitude and Phase Shift:

From the rewritten form:

- Amplitude: $\boxed{\sqrt{2}}$
- Phase shift: Solve $2x + \frac{\pi}{4} = 0 \Rightarrow x = -\frac{\pi}{8}$

So the graph is shifted $\boxed{\frac{\pi}{8}}$ units to the left.

(c) Maximum and Minimum Values:

Since cosine ranges between -1 and 1:

$$\text{Maximum} = \sqrt{2}, \quad \text{Minimum} = -\sqrt{2}$$

(d) Solve $f(x) = 0$ on $[0, 2\pi]$:

$$\cos(2x) - \sin(2x) = 0 \Rightarrow \cos(2x) = \sin(2x) \Rightarrow \tan(2x) = 1$$

$$2x = \frac{\pi}{4} + n\pi \Rightarrow x = \frac{\pi}{8} + \frac{n\pi}{2}$$

Find values in $[0, 2\pi]$:

$$x = \frac{\pi}{8}, \ \frac{5\pi}{8}, \ \frac{9\pi}{8}, \ \frac{13\pi}{8}$$

Answer:

$$\boxed{x = \frac{\pi}{8}, \ \frac{5\pi}{8}, \ \frac{9\pi}{8}, \ \frac{13\pi}{8}}$$

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Question 4

Let $f(x) = \sin(x)\cos(x) + \cos^2(x)$

(a) Express $f(x)$ in terms of a single trigonometric function or a simpler expression.

(b) Find the maximum and minimum values of $f(x)$ on the interval $[0, 2\pi]$.

(c) Determine all values of $x \in [0, 2\pi]$ where $f(x) = 1$.

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Question 4 - Solution

Simplify. Double-angle identities give

$$\boxed{f(x) = \frac{1}{2} + \frac{\sqrt{2}}{2}\sin(2x + \pi/4)}.$$

The sine attains both -1 and 1 on the given interval, so

$$\boxed{f_{\min} = \frac{1-\sqrt{2}}{2}, \qquad f_{\max} = \frac{1+\sqrt{2}}{2}}.$$

Solve without losing roots. Subtracting 1 from the original expression gives

$$f(x) - 1 = \sin x \cos x - \sin^2 x = \sin x(\cos x - \sin x).$$

Thus $\sin x = 0$ or $\cos x = \sin x$. Restricting both families to $[0, 2\pi]$ gives

$$\boxed{x = 0,\ \pi/4,\ \pi,\ 5\pi/4,\ 2\pi}.$$

Each listed value makes one factor zero, so each satisfies the original equation.

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Question 5

Let $f(x) = \frac{\sin(x)}{1+\cos(x)}$

(a) Simplify the expression $f(x)$ using trigonometric identities.

(b) Determine all $x \in (0, 2\pi)$ for which $f(x) = 1$.

(c) Determine the domain of $f(x)$.

(d) Determine whether $f(x)$ is an even function, odd function, or neither.

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Question 5 - Solution

Simplify with the domain retained. Half-angle identities give

$$\boxed{f(x) = \tan(x/2)}, \qquad x \neq (2k+1)\pi,\ k \in \mathbb{Z}.$$

Indeed $\sin x = 2\sin(x/2)\cos(x/2)$ and $1 + \cos x = 2\cos^2(x/2)$.
The alternative $\csc x - \cot x$ is valid only when $\sin x \neq 0$ and loses valid inputs such as $x = 0$.

Solve. $\tan(x/2) = 1$ implies $x = \pi/2 + 2k\pi$, so on $(0, 2\pi)$,

$$\boxed{x = \pi/2}.$$

Domain and parity. The original denominator vanishes precisely at odd multiples of π, hence

$$\boxed{D = \mathbb{R} \setminus \{(2k+1)\pi : k \in \mathbb{Z}\}}.$$

This domain is symmetric about zero, and $f(-x) = -f(x)$ on it. Therefore $\boxed{f \text{ is odd}}$.

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Question 6

Let $f(x) = \sin(x) + \sin(3x)$

(a) Use a trigonometric identity to write $f(x)$ as a product.

(b) Determine all values of $x \in [0, 2\pi]$ where $f(x) = 0$.

(c) Sketch the graph of $f(x)$ over one full period.

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Question 6 - Solution

(a) Use sum-to-product identities:

We apply:

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

Let $A = 3x$, $B = x$:

$$f(x) = \sin(x) + \sin(3x) = 2\sin\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right) = 2\sin(2x)\cos(x)$$

$$\boxed{f(x) = 2\sin(2x)\cos(x)}$$

(b) Solve $f(x) = 0$ on $[0, 2\pi]$:

$$2\sin(2x)\cos(x) = 0 \Rightarrow \sin(2x) = 0 \quad \text{or} \quad \cos(x) = 0$$

Solve $\sin(2x) = 0$:

$$2x = n\pi \Rightarrow x = \frac{n\pi}{2} \Rightarrow x = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi$$

Solve $\cos(x) = 0$:

$$x = \frac{\pi}{2},\ \frac{3\pi}{2}$$

Union of solutions:

$$\boxed{x = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi}$$

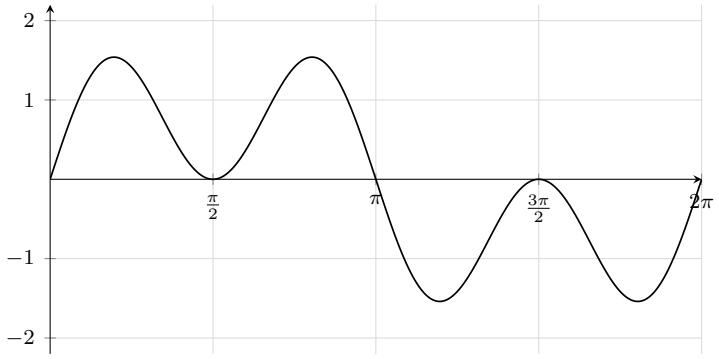
(c) Sketch the graph of $f(x) = \sin(x) + \sin(3x)$

The graph is periodic with the least common multiple of the periods of $\sin(x)$ and $\sin(3x)$.

Period of $\sin(x)$: 2π Period of $\sin(3x)$: $\frac{2\pi}{3}$ LCM of 2π and $\frac{2\pi}{3}$ is 2π

So sketch one full period over $[0, 2\pi]$. Use the product form $f(x) = 2\sin(2x)\cos(x)$ to help identify:

Zeros: Where either $\sin(2x) = 0$ or $\cos(x) = 0$ Amplitude varies due to the product of two trig functions Important points: $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$, and midpoints between for peaks/troughs



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Question 7

Let

$$f(x) = \frac{1 + \cos(2x)}{\sin(2x)}$$

(a) Simplify the expression as much as possible.

(b) Find all $x \in (0, \pi)$ for which $f(x) = 2$.

(c) Determine the domain of $f(x)$ over $[0, 2\pi]$.

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Question 7 - Solution

Simplify. Where the original function is defined,

$$f(x) = \frac{2\cos^2 x}{2\sin x \cos x} = \cot x.$$

The cancellation does not restore points where $\sin(2x) = 0$.

Solve. On $(0, \pi)$, $\cot x = 2$ gives

$$\boxed{x = \arctan(1/2) \approx 0.4636}.$$

This point is in the original domain.

Domain. All multiples of $\pi/2$ must be excluded. On $[0, 2\pi]$ the domain is

$$\boxed{(0, \pi/2) \cup (\pi/2, \pi) \cup (\pi, 3\pi/2) \cup (3\pi/2, 2\pi)}.$$

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Question 8

Let

$$f(x) = \frac{\cos(x)}{1 - \sin(x)}$$

(a) Simplify the expression $f(x)$ using trigonometric identities.

(b) Determine all values of $x \in (0, 2\pi)$ for which $f(x) = 1$.

(c) Determine whether $f(x)$ is even, odd, or neither.

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Question 8 - Solution

Simplification and domain. The original function is defined for $x \neq \pi/2 + 2k\pi$. Where $\cos x \neq 0$, rationalization gives

$$f(x) = \frac{1 + \sin x}{\cos x} = \sec x + \tan x.$$

At $x = 3\pi/2 + 2k\pi$, the original function has value 0, although this alternative expression is undefined.

Solve on the requested interval. If $f(x) = 1$, then $\cos x + \sin x = 1$, with $\sin x \neq 1$.

Using $\cos x = 1 - \sin x$ in $\cos^2 x + \sin^2 x = 1$ gives

$$2 \sin x(\sin x - 1) = 0.$$

The case $\sin x = 1$ is excluded. The other case requires $\sin x = 0$ and $\cos x = 1$, hence $x = 2k\pi$.

Neither endpoint belongs to $(0, 2\pi)$, so

$$\boxed{\text{There are no solutions in } (0, 2\pi).}$$

Parity. The domain is not symmetric about zero: $-\pi/2$ is allowed but $\pi/2$ is not.

$$\boxed{f \text{ is neither even nor odd.}}$$

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Question 9

Let

$$f(x) = \tan(x) - \sec(x)$$

(a) Show that $f(x) = \frac{\sin(x)-1}{\cos(x)}$

(b) Determine the values of $x \in (0, 2\pi)$ for which $f(x) = 0$

(c) Analyze the behavior of $f(x)$ near $x = \frac{\pi}{2}$

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Question 9 - Solution

(a) Simplify $f(x) = \tan(x) - \sec(x)$**:**

We write both terms in terms of sine and cosine:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \sec(x) = \frac{1}{\cos(x)}$$

Then:

$$f(x) = \frac{\sin(x)}{\cos(x)} - \frac{1}{\cos(x)} = \frac{\sin(x) - 1}{\cos(x)}$$

$$\boxed{f(x) = \frac{\sin(x) - 1}{\cos(x)}}$$

(b) Solve $f(x) = 0 \Rightarrow \frac{\sin(x)-1}{\cos(x)} = 0$

This occurs when the numerator is 0 (and denominator is not 0):

$$\sin(x) - 1 = 0 \Rightarrow \sin(x) = 1 \Rightarrow x = \frac{\pi}{2}$$

But at $x = \frac{\pi}{2}$, $\cos(x) = 0$, so $f(x)$ is undefined.
Therefore, there is **no solution** to $f(x) = 0$ on $(0, 2\pi)$

$$\boxed{\text{No solution in } (0, 2\pi)}$$

(c) Behavior near $x = \frac{\pi}{2}$

As $x \to \frac{\pi}{2}^-$, we have: - $\sin(x) \to 1$ - $\cos(x) \to 0^+$ - So: $f(x) \to \frac{1-1}{0^+} = 0$
As $x \to \frac{\pi}{2}^+$, we have: - $\sin(x) \to 1$ - $\cos(x) \to 0^-$ - So: $f(x) \to \frac{1-1}{0^-} = 0$
But if we take x very close to $\frac{\pi}{2}$, with small deviation, say:

$$f(x) = \frac{\sin(x) - 1}{\cos(x)} \Rightarrow \text{Numerator } \approx 0, \text{ Denominator } \to 0$$

Apply L'Hôpital's Rule:

$$\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{\cos(x)}{-\sin(x)} = \frac{0}{-1} = 0$$

So:

$$\boxed{\lim_{x \to \frac{\pi}{2}} f(x) = 0}$$

But since the function is undefined at $x = \frac{\pi}{2}$, the limit exists but the function is discontinuous there.

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Question 10

Define the function:

$$f(x) = \frac{1 - \cos(2x)}{\sin(x)}$$

(a) Simplify $f(x)$ using appropriate identities.

(b) Determine the values of $x \in (0, \pi)$ for which $f(x) = 2\sin(x)$.

(c) State the domain of $f(x)$ on $[0, 2\pi]$.

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Question 10 - Solution

(a) Simplify using identities:

Using the identity

$$1 - \cos(2x) = 2\sin^2(x),$$

we obtain

$$f(x) = \frac{2\sin^2(x)}{\sin(x)}.$$

For $\sin(x) \neq 0$, this simplifies to

$$f(x) = 2\sin(x).$$

$$\boxed{f(x) = 2\sin(x) \quad \text{for } \sin(x) \neq 0}$$

(b) Solve $f(x) = 2\sin(x)$

From part (a), the equation

$$f(x) = 2\sin(x)$$

is an identity for all x where $f(x)$ is defined.

However, the original function is undefined when

$$\sin(x) = 0 \Rightarrow x = 0,\ \pi.$$

Since the interval is restricted to $(0, \pi)$, neither excluded value lies in the interval. Therefore, all values in $(0, \pi)$ satisfy the equation.

$$\boxed{\text{Solution set: } (0, \pi) \quad \text{(excluded: none within the interval)}}$$

(c) Domain of $f(x) = \frac{1-\cos(2x)}{\sin(x)}$ on $[0, 2\pi]$

The function is undefined when the denominator is zero:

$$\sin(x) = 0 \Rightarrow x = 0,\ \pi,\ 2\pi.$$

Removing these values from the interval $[0, 2\pi]$, the domain is

$$\boxed{(0, \pi) \cup (\pi, 2\pi)}.$$

Excluded values: $x = 0,\ \pi,\ 2\pi$

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Question 1

Solve the equation:

$$2\sin^2(x) - 3\sin(x) + 1 = 0$$

for all $x \in [0, 2\pi]$.

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Question 1 - Solution

We factor the equation:

$$2\sin^2(x) - 3\sin(x) + 1 = 0.$$

Looking for factors of $2 \cdot 1 = 2$ that add to -3, we write:

$$(2\sin(x) - 1)(\sin(x) - 1) = 0.$$

Setting each factor equal to zero:

$$2\sin(x) - 1 = 0 \quad \text{or} \quad \sin(x) - 1 = 0.$$

$$\sin(x) = \frac{1}{2} \quad \text{or} \quad \sin(x) = 1.$$

Now solve for $x \in [0, 2\pi]$:

$$\sin(x) = 1 \Rightarrow x = \frac{\pi}{2}.$$

$$\sin(x) = \frac{1}{2} \Rightarrow x = \frac{\pi}{6},\ \frac{5\pi}{6}.$$

Therefore, the solutions are:

$$\boxed{x = \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{\pi}{2}}.$$

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Question 2

Solve the equation

$$\cos(2x) = 2\cos^2(x) - 1$$

for all $x \in [0, 2\pi]$, and verify the identity used.

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Question 2 - Solution

Step 1: Recognize the identity.

The right-hand side is a known double-angle identity:

$$\cos(2x) = 2\cos^2(x) - 1$$

So this equation is always true:

$$\cos(2x) = \cos(2x)$$

Thus, it holds for all x.
But the problem is to solve:

$$\cos(2x) = 2\cos^2(x) - 1$$

Since it holds for all x, this is an identity. So the equation is true for all $x \in [0, 2\pi]$

$$\boxed{x \in [0, 2\pi]}$$

Verification of identity:
Start with RHS:

$$2\cos^2(x) - 1 = \cos(2x)$$

This is a standard double-angle identity for cosine:

$$\cos(2x) = 2\cos^2(x) - 1 \Rightarrow \text{Identity verified.}$$

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Question 3

Solve the equation

$$\sin(x)\cos(x) = \frac{1}{2}$$

for all $x \in [0, 2\pi]$.

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Question 3 - Solution

We begin by using a known identity:

$$\sin(2x) = 2\sin(x)\cos(x) \Rightarrow \sin(x)\cos(x) = \frac{1}{2}\sin(2x)$$

Substitute into the equation:

$$\frac{1}{2}\sin(2x) = \frac{1}{2} \Rightarrow \sin(2x) = 1$$

Now solve:

$$\sin(2x) = 1 \Rightarrow 2x = \frac{\pi}{2} + 2n\pi \quad \text{for integer } n \Rightarrow x = \frac{\pi}{4} + n\pi$$

Find solutions in $[0, 2\pi]$:
$n = 0 \Rightarrow x = \frac{\pi}{4}$ $n = 1 \Rightarrow x = \frac{5\pi}{4}$ $n = 2 \Rightarrow x = \frac{9\pi}{4} > 2\pi \rightarrow$ discard

$$\boxed{x = \frac{\pi}{4}, \ \frac{5\pi}{4}}$$

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Question 4

Solve the equation

$$\cos(2x) + \sin(x) = 0$$

for all $x \in [0, 2\pi]$.

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Question 4 - Solution

Step 1: Use a double-angle identity for $\cos(2x)$:

We use:

$$\cos(2x) = 1 - 2\sin^2(x)$$

Substitute into the equation:

$$1 - 2\sin^2(x) + \sin(x) = 0 \Rightarrow -2\sin^2(x) + \sin(x) + 1 = 0$$

Multiply through by -1 to simplify:

$$2\sin^2(x) - \sin(x) - 1 = 0$$

Step 2: Solve the quadratic equation in $\sin(x)$:

Let $u = \sin(x)$:

$$2u^2 - u - 1 = 0 \Rightarrow u = \frac{1 \pm \sqrt{1^2 + 4(2)(1)}}{2(2)} = \frac{1 \pm \sqrt{9}}{4} = \frac{1 \pm 3}{4}$$

So:

$$u = 1 \quad \text{or} \quad u = -\frac{1}{2}$$

Now solve $\sin(x) = 1 \Rightarrow x = \frac{\pi}{2}$

And $\sin(x) = -\frac{1}{2} \Rightarrow x = \frac{7\pi}{6}, \frac{11\pi}{6}$

Final answer:

$$\boxed{x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}}$$

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Question 5

Solve the equation

$$\sin(x) + \sqrt{3}\cos(x) = 0$$

for all $x \in [0, 2\pi]$.

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Question 5 - Solution

We start by solving:

$$\sin(x) + \sqrt{3}\cos(x) = 0 \Rightarrow \sin(x) = -\sqrt{3}\cos(x)$$

Divide both sides by $\cos(x)$, assuming $\cos(x) \neq 0$:

$$\tan(x) = -\sqrt{3}$$

Now solve:

$$\tan(x) = -\sqrt{3} \Rightarrow x = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

But we want values in $[0, 2\pi]$
So find angles with tangent $= -\sqrt{3}$
The reference angle is $\frac{\pi}{3}$, and tangent is negative in quadrants II and IV

$$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}, \quad x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

$$\boxed{x = \frac{2\pi}{3}, \ \frac{5\pi}{3}}$$

Also check if $\cos(x) = 0$
If $\cos(x) = 0 \Rightarrow \sin(x) = 0$ also must hold (from original equation)
But: - If $\cos(x) = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$ - At $x = \frac{\pi}{2}$: $\sin(x) = 1$, LHS $= 1 + 0 \neq 0$ - At $x = \frac{3\pi}{2}$: $\sin(x) = -1$, LHS $= -1 + 0 \neq 0$
So those are not valid. Final solution:

$$\boxed{x = \frac{2\pi}{3}, \ \frac{5\pi}{3}}$$

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Question 6

Solve the equation

$$2\cos^2(x) - 1 = \sin(x)$$

for all $x \in [0, 2\pi]$.

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Question 6 - Solution

We begin by using the Pythagorean identity:

$$\cos^2(x) = 1 - \sin^2(x)$$

Substitute into the equation:

$$2(1 - \sin^2(x)) - 1 = \sin(x) \Rightarrow 2 - 2\sin^2(x) - 1 = \sin(x) \Rightarrow -2\sin^2(x) - \sin(x) + 1 = 0$$

Multiply through by -1:

$$2\sin^2(x) + \sin(x) - 1 = 0$$

Let $u = \sin(x)$. Then:

$$2u^2 + u - 1 = 0 \Rightarrow u = \frac{-1 \pm \sqrt{1^2 - 4(2)(-1)}}{2(2)} = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4}$$

$$u = \frac{1}{2}, \quad u = -1$$

So:

$$\sin(x) = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \ \frac{5\pi}{6}$$

$$\sin(x) = -1 \Rightarrow x = \frac{3\pi}{2}$$

Final solution:

$$\boxed{x = \frac{\pi}{6}, \ \frac{5\pi}{6}, \ \frac{3\pi}{2}}$$

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Question 7

Solve the equation

$$\sin(2x) = \sqrt{3}\cos(x)$$

for all $x \in [0, 2\pi]$.

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Question 7 - Solution

Use the identity:

$$\sin(2x) = 2\sin(x)\cos(x)$$

Substitute into the equation:

$$2\sin(x)\cos(x) = \sqrt{3}\cos(x)$$

Subtract $\sqrt{3}\cos(x)$ from both sides:

$$2\sin(x)\cos(x) - \sqrt{3}\cos(x) = 0 \Rightarrow \cos(x)\left(2\sin(x) - \sqrt{3}\right) = 0$$

Now solve each factor:
Case 1: $\cos(x) = 0 \Rightarrow x = \frac{\pi}{2},\ \frac{3\pi}{2}$
Case 2: $2\sin(x) - \sqrt{3} = 0 \Rightarrow \sin(x) = \frac{\sqrt{3}}{2}$

$$\sin(x) = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{3},\ \frac{2\pi}{3}$$

Final Solution:

$$\boxed{x = \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{\pi}{2},\ \frac{3\pi}{2}}$$

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Question 8

Solve the equation

$$\sin^2(x) - \cos(x) = 0$$

for all $x \in [0, 2\pi]$.

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Question 8 - Solution

We start with the identity:

$$\sin^2(x) = 1 - \cos^2(x)$$

Substitute into the equation:

$$1 - \cos^2(x) - \cos(x) = 0 \Rightarrow -\cos^2(x) - \cos(x) + 1 = 0 \Rightarrow \cos^2(x) + \cos(x) - 1 = 0$$

Let $u = \cos(x)$, then:

$$u^2 + u - 1 = 0 \Rightarrow u = \frac{-1 \pm \sqrt{1^2 + 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$$

So:

$$\cos(x) = \frac{-1 + \sqrt{5}}{2} \approx 0.618, \quad \cos(x) = \frac{-1 - \sqrt{5}}{2} \approx -1.618$$

Only $\frac{-1+\sqrt{5}}{2}$ is in the range $[-1, 1]$
Now solve:

$$\cos(x) = \frac{-1 + \sqrt{5}}{2} \Rightarrow x = \cos^{-1}\left(\frac{-1 + \sqrt{5}}{2}\right)$$

Let's call $\theta = \cos^{-1}\left(\frac{-1+\sqrt{5}}{2}\right)$
Since cosine is positive in quadrants I and IV, and $\cos(x) > 0$, we get:

$$x = \theta, \quad x = 2\pi - \theta$$

So:

$$\boxed{x = \cos^{-1}\left(\frac{-1 + \sqrt{5}}{2}\right), \quad x = 2\pi - \cos^{-1}\left(\frac{-1 + \sqrt{5}}{2}\right)}$$

These are the exact answers in terms of inverse cosine.

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Question 9

Solve the equation

$$\tan^2(x) - 3\tan(x) + 2 = 0$$

for all $x \in [0, 2\pi)$.

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Question 9 - Solution

Treat this as a quadratic in $\tan(x)$. Let $u = \tan(x)$:

$$u^2 - 3u + 2 = 0 \Rightarrow u = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(2)}}{2(1)} = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} \Rightarrow u = 1,\ 2$$

So:

$$\tan(x) = 1 \quad \text{or} \quad \tan(x) = 2$$

Solve $\tan(x) = 1$:

$$x = \frac{\pi}{4},\ \frac{5\pi}{4}$$

Solve $\tan(x) = 2$:
We must use the inverse tangent:

$$x = \tan^{-1}(2) \approx 1.107 \text{ radians}$$

Tangent is positive in quadrants I and III, so:

$$x = \tan^{-1}(2),\ \pi + \tan^{-1}(2)$$

Approximate values:

$$x \approx 1.107,\ 4.249$$

Final Answer:

$$\boxed{x = \frac{\pi}{4},\ \frac{5\pi}{4},\ \tan^{-1}(2),\ \pi + \tan^{-1}(2)}$$

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Question 10

Solve the equation

$$\sec(x) = 2$$

for all $x \in [0, 2\pi]$.

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Question 10 - Solution

Step 1: Recall the definition of secant:

$$\sec(x) = \frac{1}{\cos(x)} \Rightarrow \frac{1}{\cos(x)} = 2 \Rightarrow \cos(x) = \frac{1}{2}$$

Step 2: Solve $\cos(x) = \frac{1}{2}$ **in** $[0, 2\pi]$:
The cosine is $\frac{1}{2}$ at:

$$x = \frac{\pi}{3}, \ \frac{5\pi}{3}$$

$$\boxed{x = \frac{\pi}{3}, \ \frac{5\pi}{3}}$$

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Question 1

Solve the equation

$$\sin(x) = 0.65$$

for all $x \in [0, 2\pi]$, and give your answers rounded to two decimal places.

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Question 1 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\sin(1x) = 0.65, \qquad u = 1x, \qquad 0 \le u \le 2\pi.$$

Let $\alpha = \arcsin(0.65)$. All solutions are

$$u = \alpha + 2k\pi \quad \text{or} \quad u = \pi - \alpha + 2k\pi, \qquad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 0.71,\ 2.43}.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

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Question 2

Solve the equation

$$\cos(x) = -0.34$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

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Question 2 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\cos(1x) = -0.34, \qquad u = 1x, \qquad 0 \le u \le 2\pi.$$

Let $\alpha = \arccos(-0.34)$. All solutions are

$$u = \pm\alpha + 2k\pi, \qquad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 1.92,\ 4.37}.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

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Question 3

Solve the equation

$$\tan(x) = 1.8$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

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Question 3 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\tan(1x) = 1.8, \qquad u = 1x, \qquad 0 \le u \le 2\pi.$$

Let $\alpha = \arctan(1.8)$. All solutions are

$$u = \alpha + k\pi, \qquad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 1.06,\ 4.21}.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

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Question 4

Solve the equation

$$\sin(x) = -0.77$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

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Question 4 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\sin(1x) = -0.77, \qquad u = 1x, \qquad 0 \le u \le 2\pi.$$

Let $\alpha = \arcsin(-0.77)$. All solutions are

$$u = \alpha + 2k\pi \quad \text{or} \quad u = \pi - \alpha + 2k\pi, \qquad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 4.02,\ 5.40}.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

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Question 5

Solve the equation

$$\cos(x) = 0.29$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

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Question 5 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\cos(1x) = 0.29, \qquad u = 1x, \qquad 0 \le u \le 2\pi.$$

Let $\alpha = \arccos(0.29)$. All solutions are

$$u = \pm\alpha + 2k\pi, \qquad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 1.28,\ 5.01}.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

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Question 6

Solve the equation

$$\tan(x) = -0.52$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

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Question 6 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\tan(1x) = -0.52, \qquad u = 1x, \qquad 0 \le u \le 2\pi.$$

Let $\alpha = \arctan(-0.52)$. All solutions are

$$u = \alpha + k\pi, \qquad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 2.66,\ 5.80}.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

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Question 7

Solve the equation

$$\sin(x) = -0.42$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

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Question 7 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\sin(1x) = -0.42, \qquad u = 1x, \qquad 0 \le u \le 2\pi.$$

Let $\alpha = \arcsin(-0.42)$. All solutions are

$$u = \alpha + 2k\pi \quad \text{or} \quad u = \pi - \alpha + 2k\pi, \qquad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 3.58,\ 5.85}.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

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Question 8

Solve the equation

$$\cos(x) = -0.91$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

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Question 8 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\cos(1x) = -0.91, \qquad u = 1x, \qquad 0 \le u \le 2\pi.$$

Let $\alpha = \arccos(-0.91)$. All solutions are

$$u = \pm\alpha + 2k\pi, \qquad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 2.71,\ 3.57}.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

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Question 9

Solve the equation

$$\tan(x) = -1.25$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

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Question 9 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\tan(1x) = -1.25, \qquad u = 1x, \qquad 0 \le u \le 2\pi.$$

Let $\alpha = \arctan(-1.25)$. All solutions are

$$u = \alpha + k\pi, \qquad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 2.25,\ 5.39}.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

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Question 10

Solve the equation

$$\sin(x) = 0.83$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

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Question 10 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\sin(1x) = 0.83, \qquad u = 1x, \qquad 0 \le u \le 2\pi.$$

Let $\alpha = \arcsin(0.83)$. All solutions are

$$u = \alpha + 2k\pi \quad \text{or} \quad u = \pi - \alpha + 2k\pi, \qquad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 0.98,\ 2.16}.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

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Question 1

Solve the equation

$$2\cos(x) + 1 = 0$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places if necessary.

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Question 1 - Solution

Step 1: Isolate the cosine term

$$2\cos(x) + 1 = 0 \Rightarrow \cos(x) = -\frac{1}{2}$$

Step 2: Solve the equation using inverse cosine

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \Rightarrow \text{Reference angle} = \frac{\pi}{3}$$

Since cosine is negative, the solutions are in Quadrants II and III:
- Quadrant II: $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ - Quadrant III: $x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

$$\boxed{x = \frac{2\pi}{3}, \frac{4\pi}{3}}$$

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Question 2

Solve the equation

$$\sin(2x) = 0.6$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

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Question 2 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\sin(2x) = 0.6, \qquad u = 2x, \qquad 0 \le u \le 4\pi.$$

Let $\alpha = \arcsin(0.6)$. All solutions are

$$u = \alpha + 2k\pi \quad \text{or} \quad u = \pi - \alpha + 2k\pi, \qquad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 4\pi]$ and divide by 2. Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 0.32,\ 1.25,\ 3.46,\ 4.39}.$$

These are all 4 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

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Question 3

Solve the equation

$$\tan(2x) = -1.7$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

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Question 3 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\tan(2x) = -1.7, \qquad u = 2x, \qquad 0 \le u \le 4\pi.$$

Let $\alpha = \arctan(-1.7)$. All solutions are

$$u = \alpha + k\pi, \qquad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 4\pi]$ and divide by 2. Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 1.05,\ 2.62,\ 4.19,\ 5.76}.$$

These are all 4 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

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Question 4

Solve the equation

$$\cos(3x) = 0.2$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

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Question 4 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\cos(3x) = 0.2, \qquad u = 3x, \qquad 0 \le u \le 6\pi.$$

Let $\alpha = \arccos(0.2)$. All solutions are

$$u = \pm\alpha + 2k\pi, \qquad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 6\pi]$ and divide by 3. Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 0.46,\ 1.64,\ 2.55,\ 3.73,\ 4.65,\ 5.83}.$$

These are all 6 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

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Question 5

Solve the equation

$$2\sin(x) - 1 = 0.3$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

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Question 5 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\sin(1x) = 0.65, \qquad u = 1x, \qquad 0 \le u \le 2\pi.$$

Let $\alpha = \arcsin(0.65)$. All solutions are

$$u = \alpha + 2k\pi \quad \text{or} \quad u = \pi - \alpha + 2k\pi, \qquad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 0.71,\ 2.43}.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

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Question 6

Solve the equation

$$1 + 3\cos(x) = 0.4$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

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Question 6 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\cos(1x) = -0.2, \qquad u = 1x, \qquad 0 \le u \le 2\pi.$$

Let $\alpha = \arccos(-0.2)$. All solutions are

$$u = \pm\alpha + 2k\pi, \qquad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 1.77,\ 4.51}.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

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Question 7

Solve the equation

$$4\sin(2x) = 3$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

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Question 7 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\sin(2x) = 0.75, \qquad u = 2x, \qquad 0 \le u \le 4\pi.$$

Let $\alpha = \arcsin(0.75)$. All solutions are

$$u = \alpha + 2k\pi \quad \text{or} \quad u = \pi - \alpha + 2k\pi, \qquad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 4\pi]$ and divide by 2. Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 0.42,\ 1.15,\ 3.57,\ 4.29}.$$

These are all 4 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

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Question 8

Solve the equation

$$2\cos(2x) - 1 = 0.6$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

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Question 8 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\cos(2x) = 0.8, \qquad u = 2x, \qquad 0 \le u \le 4\pi.$$

Let $\alpha = \arccos(0.8)$. All solutions are

$$u = \pm\alpha + 2k\pi, \qquad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 4\pi]$ and divide by 2. Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 0.32,\ 2.82,\ 3.46,\ 5.96}.$$

These are all 4 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

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Question 9

Solve the equation

$$\tan(x) + 2 = 1.5$$

for all $x \in [0, 2\pi]$, and round your answers to two decimal places.

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Question 9 - Solution

Step 1: Isolate the trigonometric function. The equation becomes

$$\tan(1x) = -0.5, \qquad u = 1x, \qquad 0 \le u \le 2\pi.$$

Let $\alpha = \arctan(-0.5)$. All solutions are

$$u = \alpha + k\pi, \qquad k \in \mathbb{Z}.$$

Step 2: Restrict and convert. Keep precisely the values of u in $[0, 2\pi]$ and divide by 1. Evaluating the inverse function at full precision and rounding only the final values gives

$$\boxed{x \approx 2.68,\ 5.82}.$$

These are all 2 solutions in the stated interval, in radians. Substitution of the unrounded values verifies the original equation.

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Question 10

Solve the equation

$$\sin(x) + \cos(x) = 1.2$$

for all $x \in [0, 2\pi]$, rounding your answers to two decimal places.

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Question 10 - Solution

Use

$$\sin x + \cos x = \sqrt{2}\sin(x + \pi/4).$$

Let $\alpha = \arcsin(1.2/\sqrt{2})$. The two solution families are

$$x = \alpha - \pi/4 + 2k\pi, \qquad x = 3\pi/4 - \alpha + 2k\pi.$$

Only $k = 0$ gives values in $[0, 2\pi]$, hence

$$\boxed{x \approx 0.23,\ 1.34}.$$

Both satisfy the unsquared equation; adding π reverses the sign and does not give another solution.

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Question 1

The population of a bacterial culture is modeled by the function

$$P(t) = 500e^{0.3t}$$

where t is measured in hours.

(a) What is the population after 5 hours?

(b) After how many hours will the population reach 3000?

(c) By what factor does the population grow every 4 hours?

Round your answers to two decimal places.

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Question 1 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$P(5) = 500e^{1.5}$$

$$\boxed{2240.84 \quad \text{bacteria}}$$

(b)

$$500e^{0.3t} = 3000 \implies t = \ln(6)/0.3$$

$$\boxed{5.97 \quad \text{hours}}$$

(c)

$$P(t+4)/P(t) = e^{1.2}$$

$$\boxed{3.32 \quad \text{growth factor}}$$

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Question 2

A radioactive substance decays according to the formula

$$A(t) = A_0 e^{-kt}$$

where $A_0 = 100$ grams and the half-life of the substance is 10 hours.

(a) Find the decay constant k.

(b) How much of the substance remains after 24 hours?

(c) After how many hours will only 20 grams remain?

Round all answers to two decimal places.

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Question 2 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$e^{-10k} = 1/2 \implies k = \ln(2)/10$$

$$\boxed{0.07 \quad \text{hr}^{-1}}$$

The exact value of k above is used in later parts; its decimal value is 0.069315.

(b)

$$A(24) = 100e^{-24\ln(2)/10}$$

$$\boxed{18.95 \quad \text{grams}}$$

(c)

$$100e^{-kt} = 20 \implies t = 10\ln(5)/\ln(2)$$

$$\boxed{23.22 \quad \text{hours}}$$

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Question 3

A cup of coffee is initially at 190°F and is left to cool in a room that is maintained at 70°F. After 15 minutes, the coffee has cooled to 130°F.

Assume the temperature of the coffee follows Newton's Law of Cooling:

$$T(t) = T_{\text{room}} + (T_0 - T_{\text{room}})e^{-kt}$$

(a) Find the value of the constant k.

(b) What will the temperature be after 30 minutes?

(c) After how many minutes will the coffee reach 100°F?

Round all answers to two decimal places.

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Question 3 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$130 = 70 + 120e^{-15k} \implies k = \ln(2)/15$$

$$\boxed{0.05 \quad \text{min}^{-1}}$$

The exact value of k above is used in later parts; its decimal value is 0.046210.

(b)

$$T(30) = 70 + 120e^{-2\ln 2}$$

$$\boxed{100.00 \quad ^\circ\text{F}}$$

(c)

$$100 = 70 + 120e^{-kt} \implies t = \ln(4)/k$$

$$\boxed{30.00 \quad \text{minutes}}$$

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Question 4

The value of a car decreases exponentially over time according to the function

$$V(t) = 25000e^{-0.18t}$$

where $V(t)$ is the value in dollars and t is the number of years since purchase.

(a) What is the value of the car after 3 years?

(b) How long will it take for the car to be worth half its original value?

(c) After how many years will the car be worth \$5,000?

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Question 4 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$V(3) = 25000e^{-0.54}$$

$$\boxed{14568.71 \quad \text{dollars}}$$

(b)

$$e^{-0.18t} = 1/2 \implies t = \ln(2)/0.18$$

$$\boxed{3.85 \quad \text{years}}$$

(c)

$$25000e^{-0.18t} = 5000 \implies t = \ln(5)/0.18$$

$$\boxed{8.94 \quad \text{years}}$$

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Question 5

A scientist is growing a bacteria culture in a lab. The population of bacteria at time t hours is given by the function:

$$P(t) = 1200e^{0.55t}$$

(a) What is the population after 4 hours?

(b) After how many hours will the population reach 10,000?

(c) By what percentage does the population increase each hour?

Round all answers to two decimal places.

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Question 5 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$P(4) = 1200e^{2.2}$$

$$\boxed{10830.02 \quad \text{bacteria}}$$

(b)

$$t = \ln(10000/1200)/0.55$$

$$\boxed{3.86 \quad \text{hours}}$$

(c)

$$100(P(t+1)/P(t) - 1) = 100(e^{0.55} - 1)$$

$$\boxed{73.33 \quad \%}$$

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Question 6

A forest fire is spreading in a circular pattern and the area (in square kilometers) covered by the fire after t hours is modeled by the function:

$$A(t) = 2.5e^{0.4t}$$

(a) What is the area affected after 6 hours?

(b) After how many hours will the fire have covered 50 square kilometers?

(c) How fast is the area growing (in km^2/hr) after 6 hours?

Round all answers to two decimal places.

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Question 6 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$A(6) = 2.5e^{2.4}$$

$$\boxed{27.56 \quad \text{km}^2}$$

(b)

$$t = \ln(50/2.5)/0.4$$

$$\boxed{7.49 \quad \text{hours}}$$

(c)

$$A'(t) = e^{0.4t}, \quad A'(6) = e^{2.4}$$

$$\boxed{11.02 \quad \text{km}^2/\text{hr}}$$

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Question 7

The number of viewers watching a viral video on a social media platform can be modeled by the function

$$V(t) = 1500 \cdot e^{0.7t}$$

where $V(t)$ is the number of viewers and t is the time in days since the video was posted.

(a) How many people have viewed the video after 2 days?

(b) After how many days will the video have 50,000 viewers?

(c) What is the average rate of change of viewers between day 1 and day 3?

Round all answers to two decimal places.

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Question 7 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$V(2) = 1500e^{1.4}$$

$$\boxed{6082.80 \quad \text{viewers}}$$

(b)

$$t = \ln(50000/1500)/0.7$$

$$\boxed{5.01 \quad \text{days}}$$

(c)

$$\frac{V(3) - V(1)}{3 - 1} = 750(e^{2.1} - e^{0.7})$$

$$\boxed{4614.31 \quad \text{viewers/day}}$$

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Question 8

A certain medication is administered intravenously and metabolized by the body over time. The amount of the drug in the bloodstream (in milligrams) is modeled by the function:

$$C(t) = 80e^{-0.35t}$$

where t is the number of hours after the drug is administered.

(a) What is the amount of the drug in the bloodstream after 2 hours?

(b) After how many hours will only 10 mg remain in the bloodstream?

(c) What percentage of the drug is lost every hour?

Round all answers to two decimal places.

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Question 8 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$C(2) = 80e^{-0.7}$$

$$\boxed{39.73 \quad \text{mg}}$$

(b)

$$t = \ln(80/10)/0.35$$

$$\boxed{5.94 \quad \text{hours}}$$

(c)

$$100(1 - C(t+1)/C(t)) = 100(1 - e^{-0.35})$$

$$\boxed{29.53 \quad \%}$$

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Question 9

A savings account earns interest compounded continuously at an annual rate of 4.5

$$A(t) = 3000e^{0.045t}$$

(a) How much money will be in the account after 6 years?

(b) How long will it take for the account to reach \$5,000?

(c) How much interest is earned between years 4 and 8?

Round all answers to two decimal places.

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Question 9 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$A(6) = 3000e^{0.27}$$

$$\boxed{3929.89 \quad \text{dollars}}$$

(b)

$$t = \ln(5000/3000)/0.045$$

$$\boxed{11.35 \quad \text{years}}$$

(c)

$$A(8) - A(4) = 3000(e^{0.36} - e^{0.18})$$

$$\boxed{708.34 \quad \text{dollars}}$$

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Question 10

A certain species of insect is introduced to a new habitat. Its population is modeled by the exponential function:

$$P(t) = 500 \cdot e^{0.6t}$$

where $P(t)$ is the population size after t weeks.

(a) What will the population be after 5 weeks?

(b) After how many weeks will the population exceed 10,000?

(c) How fast is the population growing at $t = 5$ weeks?

Round all answers to two decimal places.

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Question 10 - Solution

Use the exact exponential model and keep full precision until the final rounding.

(a)

$$P(5) = 500e^3$$

$$\boxed{10042.77 \quad \text{insects}}$$

(b)

$$t_* = \ln(10000/500)/0.6$$

$$\boxed{4.99 \quad \text{weeks}}$$

(c)

$$P'(t) = 300e^{0.6t}, \quad P'(5) = 300e^3$$

$$\boxed{6025.66 \quad \text{insects/week}}$$

The population equals 10000 at $t = t_*$ and exceeds it precisely when $t > t_*$. The rounded threshold is not an exact inequality boundary.

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Question 1

Solve the equation for x, showing all steps:

$$\log_3(x+5) + \log_3(x-2) = 2$$

(a) Solve the logarithmic equation algebraically.

(b) State any restrictions on the domain.

(c) Check for extraneous solutions.

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Question 1 - Solution

(a) Combine the logs using log rules:

$$\log_3(x+5)+\log_3(x-2)=\log_3[(x+5)(x-2)]=2 \Rightarrow \log_3(x^2+3x-10)=2$$

Rewrite in exponential form:

$$x^2+3x-10=3^2=9 \Rightarrow x^2+3x-10=9 \Rightarrow x^2+3x-19=0$$

Solve with quadratic formula:

$$x=\frac{-3\pm\sqrt{3^2-4(1)(-19)}}{2(1)}=\frac{-3\pm\sqrt{9+76}}{2}=\frac{-3\pm\sqrt{85}}{2}$$

(b) Domain restrictions:
Since $\log_3(x+5)$ and $\log_3(x-2)$ are defined only if:

$$x+5>0 \Rightarrow x>-5, \quad x-2>0 \Rightarrow x>2 \Rightarrow \text{Overall domain: } x>2$$

(c) Check solutions:

$$\sqrt{85}\approx 9.22 \Rightarrow x_1=\frac{-3+9.22}{2}\approx 3.11, \quad x_2=\frac{-3-9.22}{2}\approx -6.11$$

Only $x_1\approx 3.11$ satisfies $x>2$. So:

$$\boxed{x=\frac{-3+\sqrt{85}}{2}} \quad \text{(valid solution)}$$

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Question 2

A scientist is studying the pH level of a solution, which is defined by the formula:

$$\text{pH} = -\log_{10}[H^+]$$

where $[H^+]$ is the concentration of hydrogen ions in moles per liter.

(a) If a solution has a hydrogen ion concentration of $[H^+] = 3.2 \times 10^{-5}$, find its pH.

(b) If another solution has a pH of 4.6, what is its hydrogen ion concentration?

(c) Which solution is more acidic, and by how many times is it more acidic?

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Question 2 - Solution

(a) Compute pH:

$$\text{pH} = -\log_{10}(3.2 \times 10^{-5}) = -[\log_{10}(3.2) + \log_{10}(10^{-5})]$$

$$= -[\log_{10}(3.2) - 5] \approx -[0.5051 - 5] = -(-4.4949) = \boxed{4.49}$$

(b) Compute hydrogen ion concentration from pH:

$$\text{pH} = -\log_{10}[H^+] \Rightarrow \log_{10}[H^+] = -4.6 \Rightarrow [H^+] = 10^{-4.6} \approx \boxed{2.51 \times 10^{-5}}$$

(c) Compare acidity:
Recall: lower pH = more acidic.
The first solution has pH ≈ 4.49, second has pH 4.6, so first solution is more acidic.
To find how many times more acidic:

$$\frac{[H^+]_{\text{first}}}{[H^+]_{\text{second}}} = \frac{3.2 \times 10^{-5}}{2.51 \times 10^{-5}} \approx \boxed{1.27 \text{ times more acidic}}$$

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Question 3

Consider the function:

$$f(x) = \log_2(4x^2 - 12x)$$

(a) Determine the domain of the function.

(b) Solve the equation $f(x) = 3$.

(c) State any restrictions or extraneous solutions.

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Question 3 - Solution

(a) Domain:

We must have:

$$4x^2 - 12x > 0 \Rightarrow 4x(x-3) > 0$$

Using a sign chart or test intervals:
- Critical points: $x = 0$, $x = 3$ - Positive intervals: $x < 0$ or $x > 3$

$$\text{Domain: } \boxed{(-\infty, 0) \cup (3, \infty)}$$

(b) Solve $f(x) = 3$:

$$\log_2(4x^2 - 12x) = 3 \Rightarrow 4x^2 - 12x = 2^3 = 8 \Rightarrow 4x^2 - 12x - 8 = 0$$

Divide by 4:

$$x^2 - 3x - 2 = 0 \Rightarrow x = \frac{3 \pm \sqrt{(-3)^2 + 4 \cdot 2}}{2} = \frac{3 \pm \sqrt{9+8}}{2} = \frac{3 \pm \sqrt{17}}{2}$$

$$\boxed{x = \frac{3 \pm \sqrt{17}}{2}}$$

(c) Check for extraneous solutions:

Approximate roots:

$$\sqrt{17} \approx 4.12 \Rightarrow x \approx \frac{3 \pm 4.12}{2} \Rightarrow x_1 \approx 3.56, \quad x_2 \approx -0.56$$

From domain: $x < 0$ or $x > 3$, so both are valid.

$$\boxed{\text{No extraneous solutions}}$$

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Question 4

Let $f(x) = \ln(x^2 - 6x + 10)$.

(a) Determine the domain of $f(x)$.

(b) Find the exact value of $f(3)$.

(c) Solve the equation $\ln(x^2 - 6x + 10) = 0$.

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Question 4 - Solution

(a) Domain:

We require the argument of the logarithm to be positive:

$$x^2 - 6x + 10 > 0$$

Consider the quadratic:

$$x^2 - 6x + 10 = (x-3)^2 + 1$$

Since $(x-3)^2 \geq 0$, we know:

$$(x-3)^2 + 1 \geq 1 > 0 \quad \text{for all real } x$$

$$\boxed{\text{Domain: } (-\infty, \infty)}$$

(b) Evaluate $f(3)$:

$$f(3) = \ln(3^2 - 6 \cdot 3 + 10) = \ln(9 - 18 + 10) = \ln(1) = \boxed{0}$$

(c) Solve $\ln(x^2 - 6x + 10) = 0$:

$$\ln(x^2 - 6x + 10) = 0 \Rightarrow x^2 - 6x + 10 = e^0 = 1 \Rightarrow x^2 - 6x + 9 = 0 \Rightarrow (x-3)^2 = 0 \Rightarrow \boxed{x = 3}$$

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Question 5

Let $f(x) = \log_5(x-1) + \log_5(x+3)$.

(a) Determine the domain of $f(x)$.

(b) Simplify the expression using logarithmic properties.

(c) Solve the equation $\log_5(x-1) + \log_5(x+3) = 2$.

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Question 5 - Solution

(a) Domain:

We need both logarithmic expressions to be defined:

$$x - 1 > 0 \Rightarrow x > 1, \quad x + 3 > 0 \Rightarrow x > -3$$

The more restrictive condition is $x > 1$, so:

$$\boxed{\text{Domain: } (1, \infty)}$$

(b) Use the product rule for logarithms:

$$f(x) = \log_5[(x-1)(x+3)] = \log_5(x^2 + 2x - 3)$$

(c) Solve the equation:

$$\log_5(x^2 + 2x - 3) = 2 \Rightarrow x^2 + 2x - 3 = 5^2 = 25 \Rightarrow x^2 + 2x - 28 = 0$$

Use the quadratic formula:

$$x = \frac{-2 \pm \sqrt{(2)^2 + 4 \cdot 28}}{2} = \frac{-2 \pm \sqrt{4 + 112}}{2} = \frac{-2 \pm \sqrt{116}}{2}$$

$$= \frac{-2 \pm 2\sqrt{29}}{2} = -1 \pm \sqrt{29}$$

Check for extraneous solutions:

$$-1 + \sqrt{29} \approx -1 + 5.385 \approx 4.385 \text{ (valid)}, \quad -1 - \sqrt{29} < 0 \text{ (invalid)}$$

Only the positive root satisfies $x > 1$, so:

$$\boxed{x = -1 + \sqrt{29}}$$

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Question 6

The intensity I of an earthquake is measured on the Richter scale using the formula:

$$R = \log_{10}\left(\frac{I}{I_0}\right)$$

where R is the Richter magnitude and $I_0 = 10^{-4}$ is the reference intensity.

(a) If an earthquake has intensity $I = 10^{-1.5}$, what is its magnitude R?

(b) If an earthquake has a Richter magnitude of $R = 6.2$, what is its intensity I?

(c) How many times more intense is an earthquake of magnitude 7.2 compared to one of magnitude 5.2?

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Question 6 - Solution

(a) Compute the magnitude R:

$$R = \log_{10}\left(\frac{10^{-1.5}}{10^{-4}}\right) = \log_{10}(10^{2.5}) = \boxed{2.5}$$

(b) Find the intensity for $R = 6.2$:

$$R = \log_{10}\left(\frac{I}{10^{-4}}\right) \Rightarrow 10^R = \frac{I}{10^{-4}} \Rightarrow I = 10^R \cdot 10^{-4}$$

$$I = 10^{6.2-4} = 10^{2.2} \approx \boxed{158.49}$$

(c) Compare intensities of magnitudes 7.2 and 5.2:
Let I_1 and I_2 be the intensities corresponding to magnitudes 7.2 and 5.2, respectively.

$$\log_{10}\left(\frac{I_1}{I_0}\right) = 7.2 \Rightarrow I_1 = 10^{7.2} \cdot I_0, \quad \log_{10}\left(\frac{I_2}{I_0}\right) = 5.2 \Rightarrow I_2 = 10^{5.2} \cdot I_0$$

$$\frac{I_1}{I_2} = \frac{10^{7.2} \cdot I_0}{10^{5.2} \cdot I_0} = 10^{7.2-5.2} = 10^2 = \boxed{100}$$

So the magnitude 7.2 earthquake is **100 times more intense** than the 5.2 earthquake.

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Question 7

Solve the following logarithmic equation:

$$2\log_4(x+1) - \log_4(x-3) = 1$$

(a) Solve the equation algebraically.

(b) State any domain restrictions and eliminate extraneous solutions.

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Question 7 - Solution

(a) Combine the logarithms:

Use the power rule:

$$\log_4(x+1)^2 - \log_4(x-3) = \log_4\left(\frac{(x+1)^2}{x-3}\right)$$

So the equation becomes:

$$\log_4\left(\frac{(x+1)^2}{x-3}\right) = 1 \Rightarrow \frac{(x+1)^2}{x-3} = 4^1 = 4$$

Multiply both sides by $x-3$:

$$(x+1)^2 = 4(x-3) \Rightarrow x^2 + 2x + 1 = 4x - 12 \Rightarrow x^2 - 2x + 13 = 0$$

Solve with the quadratic formula:

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(13)}}{2} = \frac{2 \pm \sqrt{4-52}}{2} = \frac{2 \pm \sqrt{-48}}{2} = \frac{2 \pm 4i\sqrt{3}}{2} = 1 \pm 2i\sqrt{3}$$

(b) Domain restrictions:

$x+1 > 0 \Rightarrow x > -1$, $x-3 > 0 \Rightarrow x > 3$

So domain is: $\boxed{x > 3}$

The resulting quadratic has no real roots, so the original real logarithmic equation has no solution.

$$\boxed{\text{No real solution.}}$$

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Question 8

Let $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$. Suppose that

$$f(2) = 1 \qquad \text{and} \qquad f(5) \approx 2.32193.$$

(a) Estimate $f(10)$ using logarithm properties.

(b) Estimate $f(5/2)$ using logarithm properties.

(c) Explain the properties used.

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Question 8 - Solution

The product and quotient laws hold for positive arguments:

$$f(10) = f(2) + f(5) \approx 1 + 2.32193 = \boxed{3.32193},$$

$$f(5/2) = f(5) - f(2) \approx 2.32193 - 1 = \boxed{1.32193}.$$

The datum $\log_b 2 = 1$ fixes $b = 2$, so the supplied approximation to $\log_b 5$ is consistent with the same base.

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Question 9

A bacteria culture grows according to the function

$$N(t) = N_0 e^{kt}$$

where N_0 is the initial population, k is a constant, and t is the time in hours.

Suppose a culture starts with 500 bacteria and grows to 2000 bacteria in 6 hours.

(a) Find the growth rate k.

(b) Use your result to estimate the population after 10 hours.

(c) After how many hours will the population reach 10,000?

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Question 9 - Solution

Growth rate. $2000 = 500e^{6k}$ gives

$$\boxed{k = \frac{\ln 4}{6} \approx 0.231049 \text{ hr}^{-1}}.$$

Population after ten hours.

$$N(10) = 500 \cdot 4^{10/6} \approx 5039.68,$$

so the estimated whole-number population is $\boxed{5040}$.

Time to reach 10000**.**

$$500e^{kt} = 10000 \quad \Longrightarrow \quad \boxed{t = \frac{6 \ln 20}{\ln 4} \approx 12.97 \text{ hours}}.$$

Substitution into the exact model verifies both target populations.

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Question 10

Solve the logarithmic equation:

$$\ln(x+2) + \ln(x-1) = \ln(3x)$$

(a) Solve the equation algebraically.

(b) State the domain and eliminate any extraneous solutions.

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Question 10 - Solution

(a) Combine the logarithmic expressions:

Use the product rule:

$$\ln[(x+2)(x-1)] = \ln(3x) \Rightarrow (x+2)(x-1) = 3x \Rightarrow x^2 + x - 2 = 3x \Rightarrow x^2 - 2x - 2 = 0$$

Use the quadratic formula:

$$x = \frac{2 \pm \sqrt{(-2)^2 + 4(1)(2)}}{2} = \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = \boxed{1 \pm \sqrt{3}}$$

(b) Domain restrictions:

We need all logarithms to have positive arguments:

$$x + 2 > 0 \Rightarrow x > -2, \quad x - 1 > 0 \Rightarrow x > 1, \quad 3x > 0 \Rightarrow x > 0$$

Combining all: $\boxed{x > 1}$

Now test each solution: - $x = 1 + \sqrt{3} \approx 2.73 > 1 \rightarrow$ valid - $x = 1 - \sqrt{3} \approx -0.73 < 1 \rightarrow$ invalid (violates domain)

Final answer: $\boxed{x = 1 + \sqrt{3}}$

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Question 1

Solve the following equation for x:

$$3^{2x-1} = 7^{x+2}$$

Instructions: Solve algebraically using logarithms and express your answer in exact form. Approximate the result to 3 decimal places.

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Question 1 - Solution

Taking natural logarithms (both sides are positive) gives

$$(2x-1)\ln 3 = (x+2)\ln 7.$$

Collect the terms in x and divide by their nonzero coefficient:

$$\boxed{x = \frac{\ln 3 + 2\ln 7}{2\ln 3 - \ln 7} \approx 19.857}.$$

The logarithm is one-to-one, so solving this equivalent linear equation introduces no extraneous solutions. The exact expression verifies the original equation.

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Question 2

Solve the following equation:

$$e^{2x} - 7e^x + 10 = 0$$

(a) Solve algebraically for all real solutions.

(b) Check each solution for validity.

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Question 2 - Solution

(a) Let $y = e^x$. Then the equation becomes:

$$y^2 - 7y + 10 = 0 \Rightarrow (y-5)(y-2) = 0 \Rightarrow y = 5 \quad \text{or} \quad y = 2$$

Now solve for x:

$$e^x = 5 \Rightarrow x = \ln(5)$$

$$e^x = 2 \Rightarrow x = \ln(2)$$

(b) Check domain/validity:

Both $\ln(2)$ and $\ln(5)$ are defined for positive inputs, and since both values came from valid solutions for $y = e^x > 0$, no extraneous solutions.

Final Answer:

$$\boxed{x = \ln(2) \quad \text{or} \quad x = \ln(5)}$$

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Question 3

Solve the following equation:

$$\log_3(x^2 - 4x) = 1 + \log_3(x - 2)$$

(a) Solve the equation algebraically.

(b) Identify any extraneous solutions.

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Question 3 - Solution

(a) Combine the logarithms:

Move the constant:

$$\log_3(x^2 - 4x) - \log_3(x - 2) = 1$$

Apply the quotient rule:

$$\log_3\left(\frac{x^2 - 4x}{x - 2}\right) = 1$$

Factor the numerator:

$$x^2 - 4x = x(x - 4) \Rightarrow \frac{x(x - 4)}{x - 2}$$

So:

$$\log_3\left(\frac{x(x - 4)}{x - 2}\right) = 1 \Rightarrow \frac{x(x - 4)}{x - 2} = 3^1 = 3$$

Multiply both sides:

$$x(x - 4) = 3(x - 2) \Rightarrow x^2 - 4x = 3x - 6 \Rightarrow x^2 - 7x + 6 = 0 \Rightarrow (x - 6)(x - 1) = 0 \Rightarrow x = 6,\ 1$$

(b) Check for extraneous solutions:

Logarithms are only defined for positive arguments.
Check domain: $x^2 - 4x > 0 \Rightarrow x(x - 4) > 0 \Rightarrow x < 0$ or $x > 4$, $x - 2 > 0 \Rightarrow x > 2$
Only values that satisfy both: $x > 4$
So: $x = 6 \rightarrow$ valid , $x = 1 \rightarrow$ not valid (fails both domain conditions)
Final Answer:

$$\boxed{x = 6}$$

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Question 4

Solve for x in the equation:

$$2^{3x+1} = 5^{x-2}$$

Give your answer in exact form using logarithms, and approximate the result to three decimal places.

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Question 4 - Solution

Taking natural logarithms (both sides are positive) gives

$$(3x+1)\ln 2 = (x-2)\ln 5.$$

Collect the terms in x and divide by their nonzero coefficient:

$$\boxed{x = \frac{-\ln 2 - 2\ln 5}{3\ln 2 - \ln 5} \approx -8.323}.$$

The logarithm is one-to-one, so solving this equivalent linear equation introduces no extraneous solutions. The exact expression verifies the original equation.

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Question 5

Solve the equation:

$$\ln(x-1)+\ln(x+1)=\ln(6)$$

(a) Solve for x algebraically.

(b) Identify and exclude any extraneous solutions.

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Question 5 - Solution

(a) Combine the logarithmic expressions using properties:

$$\ln(x-1)+\ln(x+1)=\ln[(x-1)(x+1)]=\ln(x^2-1)$$

So the equation becomes:

$$\ln(x^2-1)=\ln(6)\Rightarrow x^2-1=6\Rightarrow x^2=7\Rightarrow x=\pm\sqrt{7}$$

(b) Domain Check:
We must have:

$$x-1>0\Rightarrow x>1 \quad \text{and} \quad x+1>0\Rightarrow x>-1\Rightarrow \text{Combined: } x>1$$

So $x=\sqrt{7}\approx 2.6458$ is valid, but $x=-\sqrt{7}\approx -2.6458$ is not.

Final Answer: $\boxed{x=\sqrt{7}}$

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Question 6

Solve the equation:

$$4e^{2x} - 12e^{x} + 5 = 0$$

(a) Solve algebraically for all real solutions.

(b) Provide both exact and approximate answers.

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Question 6 - Solution

(a) Let $y = e^x$. Then the equation becomes:

$$4y^2 - 12y + 5 = 0$$

Use the quadratic formula:

$$y = \frac{12 \pm \sqrt{(-12)^2 - 4(4)(5)}}{2(4)} = \frac{12 \pm \sqrt{144 - 80}}{8} = \frac{12 \pm \sqrt{64}}{8} = \frac{12 \pm 8}{8}$$

So:

$$y = \frac{20}{8} = 2.5 \quad \text{or} \quad y = \frac{4}{8} = 0.5$$

Now solve for x:

$$e^x = 2.5 \Rightarrow x = \ln(2.5), \quad e^x = 0.5 \Rightarrow x = \ln(0.5)$$

(b) Final Answers:
Exact:

$$\boxed{x = \ln(2.5) \quad \text{or} \quad x = \ln(0.5)}$$

Approximate:

$$x \approx \ln(2.5) \approx 0.9163, \quad x \approx \ln(0.5) \approx -0.6931$$

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Question 7

Solve for all real solutions:

$$\log_2(x+3) + \log_2(x-1) = 3$$

Instructions: Solve algebraically and check for extraneous solutions.

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Question 7 - Solution

We are given:

$$\log_2(x+3) + \log_2(x-1) = 3$$

Step 1: Combine the logs using the product rule:

$$\log_2[(x+3)(x-1)] = 3$$

Step 2: Convert from logarithmic to exponential form:

$$(x+3)(x-1) = 2^3 = 8 \Rightarrow x^2 + 2x - 3 = 8 \Rightarrow x^2 + 2x - 11 = 0$$

Step 3: Solve the quadratic:

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-11)}}{2(1)} = \frac{-2 \pm \sqrt{4+44}}{2} = \frac{-2 \pm \sqrt{48}}{2} = \frac{-2 \pm 4\sqrt{3}}{2} = \boxed{-1 \pm 2\sqrt{3}}$$

Step 4: Domain check:
We need:

$$x + 3 > 0 \Rightarrow x > -3, \quad x - 1 > 0 \Rightarrow x > 1 \Rightarrow \text{So } x > 1$$

Now test the two roots: $x = -1+2\sqrt{3} \approx -1+3.464 \approx 2.464 \rightarrow$ valid , $x = -1-2\sqrt{3} \approx -1-3.464 \approx -4.464 \rightarrow$ invalid

Final Answer:

$$\boxed{x = -1 + 2\sqrt{3}}$$

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Question 8

Solve the equation:

$$5^{x+1} = 2^{3x-2}$$

Instructions: Solve the equation algebraically using logarithms. Provide both an exact expression and an approximate value rounded to 3 decimal places.

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Question 8 - Solution

Taking natural logarithms (both sides are positive) gives

$$(x+1)\ln 5 = (3x-2)\ln 2.$$

Collect the terms in x and divide by their nonzero coefficient:

$$\boxed{x = \frac{-\ln 5 - 2\ln 2}{\ln 5 - 3\ln 2} \approx 6.374}.$$

The logarithm is one-to-one, so solving this equivalent linear equation introduces no extraneous solutions. The exact expression verifies the original equation.

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Question 9

Solve for all real solutions:

$$\log(x^2 + 1) = 1 + \log(x - 2)$$

Instructions: Solve algebraically and check for extraneous solutions.

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Question 9 - Solution

We are given:

$$\log(x^2+1) = 1 + \log(x-2)$$

Step 1: Use properties of logarithms.
Subtract $\log(x-2)$ from both sides:

$$\log(x^2+1) - \log(x-2) = 1$$

Apply the quotient rule:

$$\log\left(\frac{x^2+1}{x-2}\right) = 1$$

Step 2: Convert to exponential form.
Recall that $\log_b(A) = C \Rightarrow A = b^C$. Since no base is written, it is base 10:

$$\frac{x^2+1}{x-2} = 10^1 = 10$$

$$x^2+1 = 10(x-2) \Rightarrow x^2+1 = 10x-20 \Rightarrow x^2-10x+21 = 0 \Rightarrow (x-3)(x-7) = 0 \Rightarrow x = 3 \text{ or } x = 7$$

Step 3: Domain check.
We need: $x-2 > 0 \Rightarrow x > 2$
Both $x = 3$ and $x = 7$ satisfy the domain.
Final Answer:

$$\boxed{x = 3 \quad \text{or} \quad x = 7}$$

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Question 10

Solve the equation:

$$2^{x+1} + 2^x = 48$$

Instructions: Solve the equation algebraically. Express your answer in exact form and give an approximation rounded to three decimal places.

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Question 10 - Solution

We are given:

$$2^{x+1} + 2^x = 48$$

Step 1: Use the property $2^{x+1} = 2 \cdot 2^x$:

$$2 \cdot 2^x + 2^x = 48 \Rightarrow 3 \cdot 2^x = 48 \Rightarrow 2^x = \frac{48}{3} = 16$$

Step 2: Solve for x:

$$2^x = 16 = 2^4 \Rightarrow \boxed{x = 4}$$

Exact answer: $\boxed{x = 4}$
Approximate answer: $\boxed{x = 4.000}$

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Question 1

Sketch the graph of the function below. Identify the domain, range, intercepts, asymptotes, and any transformations applied to a basic parent function.

$$f(x) = -\frac{1}{x-2} + 3$$

Instructions: Analyze the function using transformations of the reciprocal function $f(x) = \frac{1}{x}$.

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Question 1 - Solution

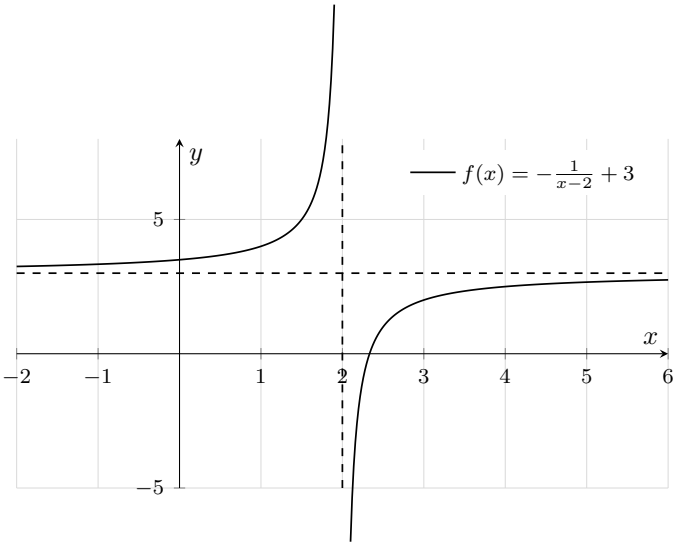
We are given:

$$f(x) = -\frac{1}{x-2} + 3$$

Parent Function: $\frac{1}{x}$
Transformations:

- Horizontal shift right by 2
- Reflection over x-axis (due to the negative sign)
- Vertical shift up by 3

Domain: $(-\infty, 2) \cup (2, \infty)$
Range: $(-\infty, 3) \cup (3, \infty)$
Vertical Asymptote: $x = 2$
Horizontal Asymptote: $y = 3$
Y-intercept: $f(0) = \frac{7}{2}$
X-intercept: $f(x) = 0 \Rightarrow x = \frac{7}{3}$



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Question 2

Sketch the graph of the function:

$$f(x) = |x + 1| - 2$$

Instructions: Identify the vertex, axis of symmetry, domain, range, and x- and y-intercepts. Then sketch the graph.

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Question 2 - Solution

We are given:

$$f(x) = |x + 1| - 2$$

Step 1: Transform the parent function $f(x) = |x|$

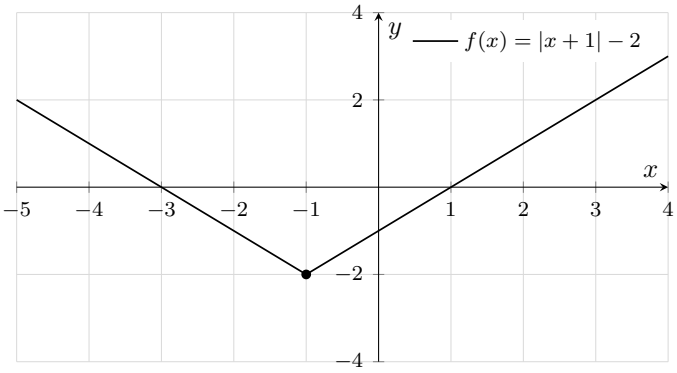
- Shift left by 1: $x \to x + 1$
- Shift down by 2: $f(x) \to f(x) - 2$

Vertex: The vertex of $f(x) = |x|$ is at $(0, 0)$, so for this function it's at $(-1, -2)$
Axis of Symmetry: $x = -1$
Domain: $(-\infty, \infty)$
Range: $[-2, \infty)$
X-intercepts:

$$|x + 1| - 2 = 0 \Rightarrow |x + 1| = 2 \Rightarrow x + 1 = \pm 2 \Rightarrow x = 1, -3$$

Y-intercept:

$$f(0) = |0 + 1| - 2 = 1 - 2 = -1$$



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Question 3

Sketch the graph of the piecewise function:

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0 \end{cases}$$

Instructions: Clearly identify continuity, domain, range, intercepts, and sketch the graph.

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Question 3 - Solution

We are given a piecewise function:

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0 \end{cases}$$

Step 1: Domain
Both pieces are defined on their respective intervals, so:

$$\boxed{\text{Domain: } (-\infty, \infty)}$$

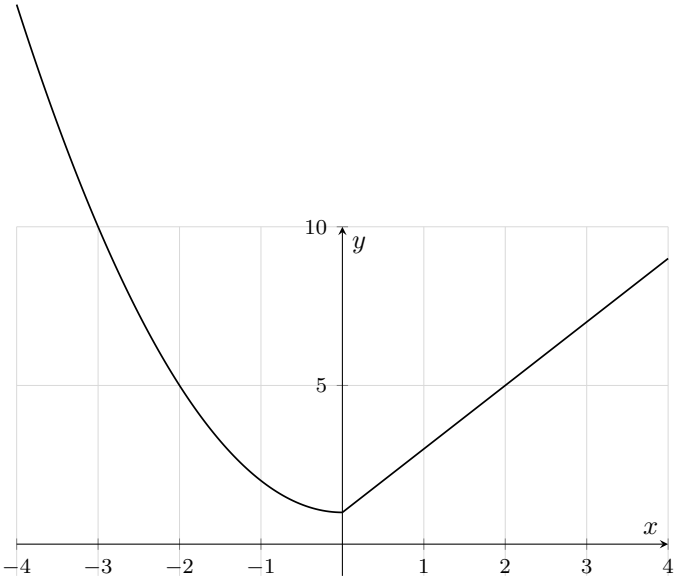
Step 2: Range
For $x < 0$: $f(x) = x^2 + 1 \Rightarrow f(x) > 1$, For $x \geq 0$: $f(x) = 2x + 1 \Rightarrow f(x) \geq 1$
So the function reaches its minimum at $f(0) = 1$, and increases in both directions:

$$\boxed{\text{Range: } [1, \infty)}$$

Step 3: Continuity at $x = 0$

$$\lim_{x \to 0^-} f(x) = 0^2 + 1 = 1, \quad \lim_{x \to 0^+} f(x) = 2(0) + 1 = 1 \Rightarrow \text{Function is continuous at } x = 0$$

Step 4: Intercepts
Y-intercept: $f(0) = 1$ **X-intercept:** Solve $f(x) = 0$: , $x^2 + 1 = 0 \Rightarrow$ no real solution , $2x + 1 = 0 \Rightarrow x = -\frac{1}{2} \notin [0, \infty) \Rightarrow$ no x-intercept
Conclusion: Domain: $(-\infty, \infty)$, Range: $[1, \infty)$, No x-intercepts, y-intercept at $(0, 1)$, Continuous everywhere



—— $f(x) = \begin{cases} x^2 + 1 & x < 0 \\ 2x + 1 & x \geq 0 \end{cases}$

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Question 4

Sketch the graph of the function:

$$f(x) = \sqrt{x+2} - 1$$

Instructions: Identify the domain, range, intercepts, and transformations applied to the parent square root function. Then provide a sketch.

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Question 4 - Solution

We are given:

$$f(x) = \sqrt{x+2} - 1$$

Step 1: Parent function: $f(x) = \sqrt{x}$
Transformations:

- Horizontal shift left by 2: $x \to x + 2$
- Vertical shift down by 1

Step 2: Domain:

$$x + 2 \geq 0 \Rightarrow x \geq -2 \Rightarrow \boxed{\text{Domain: } [-2, \infty)}$$

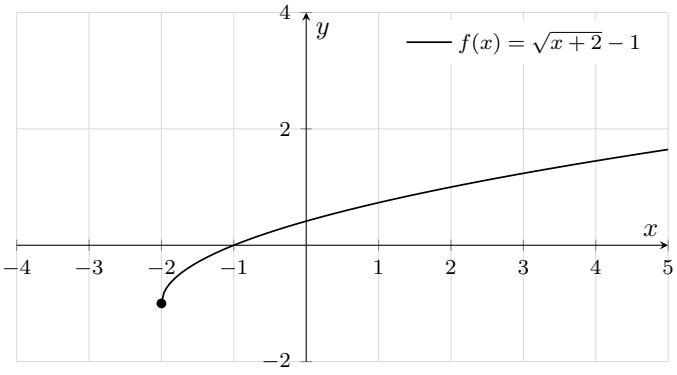
Step 3: Range: Since square root outputs ≥ 0, and we subtract 1:

$$\boxed{\text{Range: } [-1, \infty)}$$

Step 4: Intercepts

- **Y-intercept:** $f(0) = \sqrt{2} - 1 \approx 0.414$
- **X-intercept:** Set $f(x) = 0 \Rightarrow \sqrt{x+2} = 1 \Rightarrow x + 2 = 1 \Rightarrow x = -1$

Key Point: The graph starts at $(-2, -1)$



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Question 5

Sketch the graph of the function:

$$f(x) = -|x - 3| + 2$$

Instructions: Identify the vertex, axis of symmetry, domain, range, and intercepts. Then sketch the graph of the function using transformations.

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Question 5 - Solution

We are given:

$$f(x) = -|x-3| + 2$$

Step 1: Parent function: $f(x) = |x|$
Transformations:

- Shift right by 3: $x \to x - 3$
- Reflect over x-axis: negative sign
- Shift up by 2

Step 2: Vertex:

$$\text{Vertex at } (3, 2)$$

Step 3: Axis of symmetry:

$$x = 3$$

Step 4: Domain and Range

$$\text{Domain: } (-\infty, \infty)$$

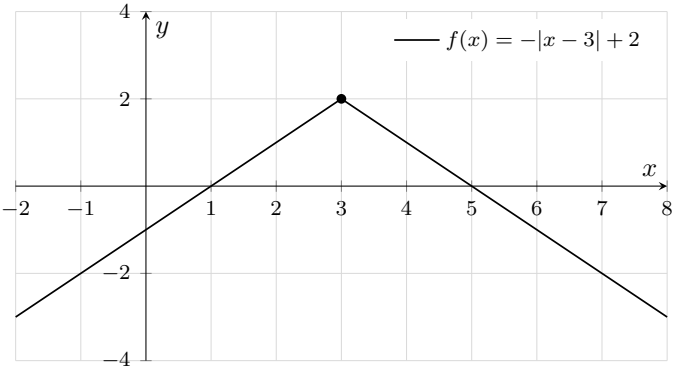
$$\text{Range: } (-\infty, 2]$$

Step 5: Intercepts
- **Y-intercept:** Set $x = 0 \Rightarrow f(0) = -|0-3| + 2 = -3 + 2 = -1$ - **X-intercepts:** Solve $f(x) = 0$

$$-|x-3| + 2 = 0 \Rightarrow |x-3| = 2 \Rightarrow x - 3 = \pm 2 \Rightarrow x = 1,\ 5$$

Final Answer Summary:

- Vertex: $(3, 2)$
- Axis of symmetry: $x = 3$
- Domain: $(-\infty, \infty)$
- Range: $(-\infty, 2]$
- X-intercepts: $x = 1,\ 5$
- Y-intercept: $y = -1$



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Question 6

Sketch the graph of the rational function:

$$f(x) = \frac{x^2 - 1}{x^2 - 4}$$

Instructions: Identify all intercepts, vertical and horizontal asymptotes, domain, and any removable discontinuities. Then sketch the graph based on your analysis.

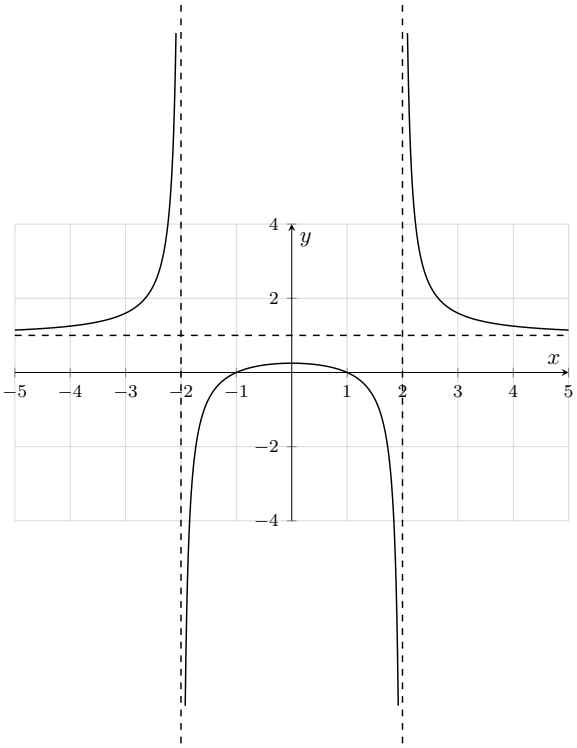
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Question 6 - Solution

We are given:

$$f(x) = \frac{x^2 - 1}{x^2 - 4} = \frac{(x-1)(x+1)}{(x-2)(x+2)}$$

Domain: $x \neq \pm 2 \quad \Rightarrow \boxed{(-\infty, -2) \cup (-2, 2) \cup (2, \infty)}$
Vertical Asymptotes: $x = -2$ and $x = 2$
Horizontal Asymptote: Degrees of numerator and denominator are equal: $y = \frac{1}{1} = \boxed{y = 1}$
X-intercepts: Set numerator $= 0$: $x = \pm 1$
Y-intercept: $f(0) = \frac{-1}{-4} = \frac{1}{4}$



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Question 7

Sketch the graph of the function:

$$f(x) = \frac{1}{x^2 + 1}$$

Instructions: Identify the domain, range, intercepts, end behavior, and any symmetry of the function. Then sketch the graph.

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Question 7 - Solution

We are given:

$$f(x) = \frac{1}{x^2 + 1}$$

Step 1: Domain
Since $x^2 + 1 > 0$ for all real x, the domain is:

$$\boxed{(-\infty, \infty)}$$

Step 2: Y-intercept
Set $x = 0$:

$$f(0) = \frac{1}{0^2 + 1} = 1$$

Step 3: X-intercepts
Set $f(x) = 0$:

$$\frac{1}{x^2 + 1} = 0 \Rightarrow \text{no solution}$$

So, **no x-intercepts**.

Step 4: Range
Since the denominator is always ≥ 1, the maximum occurs at $x = 0$, and the function decreases on both sides. Therefore:

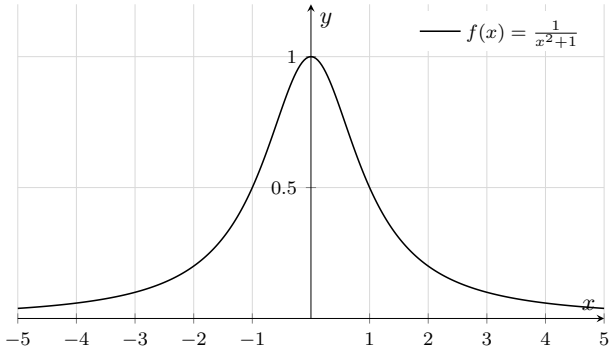
$$\boxed{\text{Range: } (0, 1]}$$

Step 5: Symmetry

$$f(-x) = \frac{1}{(-x)^2 + 1} = \frac{1}{x^2 + 1} = f(x)$$

So the function is **even**, symmetric about the y-axis.

Step 6: End Behavior
As $x \to \pm\infty$, $f(x) \to 0$



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Question 8

Sketch the graph of the piecewise function:

$$f(x) = \begin{cases} x+2 & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x \leq 2 \\ 4-x & \text{if } x > 2 \end{cases}$$

Instructions: Clearly indicate open and closed points on the graph. Identify points of discontinuity, if any.

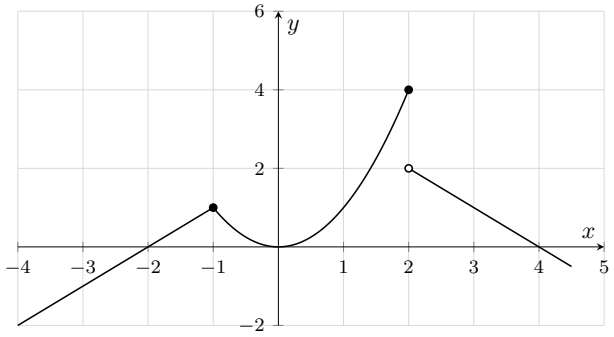
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Question 8 - Solution

We are given a piecewise function:

$$f(x) = \begin{cases} x+2 & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x \leq 2 \\ 4-x & \text{if } x > 2 \end{cases}$$

Step 1: Evaluate endpoints to check continuity.
From the left of $x = -1$, $\lim_{x \to -1^-} f(x) = -1 + 2 = 1$
At $x = -1$, $f(-1) = (-1)^2 = 1 \Rightarrow$ continuous
From the right of $x = 2$, $\lim_{x \to 2^+} f(x) = 4 - 2 = 2$
At $x = 2$, $f(2) = 2^2 = 4 \Rightarrow$ discontinuous at $x = 2$
Discontinuity: Jump discontinuity at $x = 2$



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Question 9

Sketch the graph of the function:

$$f(x) = \ln(x^2 + 1)$$

Instructions: Identify the domain, intercepts, end behavior, and any symmetry. Then sketch the graph.

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Question 9 - Solution

We are given:

$$f(x) = \ln(x^2 + 1)$$

Domain: Since $x^2 + 1 > 0$ for all real x,

$$\boxed{\text{Domain: } (-\infty, \infty)}$$

Y-intercept:

$$f(0) = \ln(0^2 + 1) = \ln(1) = 0$$

X-intercepts: Set $f(x) = 0$:

$$\ln(x^2 + 1) = 0 \Rightarrow x^2 + 1 = 1 \Rightarrow x^2 = 0 \Rightarrow x = 0$$

So the only intercept is at the origin.

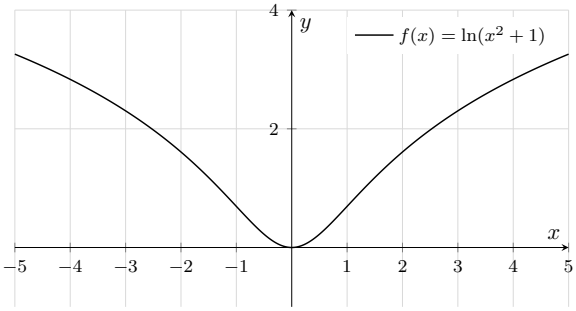
Symmetry:

$$f(-x) = \ln((-x)^2 + 1) = \ln(x^2 + 1) = f(x) \Rightarrow \text{even function}$$

End behavior: As $x \to \pm\infty$, $x^2 + 1 \to \infty \Rightarrow f(x) \to \infty$

Range: Minimum value is at $x = 0$: $\ln(1) = 0$, and it increases without bound.

$$\boxed{\text{Range: } [0, \infty)}$$



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Question 10

Sketch the graph of the function:

$$f(x) = \frac{x}{\sqrt{x^2 + 1}}$$

Instructions: Identify the domain, range, intercepts, asymptotic behavior, and any symmetry. Then sketch the graph clearly.

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Question 10 - Solution

We are given:

$$f(x) = \frac{x}{\sqrt{x^2+1}}$$

Step 1: Domain
The denominator is always positive and defined for all real x:

$$\boxed{\text{Domain: } (-\infty, \infty)}$$

Step 2: Intercepts
Y-intercept:

$$f(0) = \frac{0}{\sqrt{0^2+1}} = 0$$

X-intercept:

$$f(x) = 0 \Rightarrow \frac{x}{\sqrt{x^2+1}} = 0 \Rightarrow x = 0$$

Step 3: Symmetry
Check if the function is odd:

$$f(-x) = \frac{-x}{\sqrt{(-x)^2+1}} = \frac{-x}{\sqrt{x^2+1}} = -f(x)$$

So the function is **odd**, symmetric about the origin.
Step 4: End Behavior
As $x \to \pm\infty$:

$$f(x) = \frac{x}{\sqrt{x^2+1}} \to \pm 1$$

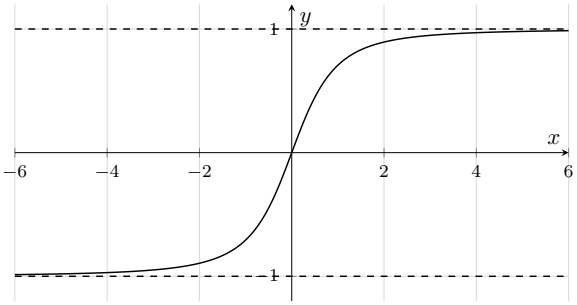
So horizontal asymptotes:

$$\boxed{y = 1} \quad \text{as } x \to \infty \quad \text{and} \quad \boxed{y = -1} \quad \text{as } x \to -\infty$$

Step 5: Range

$$\boxed{\text{Range: } (-1, 1)}$$

The function approaches 1 and -1 but never reaches them.



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Question 1

Let $f(x) = \sqrt{x+4}$. Find the slope of the secant line through the points where $x = 0$ and $x = 5$, and compare it with the slope of the tangent line at $x = 2$. Then, write the equation of the tangent line at $x = 2$.

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Question 1 - Solution

We are given:

$$f(x) = \sqrt{x+4}$$

Step 1: Slope of secant line from $x = 0$ to $x = 5$.

$$f(0) = \sqrt{4} = 2, \quad f(5) = \sqrt{9} = 3$$

$$\text{Secant slope} = \frac{3-2}{5-0} = \frac{1}{5}$$

Step 2: Derivative of $f(x)$.

$$f'(x) = \frac{1}{2\sqrt{x+4}}$$

$$f'(2) = \frac{1}{2\sqrt{6}} \approx 0.204$$

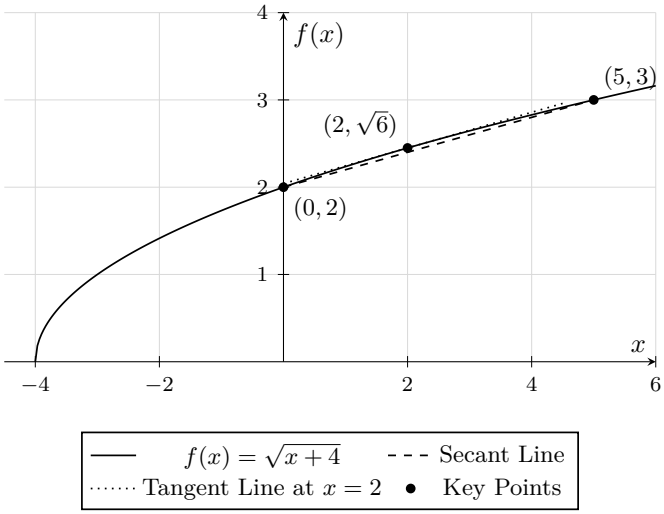
Step 3: Point at $x = 2$.

$$f(2) = \sqrt{6}$$

Step 4: Tangent line at $x = 2$:

$$y - \sqrt{6} = \frac{1}{2\sqrt{6}}(x-2)$$

Comparison: The slope of the tangent line at $x = 2$ is slightly greater than the secant slope over the interval, showing that the graph is concave down.



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Question 2

The displacement of a car traveling along a straight road is given by

$$s(t) = 10t - 0.5t^2$$

where s is in meters and t is in seconds.

(a) Find the average velocity of the car over the interval $[2, 6]$.

(b) Find the instantaneous velocity at $t = 2$.

(c) Sketch the displacement function and the secant and tangent lines described above.

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Question 2 - Solution

We are given:

$$s(t) = 10t - 0.5t^2$$

(a) Average velocity on $[2, 6]$:

$$s(2) = 10(2) - 0.5(4) = 20 - 2 = 18$$

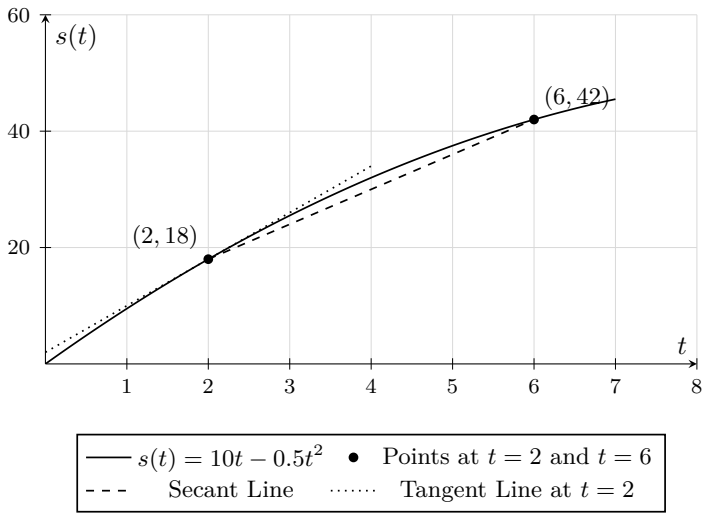
$$s(6) = 10(6) - 0.5(36) = 60 - 18 = 42$$

$$\text{Average velocity} = \frac{42 - 18}{6 - 2} = \frac{24}{4} = 6 \text{ m/s}$$

(b) Instantaneous velocity at $t = 2$:

$$s'(t) = 10 - t \quad \Rightarrow \quad s'(2) = 10 - 2 = 8 \text{ m/s}$$

(c) Interpretation: The average velocity between $t = 2$ and $t = 6$ is 6 m/s. At $t = 2$, the car is moving faster than average at 8 m/s. We now sketch the curve with the secant line between $t = 2$ and $t = 6$, and the tangent line at $t = 2$.



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Question 3

A population of bacteria in a lab experiment is modeled by the function

$$P(t) = 200e^{0.3t},$$

where t is the time in hours since the start of the experiment and $P(t)$ is the number of bacteria.

(a) Estimate the average rate of growth of the population from $t = 2$ to $t = 5$.

(b) Find the instantaneous rate of growth at $t = 2$.

(c) Sketch the population function along with the secant and tangent lines.

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Question 3 - Solution

We are given:

$$P(t) = 200e^{0.3t}$$

(a) Average rate of change on $[2, 5]$:

$$P(2) = 200e^{0.6} \approx 200(1.8221) = 364.42$$

$$P(5) = 200e^{1.5} \approx 200(4.4817) = 896.34$$

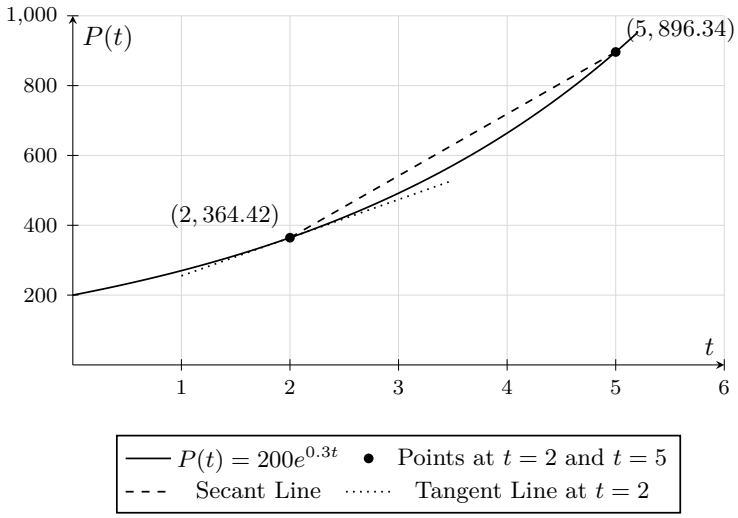
$$\text{Average rate} = \frac{896.34 - 364.42}{5 - 2} \approx \frac{531.92}{3} \approx 177.31 \text{ bacteria/hr}$$

(b) Instantaneous rate at $t = 2$:

$$P'(t) = 200 \cdot 0.3e^{0.3t} = 60e^{0.3t}$$

$$P'(2) = 60e^{0.6} \approx 60(1.8221) = 109.33 \text{ bacteria/hr}$$

(c) Interpretation: The population grows faster over the longer interval than it does exactly at $t = 2$, illustrating exponential acceleration.



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Question 4

A ball is thrown vertically upward from the ground with an initial velocity of $v_0 = 20$ m/s. Its height at time t seconds is modeled by the function

$$h(t) = 20t - 4.9t^2.$$

(a) Compute the average velocity over the interval $[1, 2]$.

(b) Find the instantaneous velocity at $t = 1$.

(c) Sketch the height function and both the secant and tangent lines involved.

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Question 4 - Solution

Given:

$$h(t) = 20t - 4.9t^2$$

(a) Average velocity on $[1, 2]$:

$$h(1) = 20(1) - 4.9(1)^2 = 20 - 4.9 = 15.1$$

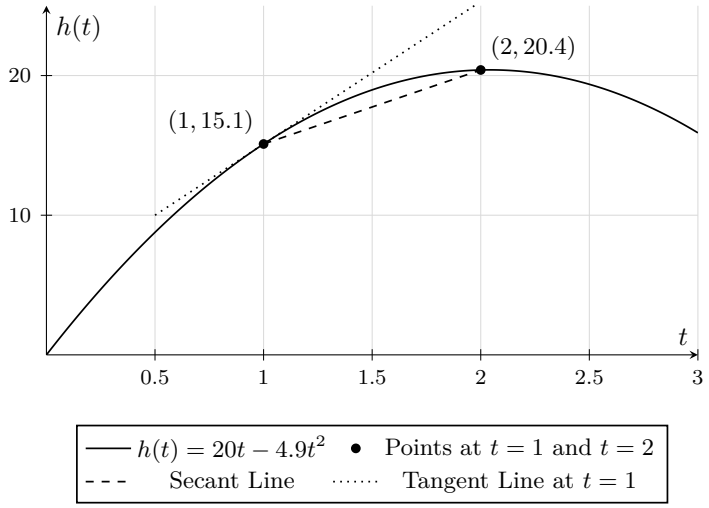
$$h(2) = 20(2) - 4.9(4) = 40 - 19.6 = 20.4$$

$$\text{Average velocity} = \frac{20.4 - 15.1}{2 - 1} = 5.3 \text{ m/s}$$

(b) Instantaneous velocity at $t = 1$:

$$h'(t) = 20 - 9.8t \quad \Rightarrow \quad h'(1) = 20 - 9.8 = 10.2 \text{ m/s}$$

(c) Interpretation: The ball's velocity at exactly $t = 1$ is greater than the average velocity over the interval, indicating it is still on the way up. Below is the sketch of the graph.



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Question 5

The position of a particle moving along a straight path is given by the function

$$s(t) = t^3 - 6t^2 + 9t,$$

where $s(t)$ is in meters and t is in seconds.

(a) Compute the average velocity of the particle on the interval $[1, 3]$.

(b) Find the instantaneous velocity at $t = 1$.

(c) Sketch the position function and include both the secant line over $[1, 3]$ and the tangent line at $t = 1$.

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Question 5 - Solution

Given:

$$s(t) = t^3 - 6t^2 + 9t$$

(a) Average velocity on $[1, 3]$:

$$s(1) = 1^3 - 6(1)^2 + 9(1) = 1 - 6 + 9 = 4$$

$$s(3) = 27 - 54 + 27 = 0$$

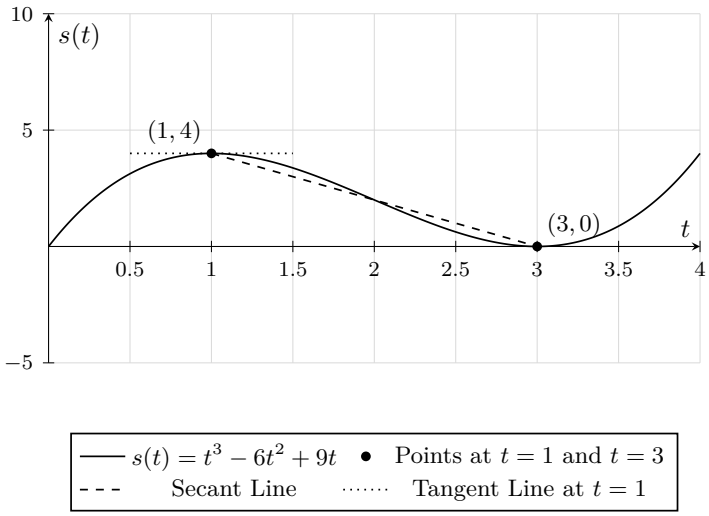
$$\text{Average velocity} = \frac{0 - 4}{3 - 1} = \frac{-4}{2} = -2 \text{ m/s}$$

(b) Instantaneous velocity at $t = 1$:

$$s'(t) = 3t^2 - 12t + 9$$

$$s'(1) = 3(1)^2 - 12(1) + 9 = 3 - 12 + 9 = 0 \text{ m/s}$$

(c) Interpretation: The particle is momentarily at rest at $t = 1$ (instantaneous velocity is 0), even though its average velocity between $t = 1$ and $t = 3$ is negative, indicating motion reversal.



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Question 6

The height h (in meters) of a rocket launched vertically is given by the function

$$h(t) = 100 + 40t - 5t^2,$$

where t is time in seconds after launch.

(a) Compute the average velocity of the rocket over the time interval $[2, 4]$.

(b) Find the instantaneous velocity at $t = 2$.

(c) Sketch the height function, the secant line over $[2, 4]$, and the tangent line at $t = 2$.

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Question 6 - Solution

Given:

$$h(t) = 100 + 40t - 5t^2$$

(a) Average velocity on $[2, 4]$:

$$h(2) = 100 + 80 - 20 = 160$$

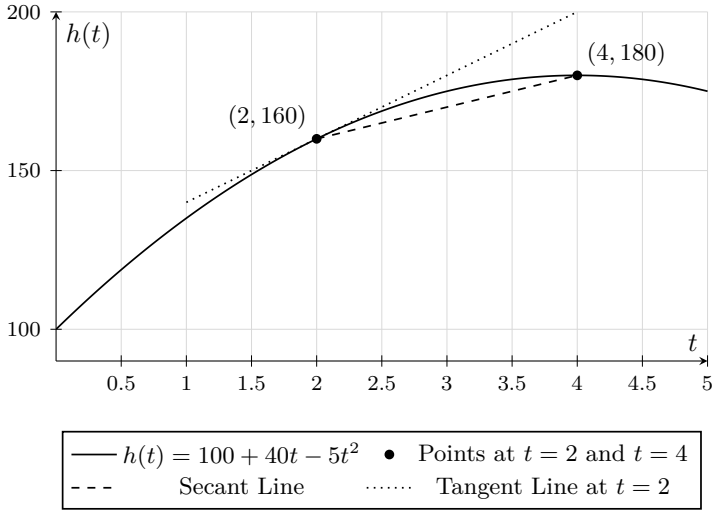
$$h(4) = 100 + 160 - 80 = 180$$

$$\text{Average velocity} = \frac{180 - 160}{4 - 2} = \frac{20}{2} = 10 \text{ m/s}$$

(b) Instantaneous velocity at $t = 2$:

$$h'(t) = 40 - 10t \quad \Rightarrow \quad h'(2) = 40 - 20 = 20 \text{ m/s}$$

(c) Interpretation: At $t = 2$, the rocket is climbing at a rate of 20 m/s, which is faster than its average velocity between $t = 2$ and $t = 4$.



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Question 7

The temperature T (in degrees Celsius) of a cooling object is modeled by

$$T(t) = 25 + 60e^{-0.4t},$$

where t is time in minutes.

(a) Compute the average rate of change of the temperature over the interval $[1, 4]$.

(b) Find the instantaneous rate of change of the temperature at $t = 1$.

(c) Sketch the temperature function, the secant line over $[1, 4]$, and the tangent line at $t = 1$.

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Question 7 - Solution

Given:

$$T(t) = 25 + 60e^{-0.4t}$$

(a) Average rate of change on $[1, 4]$:

$$T(1) = 25 + 60e^{-0.4} \approx 25 + 60(0.6703) \approx 65.22$$

$$T(4) = 25 + 60e^{-1.6} \approx 25 + 60(0.2019) \approx 37.11$$

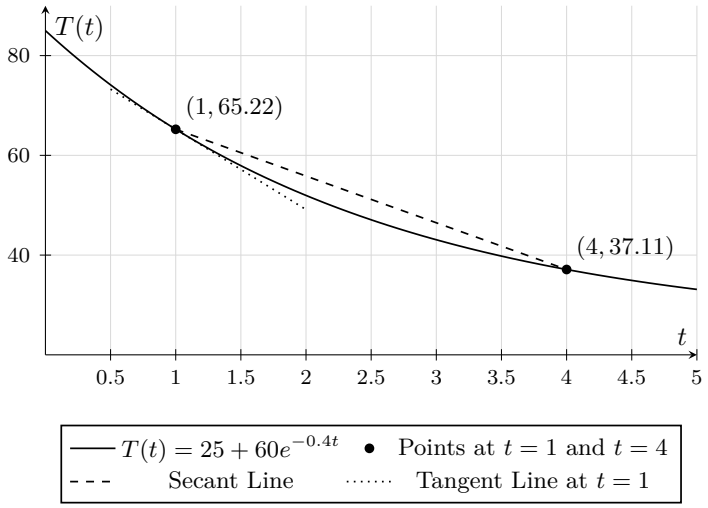
$$\text{Average rate} = \frac{37.11 - 65.22}{4 - 1} \approx \frac{-28.11}{3} \approx -9.37 \text{ °C/min}$$

(b) Instantaneous rate of change at $t = 1$:

$$T'(t) = 60(-0.4)e^{-0.4t} = -24e^{-0.4t}$$

$$T'(1) = -24e^{-0.4} \approx -24(0.6703) \approx -16.09 \text{ °C/min}$$

(c) Interpretation: At $t = 1$, the object is cooling at a faster rate than the average cooling over the interval $[1, 4]$, consistent with exponential decay.



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Question 8

The population P (in thousands) of a bacteria culture after t hours is modeled by

$$P(t) = \frac{100t}{1 + 0.5t}.$$

(a) Compute the average rate of change of the population on the interval $[1, 3]$.

(b) Find the instantaneous rate of change at $t = 1$.

(c) Sketch the population function, the secant line over $[1, 3]$, and the tangent line at $t = 1$.

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Question 8 - Solution

Given:

$$P(t) = \frac{100t}{1 + 0.5t}$$

(a) Average rate of change on $[1, 3]$:

$$P(1) = \frac{100(1)}{1 + 0.5(1)} = \frac{100}{1.5} \approx 66.67$$

$$P(3) = \frac{100(3)}{1 + 0.5(3)} = \frac{300}{2.5} = 120$$

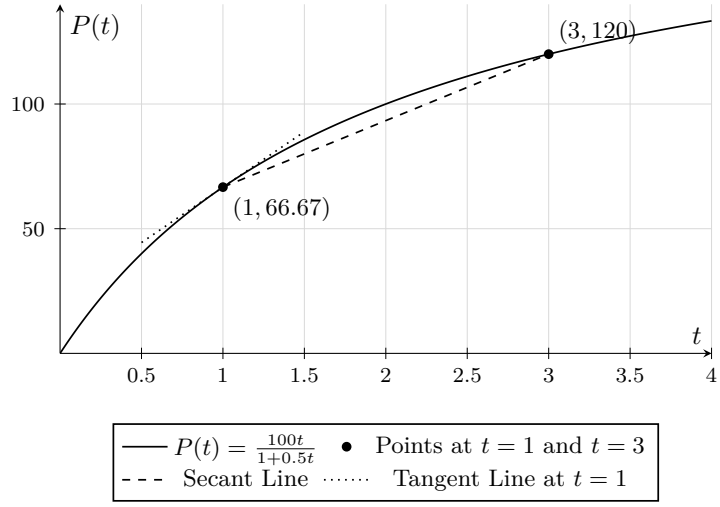
$$\text{Average rate} = \frac{120 - 66.67}{3 - 1} = \frac{53.33}{2} = 26.67 \text{ thousand/hour}$$

(b) Instantaneous rate of change at $t = 1$:
Use the quotient rule:

$$P'(t) = \frac{(1 + 0.5t)(100) - 100t(0.5)}{(1 + 0.5t)^2} = \frac{100 + 50t - 50t}{(1 + 0.5t)^2} = \frac{100}{(1 + 0.5t)^2}$$

$$P'(1) = \frac{100}{(1 + 0.5)^2} = \frac{100}{2.25} \approx 44.44 \text{ thousand/hour}$$

(c) Interpretation: The population is increasing more rapidly at $t = 1$ than the average over the interval. The tangent line shows the instantaneous trend, and the secant line reflects the average change.



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Question 9

The revenue R (in dollars) generated by selling t hundred items is given by the function

$$R(t) = 200t - 3t^2.$$

(a) Compute the average rate of change of revenue on the interval $[2, 6]$.

(b) Find the instantaneous rate of change at $t = 2$.

(c) Sketch the revenue function, the secant line over $[2, 6]$, and the tangent line at $t = 2$.

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Question 9 - Solution

Given:

$$R(t) = 200t - 3t^2$$

(a) Average rate of change on $[2, 6]$:

$$R(2) = 200(2) - 3(2)^2 = 400 - 12 = 388$$

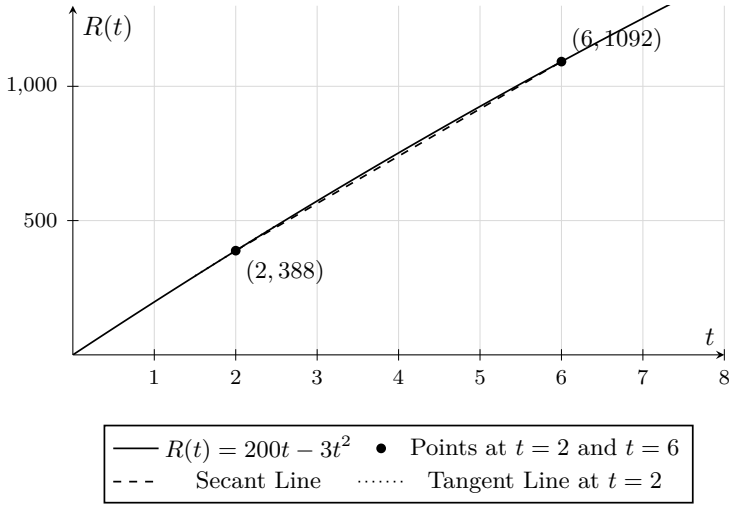
$$R(6) = 200(6) - 3(6)^2 = 1200 - 108 = 1092$$

$$\text{Average rate} = \frac{1092 - 388}{6 - 2} = \frac{704}{4} = 176 \text{ dollars per hundred items}$$

(b) Instantaneous rate of change at $t = 2$:

$$R'(t) = 200 - 6t \quad \Rightarrow \quad R'(2) = 200 - 12 = 188$$

(c) Interpretation: At $t = 2$, revenue is increasing faster than the average over the interval $[2, 6]$. The tangent line shows this steeper slope.



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Question 10

The displacement $s(t)$ (in meters) of a particle moving along a straight line is given by

$$s(t) = t^3 - 6t^2 + 9t,$$

where t is time in seconds.

(a) Find the average velocity of the particle over the interval $[1, 4]$.

(b) Find the instantaneous velocity of the particle at $t = 1$.

(c) Sketch the displacement function, the secant line over $[1, 4]$, and the tangent line at $t = 1$.

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Question 10 - Solution

Given:

$$s(t) = t^3 - 6t^2 + 9t$$

(a) Average velocity on $[1, 4]$:

$$s(1) = 1^3 - 6(1)^2 + 9(1) = 1 - 6 + 9 = 4$$

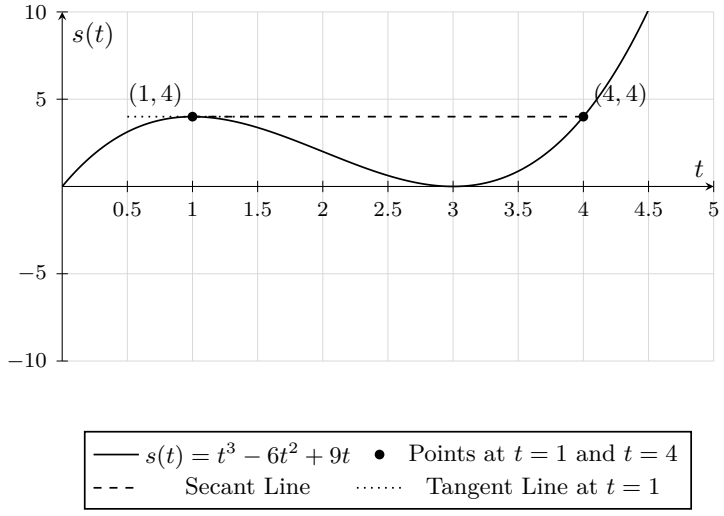
$$s(4) = 64 - 96 + 36 = 4$$

$$\text{Average velocity} = \frac{4 - 4}{4 - 1} = 0 \text{ m/s}$$

(b) Instantaneous velocity at $t = 1$:

$$s'(t) = 3t^2 - 12t + 9 \quad \Rightarrow \quad s'(1) = 3(1)^2 - 12(1) + 9 = 3 - 12 + 9 = 0 \text{ m/s}$$

(c) Interpretation: The average and instantaneous velocities are both zero, indicating the particle returns to its starting displacement at $t = 4$, and is momentarily at rest at $t = 1$.



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Question 1

Evaluate the following limit, and explain your reasoning clearly:

$$\lim_{x \to 1} \frac{x^2 - 1}{\sqrt{x} - 1}$$

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Question 1 - Solution

We are asked to evaluate:

$$\lim_{x\to 1} \frac{x^2-1}{\sqrt{x}-1}$$

This limit gives an indeterminate form $\frac{0}{0}$, so we must simplify.
Start by factoring the numerator:

$$x^2-1=(x-1)(x+1)$$

Now rewrite the expression:

$$\frac{(x-1)(x+1)}{\sqrt{x}-1}$$

To simplify, multiply numerator and denominator by the **conjugate** of the denominator:

$$\frac{(x-1)(x+1)}{\sqrt{x}-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}$$

This gives:

$$\frac{(x-1)(x+1)(\sqrt{x}+1)}{x-1}$$

Now cancel $x-1$ (which is nonzero near $x=1$):

$$(x+1)(\sqrt{x}+1)$$

Now substitute $x=1$:

$$(1+1)(\sqrt{1}+1)=2\cdot 2=\boxed{4}$$

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Question 2

Let the function be defined as:

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x < 2 \\ 3x - 2, & x \geq 2 \end{cases}$$

Determine whether $\lim_{x \to 2} f(x)$ exists. If it does, find its value. Justify your answer.

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Question 2 - Solution

We are asked to evaluate:

$$\lim_{x\to 2} f(x)$$

where

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x < 2 \\ 3x-2, & x \geq 2 \end{cases}$$

Step 1: Evaluate the Left-Hand Limit
For $x < 2$,

$$f(x) = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2}$$

As long as $x \neq 2$, we can cancel $x-2$:

$$f(x) = x+2$$

So:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x+2) = 4$$

Step 2: Evaluate the Right-Hand Limit
For $x \geq 2$, $f(x) = 3x-2$, so:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3x-2) = 3(2)-2 = 6-2 = 4$$

Step 3: Compare One-Sided Limits

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 4 \Rightarrow \lim_{x\to 2} f(x) = \boxed{4}$$

Conclusion: The limit exists and is equal to $\boxed{4}$.

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Question 3

Evaluate the limit, if it exists:

$$\lim_{x \to 0} \frac{\sqrt{1+3x} - \sqrt{1-x}}{x}$$

Explain your reasoning clearly.

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Question 3 - Solution

We are asked to evaluate:

$$\lim_{x\to 0} \frac{\sqrt{1+3x}-\sqrt{1-x}}{x}$$

Direct substitution gives an indeterminate form $\frac{0}{0}$, so we simplify.
Multiply the numerator and denominator by the conjugate of the numerator:

$$\frac{\sqrt{1+3x}-\sqrt{1-x}}{x} \cdot \frac{\sqrt{1+3x}+\sqrt{1-x}}{\sqrt{1+3x}+\sqrt{1-x}}$$

This yields:

$$\frac{(1+3x)-(1-x)}{x\left(\sqrt{1+3x}+\sqrt{1-x}\right)} = \frac{4x}{x\left(\sqrt{1+3x}+\sqrt{1-x}\right)}$$

Cancel x:

$$\frac{4}{\sqrt{1+3x}+\sqrt{1-x}}$$

Now take the limit as $x \to 0$:

$$\frac{4}{\sqrt{1}+\sqrt{1}} = \frac{4}{2} = \boxed{2}$$

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Question 4

Evaluate the following limit:

$$\lim_{x \to 5} \frac{x^2 - 25}{\sqrt{x+1} - \sqrt{6}}$$

Provide a clear justification for each step.

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Question 4 - Solution

We want to evaluate:

$$\lim_{x\to 5} \frac{x^2-25}{\sqrt{x+1}-\sqrt{6}}$$

First, try direct substitution:

$$\frac{5^2-25}{\sqrt{5+1}-\sqrt{6}} = \frac{25-25}{\sqrt{6}-\sqrt{6}} = \frac{0}{0}$$

This is an indeterminate form, so we need to simplify the expression before evaluating the limit.
Factor the numerator:

$$x^2-25 = (x-5)(x+5)$$

So the limit becomes:

$$\lim_{x\to 5} \frac{(x-5)(x+5)}{\sqrt{x+1}-\sqrt{6}}$$

The denominator contains square roots, so we multiply by the conjugate:

$$\frac{(x-5)(x+5)}{\sqrt{x+1}-\sqrt{6}} \cdot \frac{\sqrt{x+1}+\sqrt{6}}{\sqrt{x+1}+\sqrt{6}}$$

This does not change the value of the expression because we are multiplying by 1.
Now simplify the denominator using the difference of squares formula:

$$(a-b)(a+b) = a^2-b^2$$

Here,

$$a = \sqrt{x+1} \qquad \text{and} \qquad b = \sqrt{6}$$

Therefore,

$$(\sqrt{x+1}-\sqrt{6})(\sqrt{x+1}+\sqrt{6}) = (\sqrt{x+1})^2-(\sqrt{6})^2$$

$$= x+1-6$$

$$= x-5$$

So the expression becomes:

$$\lim_{x\to 5} \frac{(x-5)(x+5)(\sqrt{x+1}+\sqrt{6})}{x-5}$$

Since we are evaluating the limit as x approaches 5, we may cancel the common factor $x-5$ for $x \neq 5$:

$$\lim_{x\to 5} (x+5)(\sqrt{x+1}+\sqrt{6})$$

Now substitute $x = 5$:

$$(5+5)(\sqrt{5+1}+\sqrt{6})$$

$$= 10(\sqrt{6}+\sqrt{6})$$

$$= 10(2\sqrt{6})$$

$$= 20\sqrt{6}$$

Therefore,

$$\boxed{20\sqrt{6}}$$

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Question 5

Evaluate the following limit:

$$\lim_{x \to 0} \frac{\sin(3x)}{\tan(5x)}$$

Give a full explanation of your steps and reasoning.

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Question 5 - Solution

We want to evaluate:

$$\lim_{x\to 0} \frac{\sin(3x)}{\tan(5x)}$$

First, try direct substitution:

$$\frac{\sin(3\cdot 0)}{\tan(5\cdot 0)} = \frac{\sin(0)}{\tan(0)} = \frac{0}{0}$$

This is an indeterminate form, so we need to rewrite the expression before evaluating the limit.
We will use the standard trigonometric limits:

$$\lim_{u\to 0} \frac{\sin u}{u} = 1$$

and

$$\lim_{u\to 0} \frac{\tan u}{u} = 1$$

Since $3x \to 0$ and $5x \to 0$ as $x \to 0$, we can apply these limits to $\sin(3x)$ and $\tan(5x)$.
Start with the original expression:

$$\frac{\sin(3x)}{\tan(5x)}$$

We want to create the forms

$$\frac{\sin(3x)}{3x} \qquad \text{and} \qquad \frac{\tan(5x)}{5x}$$

To do this, multiply and divide by the needed factors:

$$\frac{\sin(3x)}{\tan(5x)} = \frac{\sin(3x)}{3x} \cdot \frac{5x}{\tan(5x)} \cdot \frac{3x}{5x}$$

This rewriting is valid because the factors we introduced balance each other out.
Now take the limit of each factor:

$$\lim_{x\to 0} \frac{\sin(3x)}{3x} = 1$$

Also,

$$\lim_{x\to 0} \frac{5x}{\tan(5x)} = 1$$

This is because

$$\lim_{x\to 0} \frac{\tan(5x)}{5x} = 1$$

so its reciprocal also approaches 1.
Finally,

$$\frac{3x}{5x} = \frac{3}{5}$$

for $x \neq 0$. Since limits only depend on values near 0, not necessarily at 0, this cancellation is allowed.
Therefore,

$$\lim_{x\to 0} \frac{\sin(3x)}{\tan(5x)} = 1 \cdot 1 \cdot \frac{3}{5}$$

$$= \boxed{\frac{3}{5}}$$

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Question 6

Evaluate the limit:

$$\lim_{x \to 2} \frac{x^2 - 4}{x^3 - 8}$$

Provide detailed steps and justification for each transformation you make.

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Question 6 - Solution

We want to evaluate:

$$\lim_{x\to 2} \frac{x^2-4}{x^3-8}$$

First, try direct substitution:

$$\frac{2^2-4}{2^3-8} = \frac{4-4}{8-8} = \frac{0}{0}$$

This is an indeterminate form, so we need to simplify the expression before evaluating the limit.
Now factor the numerator:

$$x^2-4 = (x-2)(x+2)$$

This is a difference of squares because:

$$x^2-4 = x^2-2^2$$

Next, factor the denominator:

$$x^3-8 = x^3-2^3$$

This is a difference of cubes. Recall the formula:

$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$

Here,

$$a = x \qquad \text{and} \qquad b = 2$$

So:

$$x^3-8 = (x-2)(x^2+2x+4)$$

Now rewrite the original limit using these factorizations:

$$\lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x^2+2x+4)}$$

Both the numerator and denominator contain the factor $x-2$. Since we are finding the limit as x approaches 2, we only need to consider values of x near 2, but not equal to 2. Therefore, for $x \neq 2$, we may cancel the common factor $x-2$:

$$\lim_{x\to 2} \frac{x+2}{x^2+2x+4}$$

Now the expression is no longer indeterminate, so we can substitute $x = 2$:

$$\frac{2+2}{2^2+2(2)+4}$$

$$= \frac{4}{4+4+4}$$

$$= \frac{4}{12}$$

$$= \frac{1}{3}$$

Therefore,

$$\boxed{\frac{1}{3}}$$

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Question 7

Let

$$f(x) = \begin{cases} \dfrac{\sin(5x)}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

Determine the value of k that makes $f(x)$ continuous at $x = 0$.

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Question 7 - Solution

We are given a piecewise function:

$$f(x) = \begin{cases} \dfrac{\sin(5x)}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

To make $f(x)$ continuous at $x = 0$, we must have:

$$\lim_{x \to 0} f(x) = f(0) = k$$

We now compute:

$$\lim_{x \to 0} \frac{\sin(5x)}{x}$$

We use the identity:

$$\lim_{x \to 0} \frac{\sin(ax)}{x} = a$$

So:

$$\lim_{x \to 0} \frac{\sin(5x)}{x} = 5$$

Thus, for continuity at $x = 0$, we must have:

$$k = \boxed{5}$$

Conclusion: The function is continuous at $x = 0$ if $\boxed{k = 5}$.

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Question 8

Evaluate the following limit:

$$\lim_{x \to 0} \frac{e^{2x} - 1}{\sin(3x)}$$

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Question 8 - Solution

We are asked to compute:

$$\lim_{x\to 0} \frac{e^{2x}-1}{\sin(3x)}$$

First, substitute $x = 0$ into the numerator:

$$e^{2(0)} - 1 = e^0 - 1 = 1 - 1 = 0.$$

Now substitute $x = 0$ into the denominator:

$$\sin(3 \cdot 0) = \sin(0) = 0.$$

So the expression has the indeterminate form

$$\frac{0}{0}.$$

Since both the numerator and denominator approach 0, we can use L'Hopital's Rule.
Differentiate the numerator:

$$\frac{d}{dx}\left(e^{2x}-1\right) = 2e^{2x}.$$

Differentiate the denominator:

$$\frac{d}{dx}\left(\sin(3x)\right) = 3\cos(3x).$$

Therefore,

$$\lim_{x\to 0} \frac{e^{2x}-1}{\sin(3x)} = \lim_{x\to 0} \frac{2e^{2x}}{3\cos(3x)}.$$

Now substitute $x = 0$:

$$\frac{2e^{2(0)}}{3\cos(3 \cdot 0)} = \frac{2e^0}{3\cos(0)}.$$

Since

$$e^0 = 1$$

and

$$\cos(0) = 1,$$

we get

$$\frac{2e^0}{3\cos(0)} = \frac{2(1)}{3(1)} = \frac{2}{3}.$$

Thus,

$$\boxed{\lim_{x\to 0} \frac{e^{2x}-1}{\sin(3x)} = \frac{2}{3}}$$

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Question 9

Evaluate the limit:

$$\lim_{x \to 4} \frac{x-4}{\sqrt{x}-2}$$

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Question 9 - Solution

We are given:

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2}$$

First, observe that:

$$x \to 4 \Rightarrow \text{numerator } x-4 \to 0, \quad \text{denominator } \sqrt{x}-2 \to 0$$

So, this is an indeterminate form $\frac{0}{0}$. We rationalize the denominator to resolve it. Multiply numerator and denominator by the conjugate of the denominator:

$$\frac{x-4}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \frac{(x-4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)} = \frac{(x-4)(\sqrt{x}+2)}{x-4}$$

Now cancel the common factor $x-4$, assuming $x \neq 4$:

$$= \sqrt{x}+2$$

Now take the limit:

$$\lim_{x\to 4} \sqrt{x}+2 = \sqrt{4}+2 = 2+2 = \boxed{4}$$

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Question 10

Evaluate the limit:

$$\lim_{x \to 0} \frac{\sin(3x)}{x}$$

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Question 10 - Solution

We are given:

$$\lim_{x\to 0} \frac{\sin(3x)}{x}$$

We want to simplify the expression to match a known limit. Recall the standard limit:

$$\lim_{u\to 0} \frac{\sin(u)}{u} = 1$$

Let $u = 3x$. As $x \to 0$, we also have $u \to 0$.
So:

$$\frac{\sin(3x)}{x} = \frac{\sin(3x)}{3x} \cdot 3$$

Now apply the limit:

$$\lim_{x\to 0} \frac{\sin(3x)}{x} = \lim_{x\to 0} \left(\frac{\sin(3x)}{3x} \cdot 3 \right) = \left(\lim_{u\to 0} \frac{\sin(u)}{u} \right) \cdot 3 = 1 \cdot 3 = \boxed{3}$$

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Question 1

Let the function f be defined as:

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & \text{if } x < 2 \\ x + 2, & \text{if } x \geq 2 \end{cases}$$

Evaluate:

$$\lim_{x \to 2^-} f(x) \quad \text{and} \quad \lim_{x \to 2^+} f(x)$$

State whether $\lim_{x \to 2} f(x)$ exists.

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Question 1 - Solution

We analyze the left-hand and right-hand limits separately.

Left-hand limit:

For $x < 2$,

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x-2}$$

For $x \neq 2$, the $x - 2$ cancels:

$$f(x) = x + 2$$

So:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x + 2) = 4$$

Right-hand limit:

For $x \geq 2$, we use the second piece:

$$f(x) = x + 2 \Rightarrow \lim_{x \to 2^+} f(x) = 4$$

Conclusion:

Since both one-sided limits exist and are equal:

$$\lim_{x \to 2} f(x) = \boxed{4}$$

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Question 2

Let the function f be defined as follows:

$$f(x) = \begin{cases} 3x + 1, & x < 1 \\ x^2, & x \geq 1 \end{cases}$$

(a) Compute $\lim_{x \to 1^-} f(x)$
(b) Compute $\lim_{x \to 1^+} f(x)$
(c) Does $\lim_{x \to 1} f(x)$ exist? Justify your answer.

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Question 2 - Solution

We are given:

$$f(x) = \begin{cases} 3x + 1, & x < 1 \\ x^2, & x \geq 1 \end{cases}$$

(a) Left-hand limit: For $x < 1$, use $f(x) = 3x + 1$:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x + 1) = 3(1) + 1 = 4$$

(b) Right-hand limit: For $x \geq 1$, use $f(x) = x^2$:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 = 1^2 = 1$$

(c) Two-sided limit: Since the left and right limits are not equal:

$$\lim_{x \to 1^-} f(x) = 4 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = 1 \Rightarrow \lim_{x \to 1} f(x) \text{ does not exist}$$

Final Answer:

$$\boxed{\lim_{x \to 1^-} f(x) = 4, \quad \lim_{x \to 1^+} f(x) = 1, \quad \lim_{x \to 1} f(x) \text{ does not exist}}$$

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Question 3

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x < 2 \\ x + 1, & x \geq 2 \end{cases}$$

(a) Compute $\lim_{x \to 2^-} f(x)$
(b) Compute $\lim_{x \to 2^+} f(x)$
(c) Is $f(x)$ continuous at $x = 2$? Explain.

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Question 3 - Solution

We are given:

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x < 2 \\ x+1, & x \geq 2 \end{cases}$$

(a) Left-hand limit: First, simplify the expression for $x < 2$:

$$f(x) = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2 \quad \text{for } x \neq 2$$

So,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x+2) = 2+2 = 4$$

(b) Right-hand limit:

$$f(x) = x+1 \quad \text{for } x \geq 2$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x+1) = 2+1 = 3$$

(c) Continuity at $x = 2$: Since

$$\lim_{x \to 2^-} f(x) = 4, \quad \lim_{x \to 2^+} f(x) = 3 \Rightarrow \lim_{x \to 2} f(x) \text{ does not exist}$$

And because the limit does not exist, the function is not continuous at $x = 2$.

Final Answer:

$$\boxed{\lim_{x \to 2^-} f(x) = 4, \quad \lim_{x \to 2^+} f(x) = 3, \quad \text{Not continuous at } x = 2}$$

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Question 4

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} x^2 - 4, & x < 1 \\ 2x + 1, & x \geq 1 \end{cases}$$

(a) Find $\lim_{x \to 1^-} f(x)$
(b) Find $\lim_{x \to 1^+} f(x)$
(c) Does $\lim_{x \to 1} f(x)$ exist? Explain.

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Question 4 - Solution

We are given:

$$f(x) = \begin{cases} x^2 - 4, & x < 1 \\ 2x + 1, & x \geq 1 \end{cases}$$

(a) Left-hand limit:
For $x < 1$, we use $f(x) = x^2 - 4$:

$$\lim_{x \to 1^-} f(x) = 1^2 - 4 = -3$$

$$\boxed{\lim_{x \to 1^-} f(x) = -3}$$

(b) Right-hand limit:
For $x \geq 1$, we use $f(x) = 2x + 1$:

$$\lim_{x \to 1^+} f(x) = 2(1) + 1 = 3$$

$$\boxed{\lim_{x \to 1^+} f(x) = 3}$$

(c) Two-sided limit:

$$\lim_{x \to 1^-} f(x) = -3 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = 3$$

Since the one-sided limits are not equal, the overall limit does not exist:

$$\boxed{\lim_{x \to 1} f(x) \text{ does not exist}}$$

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Question 5

Consider the function:

$$f(x) = \begin{cases} \frac{1}{x-3}, & x < 3 \\ \sqrt{x-3}, & x \geq 3 \end{cases}$$

(a) Find $\lim_{x \to 3^-} f(x)$
(b) Find $\lim_{x \to 3^+} f(x)$
(c) Does $\lim_{x \to 3} f(x)$ exist? Justify your answer.

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Question 5 - Solution

We are given:

$$f(x) = \begin{cases} \frac{1}{x-3}, & x < 3 \\ \sqrt{x-3}, & x \geq 3 \end{cases}$$

(a) Left-hand limit:
As $x \to 3^-$, we are approaching 3 from values less than 3. The expression is:

$$f(x) = \frac{1}{x-3}$$

As $x \to 3^-$, $x - 3 \to 0^- \Rightarrow \frac{1}{x-3} \to -\infty$

$$\boxed{\lim_{x \to 3^-} f(x) = -\infty}$$

(b) Right-hand limit:
As $x \to 3^+$, the function becomes:

$$f(x) = \sqrt{x-3}$$

As $x \to 3^+$, $x - 3 \to 0^+ \Rightarrow \sqrt{x-3} \to 0$

$$\boxed{\lim_{x \to 3^+} f(x) = 0}$$

(c) Does the limit exist?
Since

$$\lim_{x \to 3^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to 3^+} f(x) = 0$$

the two one-sided limits are not equal (and one diverges), so the limit $\lim_{x \to 3} f(x)$ does not exist.

$$\boxed{\text{Limit does not exist at } x = 3}$$

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Question 6

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} \frac{\sin(\pi x)}{x}, & x < 0 \\ \frac{1}{x+1}, & x \geq 0 \end{cases}$$

(a) Compute $\lim_{x \to 0^-} f(x)$
(b) Compute $\lim_{x \to 0^+} f(x)$
(c) Does $\lim_{x \to 0} f(x)$ exist? Explain.

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Question 6 - Solution

We are given:

$$f(x) = \begin{cases} \frac{\sin(\pi x)}{x}, & x < 0 \\ \frac{1}{x+1}, & x \geq 0 \end{cases}$$

(a) Left-hand limit:
For $x < 0$, use:

$$\lim_{x \to 0^-} \frac{\sin(\pi x)}{x}$$

Using the identity $\lim_{x \to 0} \frac{\sin(\pi x)}{x} = \pi$ (from standard limit $\lim_{u \to 0} \frac{\sin(u)}{u} = 1$):

$$\lim_{x \to 0^-} \frac{\sin(\pi x)}{x} = \pi$$

$$\boxed{\lim_{x \to 0^-} f(x) = \pi}$$

(b) Right-hand limit:
For $x \geq 0$, use:

$$\lim_{x \to 0^+} \frac{1}{x+1} = \frac{1}{1} = 1$$

$$\boxed{\lim_{x \to 0^+} f(x) = 1}$$

(c) Two-sided limit:
Since:

$$\lim_{x \to 0^-} f(x) = \pi \quad \text{and} \quad \lim_{x \to 0^+} f(x) = 1$$

These are not equal, so the limit does not exist:

$$\boxed{\lim_{x \to 0} f(x) \text{ does not exist}}$$

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Question 7

Consider the piecewise-defined function:

$$f(x) = \begin{cases} 3x + 1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$$

(a) Compute $\lim_{x \to 2^-} f(x)$.
(b) Compute $\lim_{x \to 2^+} f(x)$.
(c) Is $\lim_{x \to 2} f(x)$ defined? Explain.
(d) What is the value of $f(2)$?

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Question 7 - Solution

We are given:

$$f(x) = \begin{cases} 3x + 1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$$

(a) Left-hand limit as $x \to 2^-$:
Use the definition for $x < 2$:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (3x + 1) = 3(2) + 1 = 7$$

$$\boxed{\lim_{x \to 2^-} f(x) = 7}$$

(b) Right-hand limit as $x \to 2^+$:
Use the definition for $x \geq 2$:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 - 4) = 2^2 - 4 = 0$$

$$\boxed{\lim_{x \to 2^+} f(x) = 0}$$

(c) Two-sided limit:

$$\lim_{x \to 2^-} f(x) = 7, \quad \lim_{x \to 2^+} f(x) = 0$$

Since the one-sided limits are not equal:

$$\boxed{\lim_{x \to 2} f(x) \text{ does not exist}}$$

(d) Value of the function at 2:
Since $x = 2 \geq 2$, use the second case:

$$f(2) = 2^2 - 4 = 0$$

$$\boxed{f(2) = 0}$$

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Question 8

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{if } x < 2 \\ 3x - 4, & \text{if } x \geq 2 \end{cases}$$

(a) Find $\lim_{x \to 2^-} f(x)$.
(b) Find $\lim_{x \to 2^+} f(x)$.
(c) Determine whether $\lim_{x \to 2} f(x)$ exists.
(d) Is f continuous at $x = 2$? Justify your answer.

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Question 8 - Solution

We are given:

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{if } x < 2 \\ 3x - 4, & \text{if } x \geq 2 \end{cases}$$

(a) Left-hand limit:
Note: $x^2 - 4 = (x-2)(x+2)$, so for $x \neq 2$,

$$\frac{x^2-4}{x-2} = x + 2$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x+2) = 4$$

$$\boxed{\lim_{x \to 2^-} f(x) = 4}$$

(b) Right-hand limit:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x - 4) = 6 - 4 = 2$$

$$\boxed{\lim_{x \to 2^+} f(x) = 2}$$

(c) Two-sided limit:
The left-hand and right-hand limits are not equal:

$$\lim_{x \to 2^-} f(x) = 4 \neq 2 = \lim_{x \to 2^+} f(x)$$

$$\boxed{\lim_{x \to 2} f(x) \text{ does not exist}}$$

(d) Continuity at $x = 2$:
$\lim_{x \to 2} f(x)$ does not exist , $f(2) = 3(2) - 4 = 6 - 4 = 2$ (defined)
But since the limit does not exist, f is not continuous at $x = 2$.

$$\boxed{f \text{ is not continuous at } x = 2}$$

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Question 9

Consider the piecewise-defined function:

$$f(x) = \begin{cases} \frac{\sin(x)}{x}, & \text{if } x < 0 \\ 2x + 1, & \text{if } x \geq 0 \end{cases}$$

(a) Evaluate $\lim_{x \to 0^-} f(x)$.
(b) Evaluate $\lim_{x \to 0^+} f(x)$.
(c) Does $\lim_{x \to 0} f(x)$ exist? Explain.
(d) Find $f(0)$, if it exists.

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Question 9 - Solution

We are given:

$$f(x) = \begin{cases} \frac{\sin(x)}{x}, & \text{if } x < 0 \\ 2x + 1, & \text{if } x \geq 0 \end{cases}$$

(a) Left-hand limit as $x \to 0^-$:
Since $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, and this applies as $x \to 0^-$ as well:

$$\boxed{\lim_{x\to 0^-} f(x) = 1}$$

(b) Right-hand limit as $x \to 0^+$:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (2x + 1) = 1$$

$$\boxed{\lim_{x\to 0^+} f(x) = 1}$$

(c) Two-sided limit:
Since both one-sided limits are equal:

$$\boxed{\lim_{x\to 0} f(x) = 1}$$

(d) Value at the point:
Use the definition for $x \geq 0$:

$$f(0) = 2(0) + 1 = 1$$

$$\boxed{f(0) = 1}$$

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Question 10

Consider the piecewise-defined function:

$$f(x) = \begin{cases} x^2 - 3x + 2, & \text{if } x < 1 \\ \sqrt{x+3} - 2, & \text{if } x \geq 1 \end{cases}$$

(a) Find $\lim_{x \to 1^-} f(x)$.
(b) Find $\lim_{x \to 1^+} f(x)$.
(c) Determine whether $\lim_{x \to 1} f(x)$ exists.
(d) Is f continuous at $x = 1$? Justify your answer.

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Question 10 - Solution

We are given:

$$f(x) = \begin{cases} x^2 - 3x + 2, & \text{if } x < 1 \\ \sqrt{x+3} - 2, & \text{if } x \geq 1 \end{cases}$$

(a) Left-hand limit:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 - 3x + 2) = 1^2 - 3(1) + 2 = 0$$

$$\boxed{\lim_{x \to 1^-} f(x) = 0}$$

(b) Right-hand limit:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (\sqrt{x+3} - 2) = \sqrt{4} - 2 = 2 - 2 = 0$$

$$\boxed{\lim_{x \to 1^+} f(x) = 0}$$

(c) Two-sided limit:
Since both one-sided limits exist and are equal:

$$\boxed{\lim_{x \to 1} f(x) = 0}$$

(d) Continuity at $x = 1$:
We already have:

$$\lim_{x \to 1} f(x) = 0$$

Now evaluate the function at $x = 1$:

$$f(1) = \sqrt{1+3} - 2 = \sqrt{4} - 2 = 2 - 2 = 0$$

Since: $f(1)$ is defined , $\lim_{x \to 1} f(x) = f(1)$
We conclude:

$$\boxed{f \text{ is continuous at } x = 1}$$

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Question 1

Let the functions $f(x)$, $g(x)$, and $h(x)$ be defined by the following limits:

$$\lim_{x\to 2} f(x) = 4, \quad \lim_{x\to 2} g(x) = -3, \quad \lim_{x\to 2} h(x) = 0$$

Evaluate the following limit expressions using limit laws and properties:

(a) $\lim_{x\to 2} [2f(x) + 5g(x)]$

(b) $\lim_{x\to 2} \left[\dfrac{f(x)\cdot g(x)}{h(x)}\right]$

(c) Explain whether the given information determines a signed infinite limit in part (b).

Assume the quotient denominator is nonzero in some deleted neighborhood of the approach point. Distinguish finite limits from signed infinite limits.

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Question 1 - Solution

(a) Sum and constant-multiple laws

Using the given limits,

$$\begin{aligned}\lim_{x\to 2}\big[2f(x)+5g(x)\big] &= 2\lim_{x\to 2} f(x) + 5\lim_{x\to 2} g(x)\\ &= 2(4)+5(-3) = \boxed{-7}.\end{aligned}$$

(b) Quotient with a zero denominator limit

The product law gives

$$\lim_{x\to 2} f(x)g(x) = 4(-3) = -12, \qquad \lim_{x\to 2} h(x) = 0.$$

The quotient law does not apply because the denominator tends to zero. Since the numerator tends to a nonzero value, its magnitude stays bounded away from zero near $x = 2$. Therefore,

$$\left|\frac{f(x)g(x)}{h(x)}\right| \longrightarrow \infty.$$

The quotient has no finite limit.

(c) Is the sign of an infinite limit determined?

No. Take $f(x) = 4$ and $g(x) = -3$. Each denominator below satisfies the given limit and is nonzero for $x \neq 2$:

Denominator $h(x)$	Behavior of $-12/h(x)$
$(x-2)^2$	$\longrightarrow -\infty$
$-(x-2)^2$	$\longrightarrow +\infty$
$x-2$	opposite one-sided infinities

Thus more information about the sign of $h(x)$ is needed.

A signed infinite limit is not determined.

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Question 2

Suppose that

$$\lim_{x\to 3} f(x) = 5 \quad \text{and} \quad \lim_{x\to 3} g(x) = 0.$$

Define

$$h(x) = \frac{f(x)^2 - 25}{g(x)}.$$

(a) Explain why direct substitution does not determine $\lim_{x\to 3} h(x)$.

(b) Use algebraic manipulation to rewrite $h(x)$.

(c) State precisely what additional information would be required to evaluate $\lim_{x\to 3} h(x)$.

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Question 2 - Solution

We are given:

$$\lim_{x\to 3} f(x) = 5, \quad \lim_{x\to 3} g(x) = 0, \quad h(x) = \frac{f(x)^2 - 25}{g(x)}.$$

(a) Indeterminate form

Substituting the known limits gives:

$$\lim_{x\to 3} h(x) = \frac{5^2 - 25}{0} = \frac{0}{0},$$

which is an indeterminate form. Therefore, the limit cannot be evaluated by direct substitution.

(b) Algebraic simplification

Factor the numerator:

$$f(x)^2 - 25 = (f(x) - 5)(f(x) + 5).$$

Then,

$$h(x) = \frac{(f(x) - 5)(f(x) + 5)}{g(x)} = \left(\frac{f(x) - 5}{g(x)}\right)(f(x) + 5).$$

Taking limits where possible,

$$\lim_{x\to 3}(f(x) + 5) = 10.$$

(c) Required additional information

The limit

$$\lim_{x\to 3} h(x)$$

exists if and only if the limit

$$\lim_{x\to 3} \frac{f(x) - 5}{g(x)}$$

exists.

If we let

$$L = \lim_{x\to 3} \frac{f(x) - 5}{g(x)},$$

then

$$\lim_{x\to 3} h(x) = 10L.$$

Without information about how fast $f(x) \to 5$ relative to $g(x) \to 0$, the value of the limit cannot be determined.

The limit cannot be evaluated without additional information.

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Question 3

Given the following limits:

$$\lim_{x\to 1} f(x) = 4, \quad \lim_{x\to 1} g(x) = -2, \quad \lim_{x\to 1} h(x) = 0$$

Define:

$$k(x) = \frac{f(x)\cdot h(x) + g(x)}{h(x)}$$

Evaluate $\lim_{x\to 1} k(x)$, or explain why it does not exist.

Assume the quotient denominator is nonzero in some deleted neighborhood of the approach point. Distinguish finite limits from signed infinite limits.

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Question 3 - Solution

Step 1: Apply the product and sum laws

For the numerator,

$$\lim_{x\to 1}\big(f(x)h(x)+g(x)\big) = 4(0)+(-2) = -2.$$

The denominator satisfies $\lim_{x\to 1} h(x) = 0$.

Step 2: Determine whether a finite limit exists

The numerator tends to a nonzero number, so its magnitude stays bounded away from zero near $x = 1$. Since $h(x)$ is nonzero in a deleted neighborhood,

$$|k(x)| = \left|\frac{f(x)h(x)+g(x)}{h(x)}\right| \longrightarrow \infty.$$

There is no finite limit.

Step 3: Examine the sign

The sign depends on the denominator. For example, choose $f(x) = 4$ and $g(x) = -2$, so that $k(x) = 4 - 2/h(x)$.

Denominator $h(x)$	Behavior of $k(x)$
$(x-1)^2$	$\longrightarrow -\infty$
$-(x-1)^2$	$\longrightarrow +\infty$
$x-1$	opposite one-sided infinities

All three choices satisfy the given limits. Therefore,

A signed infinite limit is not determined.

Division by a zero limiting value is not a valid use of the quotient law.

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Question 4

Let

$$f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$

(a) Find $\lim\limits_{x \to 3} f(x)$ using limit properties.

(b) Determine the value of k such that $f(x)$ is continuous at $x = 3$.

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Question 4 - Solution

We are given:

$$f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$

(a) Find the limit:
First, factor the numerator:

$$x^2 - 9 = (x - 3)(x + 3)$$

So for $x \neq 3$, we can simplify:

$$f(x) = \frac{(x - 3)(x + 3)}{x - 3} = x + 3$$

Thus,

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} (x + 3) = 6$$

Answer: $\boxed{\lim_{x \to 3} f(x) = 6}$

(b) Value of k for Continuity:
For $f(x)$ to be continuous at $x = 3$, we must have:

$$\lim_{x \to 3} f(x) = f(3)$$

From (a), the limit is 6. Therefore, set $f(3) = k = 6$
Answer: $\boxed{k = 6}$

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Question 5

Let $f(x) = \sqrt{4x+5}$ and $g(x) = \dfrac{x^2+2x-3}{x-1}$, and assume $\lim_{x\to 1} f(x) = 3$.

(a) Find $\lim\limits_{x\to 1} g(x)$.

(b) Use the limit properties to find $\lim\limits_{x\to 1}[f(x)\cdot g(x)]$.

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Question 5 - Solution

We are given:

$$f(x) = \sqrt{4x+5}, \quad g(x) = \frac{x^2+2x-3}{x-1}, \quad \lim_{x\to 1} f(x) = 3$$

(a) Find $\lim_{x\to 1} g(x)$**:**
First, factor the numerator of $g(x)$:

$$x^2 + 2x - 3 = (x+3)(x-1)$$

$$g(x) = \frac{(x+3)(x-1)}{x-1}$$

For $x \neq 1$, cancel common factors:

$$g(x) = x + 3$$

So,

$$\lim_{x\to 1} g(x) = \lim_{x\to 1}(x+3) = 4$$

Answer: $\boxed{4}$

(b) Find $\lim_{x\to 1}[f(x)\cdot g(x)]$**:**
Use the limit product law:

$$\lim_{x\to 1}[f(x)\cdot g(x)] = \lim_{x\to 1} f(x) \cdot \lim_{x\to 1} g(x) = 3 \cdot 4 = 12$$

Answer: $\boxed{12}$

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Question 6

Let $f(x)$ and $g(x)$ be functions such that:

$$\lim_{x\to 2} f(x) = 5, \quad \lim_{x\to 2} g(x) = -3$$

Use limit laws to compute the following:

(a) $\lim_{x\to 2}[3f(x) - 2g(x)]$

(b) $\lim_{x\to 2}\left[\dfrac{f(x)}{g(x)}\right]$

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Question 6 - Solution

We are given:

$$\lim_{x\to 2} f(x) = 5, \quad \lim_{x\to 2} g(x) = -3$$

(a) Use limit laws for scalar multiplication and addition/subtraction:

$$\begin{aligned}\lim_{x\to 2}[3f(x) - 2g(x)] &= 3 \cdot \lim_{x\to 2} f(x) - 2 \cdot \lim_{x\to 2} g(x) \\ &= 3(5) - 2(-3) = 15 + 6 = \boxed{21}\end{aligned}$$

(b) Use the quotient limit law:

$$\lim_{x\to 2}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to 2} f(x)}{\lim_{x\to 2} g(x)} = \frac{5}{-3} = \boxed{-\frac{5}{3}}$$

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Question 7

Given that

$$\lim_{x\to 1} f(x) = 4, \quad \lim_{x\to 1} g(x) = 0, \quad \lim_{x\to 1} h(x) = -2$$

compute the following limits using limit laws:

(a) $\lim_{x\to 1}[f(x)^2 + 3h(x)]$

(b) $\lim_{x\to 1}\left[\dfrac{f(x)+h(x)}{g(x)^2+1}\right]$

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Question 7 - Solution

We are given:

$$\lim_{x\to 1} f(x) = 4, \quad \lim_{x\to 1} g(x) = 0, \quad \lim_{x\to 1} h(x) = -2$$

(a) Use the limit laws for powers and linear combinations:

$$\lim_{x\to 1}[f(x)^2 + 3h(x)] = (\lim_{x\to 1} f(x))^2 + 3 \cdot \lim_{x\to 1} h(x) = (4)^2 + 3(-2) = 16 - 6 = \boxed{10}$$

(b) Use limit laws for addition, powers, and quotients (denominator never zero):

$$\lim_{x\to 1}\left[\frac{f(x)+h(x)}{g(x)^2+1}\right] = \frac{\lim_{x\to 1}(f(x)+h(x))}{\lim_{x\to 1}(g(x)^2+1)} = \frac{4+(-2)}{0^2+1} = \frac{2}{1} = \boxed{2}$$

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Question 8

Assume the following limits exist:

$$\lim_{x\to -1} f(x) = 2, \qquad \lim_{x\to -1} g(x) = 3$$

Evaluate the limit:

$$\lim_{x\to -1} \left[(f(x) - g(x))^2 + 4f(x)\right]$$

Use limit laws to justify each step.

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Question 8 - Solution

We are given:

$$\lim_{x\to -1} f(x) = 2, \qquad \lim_{x\to -1} g(x) = 3$$

We evaluate:

$$\lim_{x\to -1} \left[(f(x) - g(x))^2 + 4f(x)\right]$$

Step 1: Apply limit laws to each part
First, consider the difference:

$$\lim_{x\to -1} (f(x) - g(x)) = \lim_{x\to -1} f(x) - \lim_{x\to -1} g(x) = 2 - 3 = -1$$

Now square the result:

$$\lim_{x\to -1} (f(x) - g(x))^2 = (-1)^2 = 1$$

Next, evaluate the linear term:

$$\lim_{x\to -1} 4f(x) = 4 \cdot \lim_{x\to -1} f(x) = 4 \cdot 2 = 8$$

Step 2: Combine results
Using the sum rule for limits:

$$\lim_{x\to -1} \left[(f(x) - g(x))^2 + 4f(x)\right] = 1 + 8 = \boxed{9}$$

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Question 9

Let $\lim_{x\to 3} f(x) = 5$ and $\lim_{x\to 3} g(x) = -2$.

Evaluate the following limit using limit laws:

$$\lim_{x\to 3}\left(\frac{f(x)^2 + 3g(x)}{g(x) - f(x)}\right)$$

Justify each step using properties of limits.

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Question 9 - Solution

We are given:

$$\lim_{x\to 3} f(x) = 5, \qquad \lim_{x\to 3} g(x) = -2$$

We need to evaluate:

$$\lim_{x\to 3}\left(\frac{f(x)^2 + 3g(x)}{g(x) - f(x)}\right)$$

Step 1: Use limit laws in numerator and denominator.
Numerator:

$$\lim_{x\to 3} f(x)^2 = \left(\lim_{x\to 3} f(x)\right)^2 = 5^2 = 25$$

$$\lim_{x\to 3} 3g(x) = 3 \cdot \lim_{x\to 3} g(x) = 3 \cdot (-2) = -6$$

$$\text{So, } \lim_{x\to 3}(f(x)^2 + 3g(x)) = 25 - 6 = 19$$

Denominator:

$$\lim_{x\to 3}(g(x) - f(x)) = \lim_{x\to 3} g(x) - \lim_{x\to 3} f(x) = -2 - 5 = -7$$

Step 2: Final Answer

$$\lim_{x\to 3}\left(\frac{f(x)^2 + 3g(x)}{g(x) - f(x)}\right) = \frac{19}{-7} = \boxed{-\frac{19}{7}}$$

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Question 10

Let the functions $f(x)$, $g(x)$, and $h(x)$ satisfy the inequalities:

$$g(x) \leq f(x) \leq h(x)$$

for all x near 2 (but not necessarily at 2), and suppose:

$$\lim_{x \to 2} g(x) = 4 \quad \text{and} \quad \lim_{x \to 2} h(x) = 4$$

Use this information to determine:

$$\lim_{x \to 2} f(x)$$

Explain the reasoning behind your conclusion.

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Question 10 - Solution

We are told that for all x near 2:

$$g(x) \leq f(x) \leq h(x)$$

and

$$\lim_{x\to 2} g(x) = 4, \quad \lim_{x\to 2} h(x) = 4$$

This is a perfect scenario to apply the "Squeeze Theorem", which states:
If $g(x) \leq f(x) \leq h(x)$ near $x = a$, and

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} f(x) = L$$

Applying this here:

$$\lim_{x\to 2} f(x) = \boxed{4}$$

Conclusion: By the Squeeze Theorem, since both bounding functions tend to 4 as $x \to 2$, so must $f(x)$.

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Question 1

Evaluate the following limit, showing all necessary steps and justification:

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 4} - 2}{x}$$

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Question 1 - Solution

We are given:

$$\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x}$$

At $x = 0$, this becomes:

$$\frac{\sqrt{0+4}-2}{0} = \frac{2-2}{0} = \frac{0}{0}$$

This is an indeterminate form, so we must simplify the expression.

Step 1: Multiply numerator and denominator by the conjugate.
Multiply by:

$$\frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2}$$

$$\frac{\sqrt{x^2+4}-2}{x} \cdot \frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2} = \frac{(\sqrt{x^2+4}-2)(\sqrt{x^2+4}+2)}{x(\sqrt{x^2+4}+2)}$$

The numerator is a difference of squares:

$$(\sqrt{x^2+4})^2 - 2^2 = x^2 + 4 - 4 = x^2$$

So the entire expression becomes:

$$\frac{x^2}{x(\sqrt{x^2+4}+2)} = \frac{x}{\sqrt{x^2+4}+2}$$

Step 2: Evaluate the limit.
Now:

$$\lim_{x\to 0} \frac{x}{\sqrt{x^2+4}+2} = \frac{0}{\sqrt{0+4}+2} = \frac{0}{4} = 0$$

Final Answer:

$$\boxed{0}$$

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Question 2

Evaluate the limit:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

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Question 2 - Solution

We are given:

$$\lim_{x\to 3} \frac{x^2-9}{x-3}$$

Step 1: Check for indeterminate form.
At $x = 3$, the numerator becomes:

$$x^2 - 9 = 9 - 9 = 0$$

The denominator becomes:

$$x - 3 = 0$$

So we have the indeterminate form $\frac{0}{0}$. We must simplify.

Step 2: Factor the numerator.

$$x^2 - 9 = (x-3)(x+3)$$

$$\Rightarrow \frac{(x-3)(x+3)}{x-3}$$

Cancel the common factor $x - 3$ (note: this is valid for $x \neq 3$):

$$= x + 3$$

Step 3: Take the limit of the simplified expression.

$$\lim_{x\to 3} x + 3 = 3 + 3 = \boxed{6}$$

Final Answer: $\boxed{6}$

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Question 3

Evaluate the limit:

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$$

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Question 3 - Solution

We are given:

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$$

Step 1: Check for indeterminate form.
At $x = 0$, the expression becomes:

$$\frac{\sqrt{1+0}-1}{0} = \frac{1-1}{0} = \frac{0}{0}$$

So we have the indeterminate form $\frac{0}{0}$. To resolve this, we rationalize the numerator.

Step 2: Multiply numerator and denominator by the conjugate of the numerator.

$$\frac{\sqrt{1+x}-1}{x} \cdot \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} = \frac{(1+x-1)}{x(\sqrt{1+x}+1)} = \frac{x}{x(\sqrt{1+x}+1)}$$

Cancel x from numerator and denominator:

$$= \frac{1}{\sqrt{1+x}+1}$$

Step 3: Take the limit of the simplified expression.

$$\lim_{x\to 0} \frac{1}{\sqrt{1+x}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

Final Answer: $\boxed{\frac{1}{2}}$

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Question 4

Evaluate the limit:

$$\lim_{x \to 0} \frac{\sin(3x)}{x}$$

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Question 4 - Solution

We are asked to evaluate:

$$\lim_{x \to 0} \frac{\sin(3x)}{x}$$

Step 1: Use standard limit identity
Recall the standard identity:

$$\lim_{x \to 0} \frac{\sin(kx)}{x} = k$$

This applies directly here with $k = 3$.

$$\lim_{x \to 0} \frac{\sin(3x)}{x} = 3$$

Answer: $\boxed{3}$

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Question 5

Evaluate the following limit:

$$\lim_{x \to 4} \frac{x - 4}{\sqrt{x} - 2}$$

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Question 5 - Solution

We are given:

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2}$$

Step 1: Direct Substitution
Substitute $x = 4$:

$$\frac{4-4}{\sqrt{4}-2} = \frac{0}{0}$$

This is an indeterminate form, so we need to simplify.

Step 2: Use Algebraic Manipulation
We can rationalize the denominator. Multiply the numerator and denominator by the conjugate of the denominator:

$$\frac{x-4}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \frac{(x-4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)}$$

Simplify the denominator:

$$(\sqrt{x}-2)(\sqrt{x}+2) = x-4$$

So the expression becomes:

$$\frac{(x-4)(\sqrt{x}+2)}{x-4}$$

Cancel the common factor $x-4$ (note: valid as $x \to 4$, not at $x = 4$):

$$= \sqrt{x}+2$$

Now take the limit:

$$\lim_{x\to 4} \sqrt{x}+2 = \sqrt{4}+2 = 2+2 = \boxed{4}$$

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Question 6

Evaluate the limit:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

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Question 6 - Solution

We are asked to evaluate:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

Step 1: Factor the numerator
Recall that $x^2 - 1 = (x - 1)(x + 1)$. So,

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1}$$

For $x \neq 1$, we can cancel $x - 1$ from numerator and denominator:

$$= x + 1$$

Step 2: Take the limit
Now that the expression simplifies to $x + 1$, we compute:

$$\lim_{x \to 1} (x + 1) = 2$$

Answer: $\boxed{2}$

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Question 7

Evaluate the limit:

$$\lim_{x\to 0} \frac{\sqrt{1+3x}-\sqrt{1+x}}{x}.$$

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Question 7 - Solution

We evaluate

$$\lim_{x\to 0} \frac{\sqrt{1+3x}-\sqrt{1+x}}{x}.$$

Step 1: Identify the Indeterminate Form

Substituting $x = 0$ gives

$$\frac{\sqrt{1}-\sqrt{1}}{0} = \frac{0}{0},$$

which is an indeterminate form. Algebraic simplification is required.

Step 2: Multiply by the Conjugate

Multiply the numerator and denominator by the conjugate of the numerator:

$$\frac{\sqrt{1+3x}-\sqrt{1+x}}{x} \cdot \frac{\sqrt{1+3x}+\sqrt{1+x}}{\sqrt{1+3x}+\sqrt{1+x}}.$$

This yields

$$\frac{(1+3x)-(1+x)}{x\big(\sqrt{1+3x}+\sqrt{1+x}\big)} = \frac{2x}{x\big(\sqrt{1+3x}+\sqrt{1+x}\big)}.$$

Cancel the common factor of x:

$$\frac{2}{\sqrt{1+3x}+\sqrt{1+x}}.$$

Step 3: Evaluate the Limit

Now substitute $x = 0$:

$$\lim_{x\to 0} \frac{2}{\sqrt{1+3x}+\sqrt{1+x}} = \frac{2}{1+1} = 1.$$

$$\boxed{1}$$

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Question 8

Evaluate the limit:

$$\lim_{x\to\infty}\left(\sqrt{x^2+5x}-x\right)$$

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Question 8 - Solution

We are asked to evaluate:

$$\lim_{x\to\infty}\left(\sqrt{x^2+5x}-x\right)$$

Step 1: Indeterminate Form

As $x\to\infty$, both $\sqrt{x^2+5x}$ and x grow large. The expression becomes $\infty-\infty$, an indeterminate form. We simplify.

Step 2: Multiply by the Conjugate

Multiply numerator and denominator by the conjugate:

$$\left(\sqrt{x^2+5x}-x\right)\cdot\frac{\sqrt{x^2+5x}+x}{\sqrt{x^2+5x}+x}$$

$$=\frac{(x^2+5x)-x^2}{\sqrt{x^2+5x}+x}=\frac{5x}{\sqrt{x^2+5x}+x}$$

Step 3: Divide numerator and denominator by x

$$=\frac{5}{\sqrt{1+\frac{5}{x}}+1}$$

As $x\to\infty$, $\frac{5}{x}\to 0$, so:

$$\lim_{x\to\infty}\frac{5}{\sqrt{1+\frac{5}{x}}+1}=\frac{5}{1+1}=\frac{5}{2}$$

Answer: $\boxed{\frac{5}{2}}$

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Question 9

Evaluate the limit:

$$\lim_{x \to 0} \frac{\sin(3x)}{\tan(5x)}.$$

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Question 9 - Solution

We evaluate

$$\lim_{x\to 0} \frac{\sin(3x)}{\tan(5x)}.$$

Step 1: Express tangent in terms of sine and cosine

Recall that

$$\tan(5x) = \frac{\sin(5x)}{\cos(5x)}.$$

Thus,

$$\frac{\sin(3x)}{\tan(5x)} = \frac{\sin(3x)\cos(5x)}{\sin(5x)}.$$

Step 2: Apply standard trigonometric limits

Rewrite the expression as

$$\frac{\sin(3x)}{3x} \cdot \frac{5x}{\sin(5x)} \cdot \frac{3}{5} \cdot \cos(5x).$$

Now take limits term by term as $x \to 0$:

$$\frac{\sin(3x)}{3x} \to 1, \quad \frac{5x}{\sin(5x)} \to 1, \quad \cos(5x) \to 1.$$

Therefore,

$$\lim_{x\to 0} \frac{\sin(3x)}{\tan(5x)} = 1 \cdot 1 \cdot \frac{3}{5} \cdot 1 = \frac{3}{5}.$$

$$\boxed{\frac{3}{5}}$$

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Question 10

Evaluate the limit:

$$\lim_{x \to 0} \frac{1 - \cos(4x)}{x^2}$$

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Question 10 - Solution

We are given the limit:

$$\lim_{x \to 0} \frac{1 - \cos(4x)}{x^2}$$

Step 1: Use the identity

$$1 - \cos(\theta) = 2\sin^2\left(\frac{\theta}{2}\right)$$

Apply this identity to the numerator:

$$1 - \cos(4x) = 2\sin^2(2x)$$

So the expression becomes:

$$\lim_{x \to 0} \frac{2\sin^2(2x)}{x^2}$$

Step 2: Rewrite the sine term

$$= \lim_{x \to 0} 2 \cdot \left(\frac{\sin(2x)}{x}\right)^2$$

We can write:

$$\frac{\sin(2x)}{x} = \frac{\sin(2x)}{2x} \cdot 2 \to 1 \cdot 2 = 2 \quad \text{as } x \to 0$$

So:

$$\left(\frac{\sin(2x)}{x}\right)^2 \to 4 \Rightarrow 2 \cdot 4 = 8$$

Final Answer: $\boxed{8}$

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Question 1

Evaluate the following limit and describe the behavior of the function near the point of discontinuity:

$$\lim_{x \to 1^-} \frac{3}{(x-1)^2}$$

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Question 1 - Solution

We are asked to evaluate:

$$\lim_{x \to 1^-} \frac{3}{(x-1)^2}$$

Step 1: Observe the behavior of the denominator

As $x \to 1^-$, the value of $x - 1$ becomes a small negative number. Squaring it makes it a small positive number:

$$(x-1)^2 > 0 \quad \text{but very small}$$

So the expression becomes:

$$\frac{3}{\text{small positive number}} \to +\infty$$

Conclusion:

The function grows without bound as x approaches 1 from the left.

$$\boxed{\lim_{x \to 1^-} \frac{3}{(x-1)^2} = \infty}$$

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Question 2

Evaluate the limit:

$$\lim_{x \to 0^+} \frac{\tan x}{x}.$$

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Question 2 - Solution

We evaluate

$$\lim_{x\to 0^+} \frac{\tan x}{x}.$$

Method 1: Local Approximation

As $x \to 0$, the Taylor expansion of $\tan x$ is

$$\tan x = x + \frac{x^3}{3} + \cdots.$$

Thus,

$$\frac{\tan x}{x} = \frac{x + \frac{x^3}{3} + \cdots}{x} = 1 + \frac{x^2}{3} + \cdots.$$

As $x \to 0^+$, the higher-order terms approach 0, so

$$\lim_{x\to 0^+} \frac{\tan x}{x} = 1.$$

Method 2: Rewrite Using Sine and Cosine

Recall that

$$\tan x = \frac{\sin x}{\cos x}.$$

Then,

$$\frac{\tan x}{x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x}.$$

Now evaluate each factor as $x \to 0^+$:

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1, \quad \lim_{x\to 0^+} \cos x = 1.$$

Therefore,

$$\lim_{x\to 0^+} \frac{\tan x}{x} = 1 \cdot \frac{1}{1} = 1.$$

Conclusion

$$\boxed{\lim_{x\to 0^+} \frac{\tan x}{x} = 1}$$

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Question 3

Evaluate the following limit:

$$\lim_{x \to 3^-} \frac{1}{(x-3)^2}$$

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Question 3 - Solution

We are given:

$$\lim_{x \to 3^-} \frac{1}{(x-3)^2}$$

Step 1: Analyze the behavior of the denominator

As $x \to 3^-$, we are approaching 3 from the left. The expression $x - 3$ becomes a small negative number. But since it is squared:

$$(x-3)^2 \to 0^+ \quad \text{(very small positive number)}$$

Step 2: Behavior of the function

Since we are dividing by a very small positive number, the entire expression becomes very large:

$$\frac{1}{(x-3)^2} \to +\infty$$

Conclusion:

$$\boxed{\lim_{x \to 3^-} \frac{1}{(x-3)^2} = \infty}$$

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Question 4

Evaluate the following limit:

$$\lim_{x \to 1^+} \frac{5}{x-1}$$

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Question 4 - Solution

We are given:

$$\lim_{x \to 1^+} \frac{5}{x-1}$$

Step 1: Understand the behavior of the denominator
As $x \to 1^+$, we are approaching 1 from the right. This means $x - 1$ is a small positive number.

Step 2: Behavior of the whole expression

$$\frac{5}{x-1} \to \frac{5}{\text{small positive number}} \to +\infty$$

Conclusion:

$$\boxed{\lim_{x \to 1^+} \frac{5}{x-1} = \infty}$$

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Question 5

Evaluate the following limit:

$$\lim_{x \to 0^-} \frac{1}{x^2}$$

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Question 5 - Solution

We are given:

$$\lim_{x \to 0^-} \frac{1}{x^2}$$

Step 1: Analyze the expression
As $x \to 0^-$, x is approaching 0 from the left. So x is negative, but very close to 0.
Now, consider the square of x:

$$x^2 > 0 \text{ for all } x \neq 0$$

Even though x is negative, x^2 is positive. As $x \to 0^-$, $x^2 \to 0^+$

Step 2: Behavior of the expression

$$\frac{1}{x^2} \to \frac{1}{\text{small positive number}} \to \infty$$

Conclusion:

$$\boxed{\lim_{x \to 0^-} \frac{1}{x^2} = \infty}$$

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Question 6

Evaluate the limit:

$$\lim_{x \to -2^+} \frac{4}{(x+2)^3}$$

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Question 6 - Solution

We are asked to evaluate:

$$\lim_{x \to -2^+} \frac{4}{(x+2)^3}$$

Step 1: Analyze the denominator
As $x \to -2^+$, we approach -2 from the right, so:

$$x + 2 \to 0^+ \quad \text{(a small positive number)}$$

Now consider the cube:

$$(x+2)^3 \to 0^+ \quad \text{(still positive, but very small)}$$

Step 2: Behavior of the expression
Since the numerator is positive and the denominator approaches a very small positive number:

$$\frac{4}{(x+2)^3} \to +\infty$$

Conclusion:

$$\boxed{\lim_{x \to -2^+} \frac{4}{(x+2)^3} = \infty}$$

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Question 7

Evaluate the limit:

$$\lim_{x \to 1^-} \frac{x^2 + 3x - 4}{x - 1}$$

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Question 7 - Solution

We are given:

$$\lim_{x \to 1^-} \frac{x^2 + 3x - 4}{x - 1}$$

Step 1: Factor the numerator
Factor the quadratic:

$$x^2 + 3x - 4 = (x + 4)(x - 1)$$

So the expression becomes:

$$\frac{(x + 4)(x - 1)}{x - 1}$$

We can cancel $x - 1$ when $x \neq 1$:

$$= x + 4 \quad \text{for } x \neq 1$$

So the original expression simplifies to $x + 4$, and we can evaluate the limit directly.

Step 2: Evaluate the limit
Since we simplified the expression:

$$\lim_{x \to 1^-} (x + 4) = 1 + 4 = 5$$

Conclusion:

$$\boxed{\lim_{x \to 1^-} \frac{x^2 + 3x - 4}{x - 1} = 5}$$

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Question 8

Evaluate the limit:

$$\lim_{x \to 2^+} \frac{1}{x-2}$$

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Question 8 - Solution

We are given the limit:

$$\lim_{x \to 2^+} \frac{1}{x-2}$$

Step 1: Analyze the expression near $x = 2^+$

As x approaches 2 from the right, the value of $x - 2$ becomes a very small positive number. Therefore, $\frac{1}{x-2}$ becomes a very large positive number.

$$x \to 2^+ \quad \Rightarrow \quad x - 2 \to 0^+ \quad \Rightarrow \quad \frac{1}{x-2} \to +\infty$$

Conclusion:

$$\boxed{\lim_{x \to 2^+} \frac{1}{x-2} = +\infty}$$

This indicates that the function grows without bound as x approaches 2 from the right.

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Question 9

Evaluate the limit:

$$\lim_{x \to 0^-} \frac{2x+1}{x}$$

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Question 9 - Solution

We are asked to evaluate:

$$\lim_{x \to 0^-} \frac{2x+1}{x}$$

Step 1: Behavior near zero from the left
As $x \to 0^-$, the numerator $2x+1 \to 1$, since $2x \to 0^-$ and 1 is a constant.
The denominator $x \to 0^-$, which is a small negative number.
So the entire fraction behaves like:

$$\frac{2x+1}{x} \approx \frac{1}{\text{tiny negative}} = -\infty$$

Step 2: More formal analysis (optional)
We can write:

$$\frac{2x+1}{x} = 2 + \frac{1}{x}$$

As $x \to 0^-$, $\frac{1}{x} \to -\infty$, and the constant 2 becomes negligible in comparison.

Conclusion:

$$\boxed{\lim_{x \to 0^-} \frac{2x+1}{x} = -\infty}$$

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Question 10

Evaluate the limit:

$$\lim_{x \to 3^+} \frac{5}{x-3}$$

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Question 10 - Solution

We are given:

$$\lim_{x \to 3^+} \frac{5}{x-3}$$

Step 1: Analyze the denominator
As $x \to 3^+$, the value of x is slightly greater than 3, so $x - 3$ is a small positive number.

$$x - 3 \to 0^+ \Rightarrow \frac{1}{x-3} \to +\infty$$

Multiplying by 5, we have:

$$\frac{5}{x-3} \to +\infty$$

Conclusion:

$$\boxed{\lim_{x \to 3^+} \frac{5}{x-3} = +\infty}$$

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Question 1

Evaluate the limit:

$$\lim_{x \to \infty} \frac{6x^3 - 4x^2 + 10}{3x^3 + 5x - 7}$$

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Question 1 - Solution

We are given:

$$\lim_{x\to\infty} \frac{6x^3 - 4x^2 + 10}{3x^3 + 5x - 7}$$

Step 1: Identify the dominant terms.
As $x \to \infty$, the highest degree terms dominate: Numerator: $6x^3$, Denominator: $3x^3$
So we focus on:

$$\frac{6x^3}{3x^3} = 2$$

Step 2: Divide all terms by x^3:

$$\lim_{x\to\infty} \frac{6 - \frac{4}{x} + \frac{10}{x^3}}{3 + \frac{5}{x^2} - \frac{7}{x^3}} = \frac{6}{3} = 2$$

Conclusion:

$$\boxed{\lim_{x\to\infty} \frac{6x^3 - 4x^2 + 10}{3x^3 + 5x - 7} = 2}$$

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Question 2

Evaluate the limit:

$$\lim_{x\to\infty}\left(\sqrt{x^2+5x}-x\right)$$

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Question 2 - Solution

We are given:

$$\lim_{x\to\infty}\left(\sqrt{x^2+5x}-x\right)$$

This expression is of the indeterminate form $\infty-\infty$, so we need to rationalize:
Multiply by the conjugate:

$$\left(\sqrt{x^2+5x}-x\right)\cdot\frac{\sqrt{x^2+5x}+x}{\sqrt{x^2+5x}+x}$$

This gives:

$$\frac{(\sqrt{x^2+5x})^2-x^2}{\sqrt{x^2+5x}+x}=\frac{x^2+5x-x^2}{\sqrt{x^2+5x}+x}=\frac{5x}{\sqrt{x^2+5x}+x}$$

Now divide numerator and denominator by x:

$$\frac{5}{\sqrt{1+\frac{5}{x}}+1}$$

As $x\to\infty$, $\frac{5}{x}\to 0$, so:

$$\frac{5}{\sqrt{1+0}+1}=\frac{5}{1+1}=\frac{5}{2}$$

Final Answer:

$$\boxed{\frac{5}{2}}$$

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Question 3

Evaluate the limit:

$$\lim_{x\to\infty} \frac{3x^2 - x + 4}{5x^2 + 7x - 9}$$

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Question 3 - Solution

We are given:

$$\lim_{x\to\infty} \frac{3x^2 - x + 4}{5x^2 + 7x - 9}$$

To evaluate this limit, we divide both numerator and denominator by x^2, the highest power of x:

$$\lim_{x\to\infty} \frac{3 - \frac{1}{x} + \frac{4}{x^2}}{5 + \frac{7}{x} - \frac{9}{x^2}}$$

As $x \to \infty$, the terms $\frac{1}{x}, \frac{4}{x^2}, \frac{7}{x}, \frac{9}{x^2} \to 0$, so we get:

$$\frac{3 - 0 + 0}{5 + 0 - 0} = \frac{3}{5}$$

Final Answer:

$$\boxed{\frac{3}{5}}$$

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Question 4

Evaluate the limit:

$$\lim_{x \to -\infty} \frac{6x^3 + 4x - 1}{-2x^3 + 7x^2 + 5}$$

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Question 4 - Solution

We are given:

$$\lim_{x\to-\infty} \frac{6x^3+4x-1}{-2x^3+7x^2+5}$$

We divide both the numerator and denominator by x^3, the highest power of x:

$$= \lim_{x\to-\infty} \frac{6+\frac{4}{x^2}-\frac{1}{x^3}}{-2+\frac{7}{x}+\frac{5}{x^3}}$$

As $x \to -\infty$, all terms with negative exponents approach zero:

$$= \frac{6}{-2} = -3$$

Final Answer:

$$\boxed{-3}$$

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Question 5

Determine the limit:

$$\lim_{x\to\infty} \frac{5x^4 - 3x^2 + x + 1}{2x^3 + 4x - 7}$$

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Question 5 - Solution

We are given:

$$\lim_{x\to\infty} \frac{5x^4 - 3x^2 + x + 1}{2x^3 + 4x - 7}$$

Step 1: Compare the degrees of the numerator and denominator.

Degree of numerator: 4 , Degree of denominator: 3

Since the degree of the numerator is greater than the degree of the denominator, the limit is infinite (the function diverges).

Step 2: Determine the sign of the infinity.

As $x \to \infty$, both numerator and denominator are positive:

$$\frac{5x^4 + \text{smaller terms}}{2x^3 + \text{smaller terms}} > 0$$

So the limit tends to:

$$\boxed{+\infty}$$

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Question 6

Evaluate the limit:

$$\lim_{x\to\infty} \frac{3x^5 - 2x^2 + 1}{x^5 + 4x^3 - x}$$

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Question 6 - Solution

We are given:

$$\lim_{x\to\infty} \frac{3x^5 - 2x^2 + 1}{x^5 + 4x^3 - x}$$

Step 1: Divide numerator and denominator by the highest power of x, which is x^5:

$$= \lim_{x\to\infty} \frac{3 - \frac{2}{x^3} + \frac{1}{x^5}}{1 + \frac{4}{x^2} - \frac{1}{x^4}}$$

Step 2: As $x \to \infty$, all terms with negative powers of x go to zero:

$$= \frac{3 - 0 + 0}{1 + 0 - 0} = \boxed{3}$$

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Question 7

Evaluate the limit:

$$\lim_{x \to -\infty} \frac{5x^3 + x^2 - 7x + 1}{-2x^3 + 3x - 9}$$

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Question 7 - Solution

We are given:

$$\lim_{x\to-\infty}\frac{5x^3+x^2-7x+1}{-2x^3+3x-9}$$

Step 1: Factor out the highest degree of x from both numerator and denominator:

$$=\lim_{x\to-\infty}\frac{x^3\left(5+\frac{1}{x}-\frac{7}{x^2}+\frac{1}{x^3}\right)}{x^3\left(-2+\frac{3}{x^2}-\frac{9}{x^3}\right)}$$

Cancel the common factor of x^3:

$$=\lim_{x\to-\infty}\frac{5+\frac{1}{x}-\frac{7}{x^2}+\frac{1}{x^3}}{-2+\frac{3}{x^2}-\frac{9}{x^3}}$$

Step 2: Let $x\to-\infty$. All terms with negative powers vanish:

$$=\frac{5+0-0+0}{-2+0-0}=\boxed{-\frac{5}{2}}$$

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Question 8

Determine the limit:

$$\lim_{x\to\infty} \frac{6x^2 - x + 8}{\sqrt{x^4 + 3x^2 + 1}}$$

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Question 8 - Solution

We are given:

$$\lim_{x\to\infty} \frac{6x^2 - x + 8}{\sqrt{x^4 + 3x^2 + 1}}$$

Step 1: Rewrite the denominator using properties of radicals:

$$\sqrt{x^4 + 3x^2 + 1} = x^2\sqrt{1 + \frac{3}{x^2} + \frac{1}{x^4}}$$

So the entire expression becomes:

$$\frac{6x^2 - x + 8}{x^2\sqrt{1 + \frac{3}{x^2} + \frac{1}{x^4}}} = \frac{x^2\left(6 - \frac{1}{x} + \frac{8}{x^2}\right)}{x^2\sqrt{1 + \frac{3}{x^2} + \frac{1}{x^4}}}$$

Cancel x^2:

$$= \frac{6 - \frac{1}{x} + \frac{8}{x^2}}{\sqrt{1 + \frac{3}{x^2} + \frac{1}{x^4}}}$$

Step 2: Let $x \to \infty$:

$$= \frac{6 - 0 + 0}{\sqrt{1 + 0 + 0}} = \frac{6}{1} = \boxed{6}$$

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Question 9

Evaluate the limit:

$$\lim_{x \to -\infty} \frac{2x^3 + x^2 - 5}{x^3 - 4x + 1}$$

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Question 9 - Solution

We are given:

$$\lim_{x\to-\infty} \frac{2x^3+x^2-5}{x^3-4x+1}$$

Step 1: Factor out the highest power of x (which is x^3) from both numerator and denominator:

$$= \lim_{x\to-\infty} \frac{x^3\left(2+\frac{1}{x}-\frac{5}{x^3}\right)}{x^3\left(1-\frac{4}{x^2}+\frac{1}{x^3}\right)}$$

Cancel the common x^3 terms:

$$= \lim_{x\to-\infty} \frac{2+\frac{1}{x}-\frac{5}{x^3}}{1-\frac{4}{x^2}+\frac{1}{x^3}}$$

Step 2: Take the limit as $x \to -\infty$. All terms with $\frac{1}{x^n} \to 0$:

$$= \frac{2+0-0}{1-0+0} = \boxed{2}$$

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Question 10

Evaluate the limit:

$$\lim_{x\to\infty} \frac{5x^2+3x-7}{2x^2-x+4}$$

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Question 10 - Solution

We are given:

$$\lim_{x\to\infty} \frac{5x^2+3x-7}{2x^2-x+4}$$

Step 1: Factor out the highest power of x (which is x^2) from both the numerator and denominator:

$$= \lim_{x\to\infty} \frac{x^2\left(5+\frac{3}{x}-\frac{7}{x^2}\right)}{x^2\left(2-\frac{1}{x}+\frac{4}{x^2}\right)}$$

Step 2: Cancel the common factor x^2:

$$= \lim_{x\to\infty} \frac{5+\frac{3}{x}-\frac{7}{x^2}}{2-\frac{1}{x}+\frac{4}{x^2}}$$

Step 3: Take the limit as $x \to \infty$. Terms with $\frac{1}{x^n} \to 0$:

$$= \frac{5+0-0}{2-0+0} = \boxed{\frac{5}{2}}$$

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Question 1

Evaluate the limit:

$$\lim_{x\to\infty}\left(\sqrt{x^2+5x+7}-\sqrt{x^2+x}\right).$$

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Question 1 - Solution

We evaluate

$$\lim_{x\to\infty}\left(\sqrt{x^2+5x+7}-\sqrt{x^2+x}\right).$$

This expression has the indeterminate form $\infty-\infty$. To resolve it, we multiply by the conjugate.

Step 1: Multiply by the Conjugate

$$=\lim_{x\to\infty}\left(\sqrt{x^2+5x+7}-\sqrt{x^2+x}\right)\cdot\frac{\sqrt{x^2+5x+7}+\sqrt{x^2+x}}{\sqrt{x^2+5x+7}+\sqrt{x^2+x}}.$$

This simplifies to

$$=\lim_{x\to\infty}\frac{(x^2+5x+7)-(x^2+x)}{\sqrt{x^2+5x+7}+\sqrt{x^2+x}}.$$

Step 2: Simplify the Numerator

$$(x^2+5x+7)-(x^2+x)=4x+7,$$

so the limit becomes

$$\lim_{x\to\infty}\frac{4x+7}{\sqrt{x^2+5x+7}+\sqrt{x^2+x}}.$$

Step 3: Factor Out x

Factor x from each square root:

$$=\lim_{x\to\infty}\frac{4x+7}{x\left(\sqrt{1+\frac{5}{x}+\frac{7}{x^2}}+\sqrt{1+\frac{1}{x}}\right)}.$$

Divide numerator and denominator by x:

$$=\lim_{x\to\infty}\frac{4+\frac{7}{x}}{\sqrt{1+\frac{5}{x}+\frac{7}{x^2}}+\sqrt{1+\frac{1}{x}}}.$$

Step 4: Take the Limit

As $x\to\infty$,

$$\frac{7}{x}\to 0,\quad \frac{5}{x}\to 0,\quad \frac{7}{x^2}\to 0,\quad \frac{1}{x}\to 0.$$

Thus,

$$\lim_{x\to\infty}\frac{4+\frac{7}{x}}{\sqrt{1+\frac{5}{x}+\frac{7}{x^2}}+\sqrt{1+\frac{1}{x}}}=\frac{4}{1+1}=2.$$

Final Answer

$$\boxed{2}$$

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Question 2

Evaluate the limit:

$$\lim_{x \to -\infty} \frac{3x^2 + 7x - 5}{\sqrt{x^4 + 2x^2 + 1}}.$$

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Question 2 - Solution

We evaluate

$$\lim_{x\to-\infty} \frac{3x^2+7x-5}{\sqrt{x^4+2x^2+1}}.$$

Step 1: Rewrite the denominator carefully

Note that

$$\sqrt{x^4+2x^2+1} = \sqrt{(x^2+1)^2} = |x^2+1|.$$

Since $x^2+1>0$ for all real x,

$$|x^2+1| = x^2+1.$$

Thus, the expression becomes

$$\frac{3x^2+7x-5}{x^2+1}.$$

Step 2: Divide by the highest power of x

Divide both numerator and denominator by x^2:

$$\frac{3+\frac{7}{x}-\frac{5}{x^2}}{1+\frac{1}{x^2}}.$$

Step 3: Take the limit

As $x\to-\infty$,

$$\frac{7}{x}\to 0, \quad \frac{5}{x^2}\to 0, \quad \frac{1}{x^2}\to 0.$$

Therefore,

$$\lim_{x\to-\infty} \frac{3+\frac{7}{x}-\frac{5}{x^2}}{1+\frac{1}{x^2}} = \frac{3}{1} = 3.$$

Final Answer

$$\boxed{3}$$

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Question 3

Evaluate the limit:

$$\lim_{x\to\infty}\left(\sqrt{x^2+6x+5}-\sqrt{x^2+2x+1}\right).$$

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Question 3 - Solution

We evaluate

$$\lim_{x\to\infty}\left(\sqrt{x^2+6x+5}-\sqrt{x^2+2x+1}\right).$$

This expression has the indeterminate form $\infty-\infty$, so we rationalize.

Step 1: Multiply by the Conjugate

$$=\lim_{x\to\infty}\left(\sqrt{x^2+6x+5}-\sqrt{x^2+2x+1}\right)\cdot\frac{\sqrt{x^2+6x+5}+\sqrt{x^2+2x+1}}{\sqrt{x^2+6x+5}+\sqrt{x^2+2x+1}}.$$

This simplifies to

$$=\lim_{x\to\infty}\frac{(x^2+6x+5)-(x^2+2x+1)}{\sqrt{x^2+6x+5}+\sqrt{x^2+2x+1}}.$$

Step 2: Simplify the Numerator

$$(x^2+6x+5)-(x^2+2x+1)=4x+4,$$

so the limit becomes

$$\lim_{x\to\infty}\frac{4x+4}{\sqrt{x^2+6x+5}+\sqrt{x^2+2x+1}}.$$

Step 3: Factor Out x

Factor x from each square root:

$$=\lim_{x\to\infty}\frac{4x+4}{x\left(\sqrt{1+\frac{6}{x}+\frac{5}{x^2}}+\sqrt{1+\frac{2}{x}+\frac{1}{x^2}}\right)}.$$

Divide numerator and denominator by x:

$$=\lim_{x\to\infty}\frac{4+\frac{4}{x}}{\sqrt{1+\frac{6}{x}+\frac{5}{x^2}}+\sqrt{1+\frac{2}{x}+\frac{1}{x^2}}}.$$

Step 4: Take the Limit

As $x\to\infty$, all terms involving $\frac{1}{x}$ and higher powers go to 0. Therefore,

$$\lim_{x\to\infty}\frac{4+\frac{4}{x}}{\sqrt{1+\frac{6}{x}+\frac{5}{x^2}}+\sqrt{1+\frac{2}{x}+\frac{1}{x^2}}}=\frac{4}{1+1}=2.$$

Final Answer

$$\boxed{2}$$

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Question 4

Evaluate the limit:

$$\lim_{x \to -\infty} \frac{x^3 + 2x^2 - 5x + 7}{x^3 - 4x + 1}$$

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Question 4 - Solution

We are asked to evaluate:

$$\lim_{x\to-\infty} \frac{x^3+2x^2-5x+7}{x^3-4x+1}$$

Step 1: Compare the degrees of numerator and denominator

Both numerator and denominator are degree 3 polynomials.
When the degrees are the same, the limit as $x \to \pm\infty$ is the ratio of the leading coefficients.

Step 2: Extract leading terms

Numerator leading term: x^3 , Denominator leading term: x^3
So,

$$\lim_{x\to-\infty} \frac{x^3+2x^2-5x+7}{x^3-4x+1} = \lim_{x\to-\infty} \frac{1+\frac{2}{x}-\frac{5}{x^2}+\frac{7}{x^3}}{1-\frac{4}{x^2}+\frac{1}{x^3}}$$

As $x \to -\infty$, all terms with $\frac{1}{x^k} \to 0$ for $k > 0$
Step 3: Evaluate the limit

$$\lim_{x\to-\infty} \frac{1+0-0+0}{1-0+0} = \frac{1}{1} = 1$$

Final Answer:

$$\boxed{1}$$

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Question 5

Evaluate the limit:

$$\lim_{x\to\infty}\left(\frac{x^2+3x-7}{\sqrt{x^4+2x^3+5}}\right)$$

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Question 5 - Solution

We are given:

$$\lim_{x\to\infty}\left(\frac{x^2+3x-7}{\sqrt{x^4+2x^3+5}}\right)$$

Step 1: Understand the growth of numerator and denominator
As $x\to\infty$, both numerator and denominator tend to infinity, so we need to simplify.

Step 2: Factor out highest power of x
In the numerator: highest power is x^2 In the denominator (inside square root): highest power is x^4
Factor numerator:

$$x^2\left(1+\frac{3}{x}-\frac{7}{x^2}\right)$$

Factor denominator inside square root:

$$\sqrt{x^4\left(1+\frac{2}{x}+\frac{5}{x^4}\right)}=x^2\sqrt{1+\frac{2}{x}+\frac{5}{x^4}}$$

Step 3: Substitute factored forms into the expression**

$$\frac{x^2\left(1+\frac{3}{x}-\frac{7}{x^2}\right)}{x^2\sqrt{1+\frac{2}{x}+\frac{5}{x^4}}}$$

Cancel x^2:

$$\frac{1+\frac{3}{x}-\frac{7}{x^2}}{\sqrt{1+\frac{2}{x}+\frac{5}{x^4}}}$$

Step 4: Take the limit as $x\to\infty$
As $x\to\infty$, all terms with $\frac{1}{x}$ go to 0:

$$\lim_{x\to\infty}\frac{1+\frac{3}{x}-\frac{7}{x^2}}{\sqrt{1+\frac{2}{x}+\frac{5}{x^4}}}=\frac{1+0-0}{\sqrt{1+0+0}}=\frac{1}{1}=1$$

Final Answer:

$$\boxed{1}$$

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Question 6

Evaluate the limit:

$$\lim_{x \to -\infty} \left(\frac{3x^3 + 2x^2 - x + 5}{\sqrt{x^6 + x^2 + 1}} \right)$$

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Question 6 - Solution

We are given:

$$\lim_{x\to-\infty}\left(\frac{3x^3+2x^2-x+5}{\sqrt{x^6+x^2+1}}\right)$$

Step 1: Factor out highest powers
Numerator is a cubic polynomial: factor x^3:

$$3x^3+2x^2-x+5=x^3\left(3+\frac{2}{x}-\frac{1}{x^2}+\frac{5}{x^3}\right)$$

Denominator is a square root of a degree-6 polynomial:

$$\sqrt{x^6+x^2+1}=|x^3|\sqrt{1+\frac{1}{x^4}+\frac{1}{x^6}}$$

Note: Since $x\to-\infty$, we have $|x^3|=-x^3$
So,

$$\sqrt{x^6+x^2+1}=-x^3\sqrt{1+\frac{1}{x^4}+\frac{1}{x^6}}$$

Step 2: Substitute into the expression

$$\frac{x^3\left(3+\frac{2}{x}-\frac{1}{x^2}+\frac{5}{x^3}\right)}{-x^3\sqrt{1+\frac{1}{x^4}+\frac{1}{x^6}}}$$

Cancel x^3:

$$\frac{3+\frac{2}{x}-\frac{1}{x^2}+\frac{5}{x^3}}{-\sqrt{1+\frac{1}{x^4}+\frac{1}{x^6}}}$$

Step 3: Take the limit as $x\to-\infty$
All terms with $\frac{1}{x^n}\to 0$. So:

$$\lim_{x\to-\infty}\frac{3+\frac{2}{x}-\frac{1}{x^2}+\frac{5}{x^3}}{-\sqrt{1+\frac{1}{x^4}+\frac{1}{x^6}}}=\frac{3}{-1}=-3$$

Final Answer:

$$\boxed{-3}$$

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Question 7

Evaluate the limit:

$$\lim_{x\to\infty} x\left(\sqrt{x^2+5x}-x\right).$$

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Question 7 - Solution

We evaluate

$$\lim_{x\to\infty} x\left(\sqrt{x^2+5x}-x\right).$$

At first glance, as $x\to\infty$,

$$\sqrt{x^2+5x}\to\infty \quad \text{and} \quad x\to\infty,$$

so the expression inside the parentheses has the indeterminate form $\infty-\infty$.

Step 1: Rationalize the Expression

Multiply the expression inside the parentheses by its conjugate:

$$\sqrt{x^2+5x}-x=\frac{(\sqrt{x^2+5x}-x)(\sqrt{x^2+5x}+x)}{\sqrt{x^2+5x}+x}.$$

Using the identity $(a-b)(a+b)=a^2-b^2$, we obtain

$$=\frac{(x^2+5x)-x^2}{\sqrt{x^2+5x}+x}=\frac{5x}{\sqrt{x^2+5x}+x}.$$

Thus the original limit becomes

$$\lim_{x\to\infty} x\cdot\frac{5x}{\sqrt{x^2+5x}+x}.$$

Step 2: Simplify

Rewrite the expression as

$$\lim_{x\to\infty}\frac{5x^2}{\sqrt{x^2+5x}+x}.$$

Factor x out of the square root:

$$\sqrt{x^2+5x}=x\sqrt{1+\frac{5}{x}}\quad (x>0).$$

Substitute:

$$\frac{5x^2}{x\left(\sqrt{1+\frac{5}{x}}+1\right)}=\frac{5x}{\sqrt{1+\frac{5}{x}}+1}.$$

Step 3: Take the Limit

As $x\to\infty$, $\frac{5}{x}\to 0$, so

$$\sqrt{1+\frac{5}{x}}\to 1.$$

Therefore,

$$\lim_{x\to\infty}\frac{5x}{\sqrt{1+\frac{5}{x}}+1}=\lim_{x\to\infty}\frac{5x}{2}=\infty.$$

Final Answer

$$\boxed{\text{The limit diverges to } \infty.}$$

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Question 8

Evaluate the limit:

$$\lim_{x\to\infty} x^2\left(\sqrt{x^2+4x}-x\right).$$

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Question 8 - Solution

We evaluate

$$\lim_{x\to\infty} x^2\left(\sqrt{x^2+4x}-x\right).$$

As $x\to\infty$, both $\sqrt{x^2+4x}$ and x approach infinity, so the expression inside the parentheses has the indeterminate form $\infty-\infty$. Multiplying by x^2 significantly changes the behavior of the limit, so careful analysis is required.

Step 1: Rationalize the Expression

We first rationalize the difference:

$$\sqrt{x^2+4x}-x=\frac{(\sqrt{x^2+4x}-x)(\sqrt{x^2+4x}+x)}{\sqrt{x^2+4x}+x}.$$

Using the identity $(a-b)(a+b)=a^2-b^2$, this becomes

$$=\frac{(x^2+4x)-x^2}{\sqrt{x^2+4x}+x}=\frac{4x}{\sqrt{x^2+4x}+x}.$$

Substitute into the original expression:

$$\lim_{x\to\infty} x^2\cdot\frac{4x}{\sqrt{x^2+4x}+x}=\lim_{x\to\infty}\frac{4x^3}{\sqrt{x^2+4x}+x}.$$

Step 2: Simplify

Factor x out of the square root:

$$\sqrt{x^2+4x}=x\sqrt{1+\frac{4}{x}}\quad(x>0).$$

Then

$$\frac{4x^3}{x\left(\sqrt{1+\frac{4}{x}}+1\right)}=\frac{4x^2}{\sqrt{1+\frac{4}{x}}+1}.$$

Step 3: Take the Limit

As $x\to\infty$, $\frac{4}{x}\to 0$, so

$$\sqrt{1+\frac{4}{x}}\to 1.$$

Thus,

$$\lim_{x\to\infty}\frac{4x^2}{\sqrt{1+\frac{4}{x}}+1}=\lim_{x\to\infty}\frac{4x^2}{2}=\infty.$$

Final Answer

$$\boxed{\text{The limit diverges to } \infty.}$$

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Question 9

Evaluate the limit:

$$\lim_{x \to -\infty} \frac{x^3 + 4x^2 - 7}{2x^3 - 5x + 1}$$

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Question 9 - Solution

We are given the rational function:

$$\lim_{x\to-\infty} \frac{x^3+4x^2-7}{2x^3-5x+1}$$

—

Step 1: Analyze degrees of numerator and denominator

- The highest degree in both numerator and denominator is 3. - For limits at infinity, when the degrees match, the limit is the ratio of the leading coefficients.

So:

$$\lim_{x\to-\infty} \frac{x^3+4x^2-7}{2x^3-5x+1} = \frac{1}{2}$$

—

Step 2: Justify with algebraic manipulation (optional)

Divide numerator and denominator by x^3:

$$= \frac{1+\frac{4}{x}-\frac{7}{x^3}}{2-\frac{5}{x^2}+\frac{1}{x^3}}$$

As $x \to -\infty$, the fractions go to 0:

$$\frac{1+0-0}{2-0+0} = \frac{1}{2}$$

—

Final Answer:

$$\boxed{\frac{1}{2}}$$

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Question 10

Evaluate the limit:

$$\lim_{x\to\infty} x\left(\sqrt{x^2+3x}-x\right).$$

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Question 10 - Solution

We evaluate

$$\lim_{x\to\infty} x\left(\sqrt{x^2+3x}-x\right).$$

As $x\to\infty$, both $\sqrt{x^2+3x}$ and x grow without bound, so the expression inside the parentheses has the indeterminate form $\infty-\infty$. Multiplying by x changes the rate at which the expression grows, so careful analysis is required.

Step 1: Rationalize the Expression

We first rationalize the difference:

$$\sqrt{x^2+3x}-x=\frac{(\sqrt{x^2+3x}-x)(\sqrt{x^2+3x}+x)}{\sqrt{x^2+3x}+x}.$$

Using the identity $(a-b)(a+b)=a^2-b^2$, this becomes

$$=\frac{(x^2+3x)-x^2}{\sqrt{x^2+3x}+x}=\frac{3x}{\sqrt{x^2+3x}+x}.$$

Substitute this into the original limit:

$$\lim_{x\to\infty} x\cdot\frac{3x}{\sqrt{x^2+3x}+x}=\lim_{x\to\infty}\frac{3x^2}{\sqrt{x^2+3x}+x}.$$

Step 2: Simplify

Factor x out of the square root:

$$\sqrt{x^2+3x}=x\sqrt{1+\frac{3}{x}}\quad(x>0).$$

Then

$$\frac{3x^2}{x\left(\sqrt{1+\frac{3}{x}}+1\right)}=\frac{3x}{\sqrt{1+\frac{3}{x}}+1}.$$

Step 3: Take the Limit

As $x\to\infty$, $\frac{3}{x}\to 0$, so

$$\sqrt{1+\frac{3}{x}}\to 1.$$

Therefore,

$$\lim_{x\to\infty}\frac{3x}{\sqrt{1+\frac{3}{x}}+1}=\lim_{x\to\infty}\frac{3x}{2}=\infty.$$

Final Answer

$$\boxed{\text{The limit diverges to } \infty.}$$

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Question 1

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} kx^2 + 3x + 1, & \text{if } x < 2 \\ x^3 - 3kx + c, & \text{if } x \geq 2 \end{cases}$$

Find the values of constants k and c such that $f(x)$ is continuous and differentiable at $x = 2$.

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Question 1 - Solution

We are given:

$$f(x) = \begin{cases} kx^2 + 3x + 1, & \text{if } x < 2 \\ x^3 - 3kx + c, & \text{if } x \geq 2 \end{cases}$$

We want $f(x)$ to be both:
1. Continuous at $x = 2$ 2. Differentiable at $x = 2$

Step 1: Continuity at $x = 2$
For continuity, we need:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

Left-hand limit:

$$\lim_{x \to 2^-} f(x) = k(2)^2 + 3(2) + 1 = 4k + 6 + 1 = 4k + 7$$

Right-hand limit:

$$\lim_{x \to 2^+} f(x) = (2)^3 - 3k(2) + c = 8 - 6k + c$$

Set them equal:

$$4k + 7 = 8 - 6k + c \quad (1)$$

Step 2: Differentiability at $x = 2$

Derivatives:
Left: $f'(x) = 2kx + 3 \Rightarrow f'(2^-) = 4k + 3$
Right: $f'(x) = 3x^2 - 3k \Rightarrow f'(2^+) = 12 - 3k$
Set equal:

$$4k + 3 = 12 - 3k \quad (2)$$

Step 3: Solve the system
From (2):

$$4k + 3 = 12 - 3k \Rightarrow 7k = 9 \Rightarrow k = \frac{9}{7}$$

Substitute into (1):

$$4\left(\frac{9}{7}\right) + 7 = 8 - 6\left(\frac{9}{7}\right) + c \Rightarrow \frac{36}{7} + 7 = 8 - \frac{54}{7} + c \Rightarrow \frac{85}{7} = \frac{2}{7} + c \Rightarrow c = \frac{83}{7}$$

Final Answer:

$$\boxed{k = \frac{9}{7}, \quad c = \frac{83}{7}}$$

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Question 2

For real constants a and b, let

$$f(x) = \begin{cases} \dfrac{\sin(ax)}{x}, & x \neq 0, \\ b, & x = 0. \end{cases}$$

Classify the pairs (a, b) for which f is continuous at zero and those for which it has a removable discontinuity there.

State how to remove the discontinuity.

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Question 2 - Solution

For $a \neq 0$, write

$$\frac{\sin(ax)}{x} = a\frac{\sin(ax)}{ax} \longrightarrow a.$$

For $a = 0$, the quotient is identically zero away from zero, so the same limit holds.
Thus $\lim_{x\to 0} f(x) = a$ for every real a.
Since $f(0) = b$, continuity holds exactly when $\boxed{b = a}$.
When $\boxed{b \neq a}$ the limit exists but differs from $f(0)$, so the discontinuity is removable by resetting $f(0) = a$.
A function cannot be both continuous and discontinuous at the same point.

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Question 3

Let the function $f(x)$ be defined by:

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ k, & x = 2 \end{cases}$$

Determine the value of k such that $f(x)$ is continuous at $x = 2$.

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Question 3 - Solution

We are given:

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ k, & x = 2 \end{cases}$$

To make $f(x)$ continuous at $x = 2$, the following condition must hold:

$$\lim_{x \to 2} f(x) = f(2)$$

Step 1: Compute the limit as $x \to 2$
Factor the numerator:

$$x^2 - 4 = (x-2)(x+2)$$

For $x \neq 2$, the function simplifies to:

$$f(x) = x + 2$$

So,

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x+2) = 4$$

Step 2: Match the function value
Continuity requires:

$$f(2) = k = 4$$

Final Answer:

$$\boxed{k = 4}$$

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Question 4

Consider the function

$$f(x) = \begin{cases} \frac{\sin(3x)}{x}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

Find the value of c that makes $f(x)$ continuous at $x = 0$.

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Question 4 - Solution

We are given:

$$f(x) = \begin{cases} \frac{\sin(3x)}{x}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

We want $f(x)$ to be continuous at $x = 0$. This means:

$$\lim_{x \to 0} f(x) = f(0) = c$$

Step 1: Compute the limit as $x \to 0$
We use the standard limit identity:

$$\lim_{x \to 0} \frac{\sin(kx)}{x} = k$$

So, in our case:

$$\lim_{x \to 0} \frac{\sin(3x)}{x} = 3$$

Step 2: Set the limit equal to the function value
To ensure continuity at $x = 0$, we must have:

$$c = 3$$

Final Answer:

$$\boxed{c = 3}$$

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Question 5

Let the function $f(x)$ be defined as

$$f(x) = \begin{cases} kx^2 + 2x, & x < 3 \\ x^2 - 4kx + 9, & x \geq 3 \end{cases}$$

Find the value of k that makes $f(x)$ continuous at $x = 3$.

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Question 5 - Solution

We are given:

$$f(x) = \begin{cases} kx^2 + 2x, & x < 3 \\ x^2 - 4kx + 9, & x \geq 3 \end{cases}$$

To ensure continuity at $x = 3$, the left-hand and right-hand limits must agree, and they must equal the function value:

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$$

Step 1: Left-hand limit as $x \to 3^-$
From the first piece:

$$\lim_{x \to 3^-} f(x) = k(3)^2 + 2(3) = 9k + 6$$

Step 2: Right-hand limit and $f(3)$
From the second piece:

$$f(3) = (3)^2 - 4k(3) + 9 = 9 - 12k + 9 = 18 - 12k$$

Step 3: Set the two expressions equal for continuity

$$9k + 6 = 18 - 12k$$

Solve for k:

$$9k + 12k = 18 - 621k = 12k = \frac{12}{21} = \frac{4}{7}$$

Final Answer:

$$\boxed{k = \frac{4}{7}}$$

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Question 6

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} \frac{\sin(ax)}{x}, & x \neq 0 \\ b, & x = 0 \end{cases}$$

Find the values of constants a and b such that $f(x)$ is continuous at $x = 0$.

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Question 6 - Solution

To ensure continuity at $x = 0$, we must have:

$$\lim_{x \to 0} f(x) = f(0) = b$$

We are given:

$$f(x) = \frac{\sin(ax)}{x}, \quad x \neq 0$$
$$f(0) = b$$

We evaluate the limit:

$$\lim_{x \to 0} \frac{\sin(ax)}{x} = \lim_{x \to 0} a \cdot \frac{\sin(ax)}{ax} = a \cdot \lim_{u \to 0} \frac{\sin u}{u} = a \cdot 1 = a$$

So for continuity, we must have:

$$\lim_{x \to 0} f(x) = a = b$$

Final Answer:

$$\boxed{\text{Any } a \in \mathbb{R} \text{ and } b = a}$$

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Question 7

Let the function $f(x)$ be defined by

$$f(x) = \begin{cases} x^2 - 4, & x < 2 \\ kx + 1, & x \geq 2 \end{cases}$$

Determine the value of k that makes $f(x)$ continuous at $x = 2$.

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Question 7 - Solution

We are given a piecewise function:

$$f(x) = \begin{cases} x^2 - 4, & x < 2 \\ kx + 1, & x \geq 2 \end{cases}$$

To ensure continuity at $x = 2$, we require:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

Evaluate the left-hand limit:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 - 4) = 4 - 4 = 0$$

Evaluate the right-hand limit:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (kx + 1) = 2k + 1$$

Set the two limits equal for continuity:

$$2k + 1 = 0 \quad \Rightarrow \quad k = -\frac{1}{2}$$

Final Answer:

$$\boxed{k = -\frac{1}{2}}$$

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Question 8

Let

$$f(x) = \begin{cases} \frac{\sin(3x)}{x}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

Find the value of c that makes $f(x)$ continuous at $x = 0$.

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Question 8 - Solution

To ensure continuity at $x = 0$, we require:

$$\lim_{x \to 0} f(x) = f(0) = c$$

Compute the limit:

$$\lim_{x \to 0} \frac{\sin(3x)}{x} = \lim_{x \to 0} \frac{3\sin(3x)}{3x} = 3 \cdot \lim_{x \to 0} \frac{\sin(3x)}{3x} = 3 \cdot 1 = 3$$

Thus, to make f continuous at $x = 0$, we must set:

$$\boxed{c = 3}$$

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Question 9

Let

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ k, & x = 2 \end{cases}$$

Determine the value of k such that $f(x)$ is continuous at $x = 2$.

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Question 9 - Solution

We want f to be continuous at $x = 2$, so we must have:

$$\lim_{x \to 2} f(x) = f(2) = k$$

First, compute the limit:

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2}$$

For $x \neq 2$, we can cancel the common factor:

$$f(x) = x + 2$$

Therefore:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x + 2) = 4$$

To make f continuous at $x = 2$, we set:

$$\boxed{k = 4}$$

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Question 10

Let

$$f(x) = \begin{cases} \frac{\sin(3x)}{x}, & x \neq 0 \\ a, & x = 0 \end{cases}$$

(a) Find the value of a such that $f(x)$ is continuous at $x = 0$.

(b) Is $f(x)$ differentiable at $x = 0$? Justify your answer.

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Question 10 - Solution

(a) For continuity at $x = 0$, we must have

$$\lim_{x\to 0} f(x) = f(0) = a.$$

We compute the limit:

$$\lim_{x\to 0} \frac{\sin(3x)}{x} = \lim_{x\to 0} 3 \cdot \frac{\sin(3x)}{3x} = 3 \cdot 1 = 3.$$

Thus, to make $f(x)$ continuous at $x = 0$, we must set

$$\boxed{a = 3}.$$

(b) Differentiability at $x = 0$:
We use the definition of the derivative:

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{\frac{\sin(3h)}{h} - 3}{h}.$$

Rewrite the expression:

$$\frac{\frac{\sin(3h)}{h} - 3}{h} = \frac{\sin(3h) - 3h}{h^2}.$$

Rescale by $u = 3h$ (so $u^2 = 9h^2$):

$$\frac{\sin(3h) - 3h}{h^2} = 9 \cdot \frac{\sin(3h) - 3h}{(3h)^2}.$$

Let $u = 3h$. Then as $h \to 0$, $u \to 0$, and the limit becomes

$$f'(0) = 9 \lim_{u\to 0} \frac{\sin u - u}{u^2}.$$

Using the standard limit

$$\lim_{u\to 0} \frac{\sin u - u}{u^2} = 0,$$

we conclude that

$$f'(0) = 0.$$

Conclusion: The limit exists, so $f(x)$ is differentiable at $x = 0$, and

$$\boxed{f'(0) = 0}.$$

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Question 1

Let

$$f(x) = 4x - 1$$

Use the ε-δ definition of the limit to prove that:

$$\lim_{x \to 2} f(x) = 7$$

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Question 1 - Solution

We are given:

$$f(x) = 4x - 1, \quad \lim_{x \to 2} f(x) = 7$$

We will prove this using the ε-δ definition:
Goal: For every $\varepsilon > 0$, we must find $\delta > 0$ such that:

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |f(x) - 7| < \varepsilon$$

Step 1: Write out $|f(x) - 7|$:

$$|f(x) - 7| = |4x - 1 - 7| = |4x - 8| = 4|x - 2|$$

Step 2: Make $4|x - 2| < \varepsilon$:
We can achieve this by choosing:

$$|x - 2| < \frac{\varepsilon}{4}$$

Step 3: Let $\delta = \frac{\varepsilon}{4}$.
Then, for all x such that $0 < |x - 2| < \delta$, we have:

$$|f(x) - 7| = 4|x - 2| < 4\delta = 4 \cdot \frac{\varepsilon}{4} = \varepsilon$$

Conclusion:

Therefore, $\lim_{x \to 2} f(x) = 7$ by the ε-δ definition.

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Question 2

Let

$$f(x) = x^2 + 3x$$

Use the ε-δ definition of a limit to prove that:

$$\lim_{x \to 1} f(x) = 4$$

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Question 2 - Solution

We are given:

$$f(x) = x^2 + 3x, \quad \lim_{x \to 1} f(x) = 4$$

We want to prove using the ε-δ definition:
Goal: For every $\varepsilon > 0$, we must find $\delta > 0$ such that:

$$0 < |x - 1| < \delta \quad \Rightarrow \quad |f(x) - 4| < \varepsilon$$

Step 1: Write the expression $|f(x) - 4|$:

$$|f(x) - 4| = |x^2 + 3x - 4|$$

Factor the expression:

$$x^2 + 3x - 4 = (x - 1)(x + 4) \Rightarrow |f(x) - 4| = |x - 1||x + 4|$$

We want:

$$|x - 1||x + 4| < \varepsilon$$

Step 2: Estimate $|x + 4|$:
Assume $|x - 1| < 1 \Rightarrow x \in (0, 2) \Rightarrow x + 4 \in (4, 6) \Rightarrow |x + 4| < 6$
Then:

$$|f(x) - 4| = |x - 1||x + 4| < 6|x - 1|$$

To make $|f(x) - 4| < \varepsilon$, it is enough to make:

$$6|x - 1| < \varepsilon \Rightarrow |x - 1| < \frac{\varepsilon}{6}$$

Step 3: Let $\delta = \min\left\{1, \frac{\varepsilon}{6}\right\}$
Then if $0 < |x - 1| < \delta$, we have:

$$|f(x) - 4| < \varepsilon$$

Conclusion:

$$\text{Therefore, } \lim_{x \to 1}(x^2 + 3x) = 4 \text{ by the } \varepsilon\text{-}\delta \text{ definition.}$$

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Question 3

Let

$$f(x) = 5x - 1$$

Prove, using the ε-δ definition of a limit, that:

$$\lim_{x \to 2} f(x) = 9$$

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Question 3 - Solution

We are given:

$$f(x) = 5x - 1, \quad \lim_{x \to 2} f(x) = 9$$

We will use the ε-δ definition of a limit:
Goal: For every $\varepsilon > 0$, find $\delta > 0$ such that:

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |f(x) - 9| < \varepsilon$$

Step 1: Express $|f(x) - 9|$:

$$|f(x) - 9| = |5x - 1 - 9| = |5x - 10| = 5|x - 2|$$

We want:

$$5|x - 2| < \varepsilon \quad \Rightarrow \quad |x - 2| < \frac{\varepsilon}{5}$$

Step 2: Choose δ:
Let:

$$\delta = \frac{\varepsilon}{5}$$

Then if $0 < |x - 2| < \delta$, we have:

$$|f(x) - 9| = 5|x - 2| < 5\delta = \varepsilon$$

Conclusion:

$$\text{Therefore, } \lim_{x \to 2}(5x - 1) = 9 \text{ by the } \varepsilon\text{-}\delta \text{ definition.}$$

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Question 4

Let

$$f(x) = \frac{1}{x}$$

Prove, using the ε-δ definition of a limit, that:

$$\lim_{x \to 1} \frac{1}{x} = 1$$

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Question 4 - Solution

We are given:

$$f(x) = \frac{1}{x}, \quad \lim_{x \to 1} f(x) = 1$$

We will prove this using the ε-δ definition of the limit.
Goal: For every $\varepsilon > 0$, find $\delta > 0$ such that:

$$0 < |x - 1| < \delta \quad \Rightarrow \quad \left| \frac{1}{x} - 1 \right| < \varepsilon$$

Step 1: Manipulate the expression
We begin by simplifying:

$$\left| \frac{1}{x} - 1 \right| = \left| \frac{1 - x}{x} \right| = \frac{|x - 1|}{|x|}$$

So we want:

$$\frac{|x - 1|}{|x|} < \varepsilon$$

Step 2: Bound $|x|$ away from zero
Choose $\delta \leq \frac{1}{2}$, so $|x - 1| < \frac{1}{2} \Rightarrow \frac{1}{2} < x < \frac{3}{2}$
This gives us a lower bound: $|x| > \frac{1}{2}$
So:

$$\frac{|x - 1|}{|x|} < \frac{|x - 1|}{1/2} = 2|x - 1|$$

Step 3: Ensure $2|x - 1| < \varepsilon$
To make this less than ε, we require:

$$2|x - 1| < \varepsilon \quad \Rightarrow \quad |x - 1| < \frac{\varepsilon}{2}$$

Step 4: Choose δ
Let:

$$\delta = \min\left(\frac{1}{2}, \frac{\varepsilon}{2}\right)$$

Then whenever $0 < |x - 1| < \delta$, we get:

$$\left| \frac{1}{x} - 1 \right| = \frac{|x - 1|}{|x|} < 2|x - 1| < 2\delta \leq \varepsilon$$

Conclusion:
Thus, by the ε-δ definition of a limit,

$$\boxed{\lim_{x \to 1} \frac{1}{x} = 1}$$

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Question 5

Let

$$f(x) = 5x - 3$$

Prove, using the ε-δ definition of a limit, that:

$$\lim_{x \to 2}(5x - 3) = 7$$

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Question 5 - Solution

We are given:

$$f(x) = 5x - 3$$

We want to prove:

$$\lim_{x \to 2}(5x - 3) = 7$$

using the formal ε-δ definition of the limit.

Definition: For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |(5x - 3) - 7| < \varepsilon$$

Step 1: Simplify the expression

$$|(5x - 3) - 7| = |5x - 10| = 5|x - 2|$$

So we want:

$$5|x - 2| < \varepsilon \quad \Rightarrow \quad |x - 2| < \frac{\varepsilon}{5}$$

Step 2: Choose δ
Let:

$$\delta = \frac{\varepsilon}{5}$$

Step 3: Conclusion
Then whenever $0 < |x - 2| < \delta$, we have:

$$|(5x - 3) - 7| = 5|x - 2| < 5\delta = \varepsilon$$

Therefore, by the ε-δ definition,

$$\boxed{\lim_{x \to 2}(5x - 3) = 7}$$

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Question 6

Let

$$f(x) = \frac{1}{2}x + 4$$

Prove using the ε-δ definition of a limit that:

$$\lim_{x \to -2} \left(\frac{1}{2}x + 4 \right) = 3$$

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Question 6 - Solution

We are given:

$$f(x) = \frac{1}{2}x + 4$$

and want to prove:

$$\lim_{x \to -2} \left(\frac{1}{2}x + 4 \right) = 3$$

using the formal ε-δ definition of a limit.

Definition: For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x + 2| < \delta \quad \Rightarrow \quad \left| \left(\frac{1}{2}x + 4 \right) - 3 \right| < \varepsilon$$

Step 1: Simplify the expression

$$\left| \left(\frac{1}{2}x + 4 \right) - 3 \right| = \left| \frac{1}{2}x + 1 \right| = \left| \frac{1}{2}(x + 2) \right| = \frac{1}{2}|x + 2|$$

So we want:

$$\frac{1}{2}|x + 2| < \varepsilon \quad \Rightarrow \quad |x + 2| < 2\varepsilon$$

Step 2: Choose δ

Let:

$$\delta = 2\varepsilon$$

Step 3: Conclusion

Then whenever $0 < |x + 2| < \delta$, we have:

$$\left| \left(\frac{1}{2}x + 4 \right) - 3 \right| = \frac{1}{2}|x + 2| < \frac{1}{2} \cdot \delta = \frac{1}{2}(2\varepsilon) = \varepsilon$$

Therefore, by the ε-δ definition,

$$\boxed{\lim_{x \to -2} \left(\frac{1}{2}x + 4 \right) = 3}$$

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Question 7

Let

$$f(x) = 3x - 1$$

Use the ε-δ definition of a limit to prove that:

$$\lim_{x \to 2} f(x) = 5$$

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Question 7 - Solution

We are given the function:

$$f(x) = 3x - 1$$

and we are asked to prove:

$$\lim_{x \to 2}(3x - 1) = 5$$

using the ε-δ definition.

Step 1: Recall the definition.

We must show that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |(3x - 1) - 5| < \varepsilon$$

Step 2: Simplify the expression inside the absolute value.

$$|(3x - 1) - 5| = |3x - 6| = 3|x - 2|$$

So we require:

$$3|x - 2| < \varepsilon \quad \Rightarrow \quad |x - 2| < \frac{\varepsilon}{3}$$

Step 3: Choose δ.

Let

$$\delta = \frac{\varepsilon}{3}$$

Step 4: Verification.

If $0 < |x - 2| < \delta = \frac{\varepsilon}{3}$, then:

$$|f(x) - 5| = |3x - 6| = 3|x - 2| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon$$

Conclusion: By the ε-δ definition, we have:

$$\boxed{\lim_{x \to 2}(3x - 1) = 5}$$

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Question 8

Let

$$f(x) = x^2$$

Use the ε-δ definition of a limit to prove that:

$$\lim_{x \to 3} f(x) = 9$$

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Question 8 - Solution

We are given:

$$f(x) = x^2, \quad \lim_{x \to 3} x^2 = 9$$

We want to prove that:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - 3| < \delta \Rightarrow |x^2 - 9| < \varepsilon$$

Step 1: Manipulate the expression.

$$|x^2 - 9| = |x - 3||x + 3|$$

We want to bound this expression. Assume $|x - 3| < 1$. Then:

$$2 < x < 4 \Rightarrow 5 < x + 3 < 7 \Rightarrow |x + 3| < 7$$

So:

$$|x^2 - 9| = |x - 3||x + 3| < |x - 3| \cdot 7$$

We want:

$$|x - 3| \cdot 7 < \varepsilon \Rightarrow |x - 3| < \frac{\varepsilon}{7}$$

Step 2: Choose δ.
Let:

$$\delta = \min\left(1, \frac{\varepsilon}{7}\right)$$

Step 3: Verify.
If $0 < |x - 3| < \delta$, then:

$$|x^2 - 9| = |x - 3||x + 3| < \delta \cdot 7 \leq \frac{\varepsilon}{7} \cdot 7 = \varepsilon$$

Conclusion: By the ε-δ definition of the limit:

$$\boxed{\lim_{x \to 3} x^2 = 9}$$

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Question 9

Let

$$f(x) = 5x - 1$$

Use the ε-δ definition of a limit to prove that:

$$\lim_{x \to 2} f(x) = 9$$

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Question 9 - Solution

We are given:

$$f(x) = 5x - 1, \quad \lim_{x \to 2}(5x - 1) = 9$$

We want to show that:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that if } 0 < |x - 2| < \delta, \text{ then } |f(x) - 9| < \varepsilon$$

Step 1: Work with the expression $|f(x) - 9|$

$$|f(x) - 9| = |5x - 1 - 9| = |5x - 10| = 5|x - 2|$$

We want:

$$5|x - 2| < \varepsilon \Rightarrow |x - 2| < \frac{\varepsilon}{5}$$

Step 2: Choose δ
Let:

$$\delta = \frac{\varepsilon}{5}$$

Step 3: Conclusion
Then for all $\varepsilon > 0$, if $0 < |x - 2| < \delta$, we have:

$$|f(x) - 9| = 5|x - 2| < 5 \cdot \delta = \varepsilon$$

Therefore, by the ε-δ definition of the limit:

$$\boxed{\lim_{x \to 2}(5x - 1) = 9}$$

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Question 10

Let

$$f(x) = \frac{x^2 - 1}{x - 1}$$

Use the ε-δ definition of a limit to prove:

$$\lim_{x \to 1} f(x) = 2$$

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Question 10 - Solution

We are given:

$$f(x) = \frac{x^2 - 1}{x - 1}$$

Note that:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1 \quad \text{for } x \neq 1$$

So for $x \neq 1$, $f(x) = x + 1$, and:

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} (x + 1) = 2$$

We now prove this using the ε-δ definition.
We want to show:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - 1| < \delta \Rightarrow |f(x) - 2| < \varepsilon$$

Since $f(x) = x + 1$ for $x \neq 1$, we compute:

$$|f(x) - 2| = |x + 1 - 2| = |x - 1|$$

So:

$$|f(x) - 2| < \varepsilon \quad \text{whenever} \quad |x - 1| < \varepsilon$$

Step: Choose $\delta = \varepsilon$
Then if $0 < |x - 1| < \delta$, we get:

$$|f(x) - 2| = |x - 1| < \varepsilon$$

Conclusion: By the ε-δ definition, we have:

$$\boxed{\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2}$$

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Question 1

Let

$$f(x) = \frac{1}{x+2}$$

Use the definition of the derivative to compute $f'(1)$. Show all steps.

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Question 1 - Solution

We are given:

$$f(x) = \frac{1}{x+2}$$

Using the definition of the derivative:

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

Compute:

$$f(1+h) = \frac{1}{(1+h)+2} = \frac{1}{h+3}, \quad f(1) = \frac{1}{1+2} = \frac{1}{3}$$

So:

$$f'(1) = \lim_{h\to 0} \frac{\frac{1}{h+3} - \frac{1}{3}}{h}$$

Find a common denominator for the numerator:

$$\frac{1}{h+3} - \frac{1}{3} = \frac{3-(h+3)}{3(h+3)} = \frac{-h}{3(h+3)}$$

Now plug back into the limit:

$$f'(1) = \lim_{h\to 0} \frac{-h}{3(h+3)} \cdot \frac{1}{h} = \lim_{h\to 0} \frac{-1}{3(h+3)}$$

Now evaluate the limit:

$$f'(1) = \frac{-1}{3(0+3)} = \boxed{-\frac{1}{9}}$$

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Question 2

Let

$$f(x) = \sqrt{4x + 1}$$

Use the definition of the derivative to compute $f'(2)$. Show all steps clearly without using shortcut rules.

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Question 2 - Solution

We are given:

$$f(x) = \sqrt{4x+1}$$

Using the limit definition of the derivative:

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

Compute:

$$f(2+h) = \sqrt{4(2+h)+1} = \sqrt{8+4h+1} = \sqrt{4h+9}$$

$$f(2) = \sqrt{4(2)+1} = \sqrt{9} = 3$$

So,

$$f'(2) = \lim_{h\to 0} \frac{\sqrt{4h+9} - 3}{h}$$

To evaluate this limit, multiply numerator and denominator by the conjugate of the numerator:

$$f'(2) = \lim_{h\to 0} \frac{\sqrt{4h+9} - 3}{h} \cdot \frac{\sqrt{4h+9} + 3}{\sqrt{4h+9} + 3}$$

Apply difference of squares in the numerator:

$$= \lim_{h\to 0} \frac{(4h+9) - 9}{h(\sqrt{4h+9} + 3)} = \lim_{h\to 0} \frac{4h}{h(\sqrt{4h+9} + 3)}$$

Cancel h in numerator and denominator:

$$= \lim_{h\to 0} \frac{4}{\sqrt{4h+9} + 3}$$

Now evaluate the limit:

$$f'(2) = \frac{4}{\sqrt{9}+3} = \frac{4}{3+3} = \boxed{\frac{2}{3}}$$

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Question 3

Let

$$f(x) = \frac{1}{x+3}$$

Use the definition of the derivative to compute $f'(1)$. Show all steps clearly without using differentiation shortcuts.

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Question 3 - Solution

We are given:

$$f(x) = \frac{1}{x+3}$$

Using the definition of the derivative:

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

Compute:

$$f(1+h) = \frac{1}{1+h+3} = \frac{1}{h+4}, \quad f(1) = \frac{1}{4}$$

So,

$$f'(1) = \lim_{h\to 0} \frac{\frac{1}{h+4} - \frac{1}{4}}{h}$$

Combine the terms in the numerator:

$$= \lim_{h\to 0} \frac{\frac{4-(h+4)}{4(h+4)}}{h} = \lim_{h\to 0} \frac{\frac{-h}{4(h+4)}}{h}$$

Simplify:

$$= \lim_{h\to 0} \frac{-1}{4(h+4)}$$

Now take the limit:

$$f'(1) = \frac{-1}{4(4)} = \boxed{-\frac{1}{16}}$$

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Question 4

Let

$$f(x) = x^2$$

Use the definition of the derivative to compute $f'(3)$. Do not use differentiation rules.

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Question 4 - Solution

We are given:

$$f(x) = x^2$$

Using the definition of the derivative at $x = 3$:

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

Compute each term:

$$f(3+h) = (3+h)^2 = 9 + 6h + h^2$$

$$f(3) = 3^2 = 9$$

Substitute into the limit:

$$f'(3) = \lim_{h \to 0} \frac{(9 + 6h + h^2) - 9}{h} = \lim_{h \to 0} \frac{6h + h^2}{h}$$

Factor h from the numerator:

$$= \lim_{h \to 0} \frac{h(6+h)}{h}$$

Cancel h:

$$= \lim_{h \to 0} (6+h)$$

Now take the limit:

$$f'(3) = \boxed{6}$$

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Question 5

Let

$$f(x) = \frac{1}{x}$$

Use the definition of the derivative to find an expression for $f'(x)$. Then evaluate $f'(2)$ using your expression.

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Question 5 - Solution

We are given:

$$f(x) = \frac{1}{x}$$

Use the definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

Combine the terms in the numerator:

$$= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{\frac{-h}{x(x+h)}}{h}$$

Simplify:

$$= \lim_{h\to 0} \frac{-1}{x(x+h)}$$

Take the limit:

$$f'(x) = \frac{-1}{x^2}$$

Now evaluate at $x = 2$:

$$f'(2) = \frac{-1}{2^2} = \frac{-1}{4}$$

Final Answer:

$$f'(x) = \frac{-1}{x^2}, \quad f'(2) = \boxed{-\frac{1}{4}}$$

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Question 6

Let

$$f(x) = \sqrt{x+1}$$

Use the definition of the derivative to find $f'(x)$. Show all steps clearly without using derivative shortcuts.

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Question 6 - Solution

We are given:

$$f(x) = \sqrt{x+1}$$

Use the definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$$

To simplify the numerator, multiply by the conjugate:

$$\lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}}$$

This becomes:

$$= \lim_{h\to 0} \frac{(x+h+1) - (x+1)}{h\left(\sqrt{x+h+1} + \sqrt{x+1}\right)} = \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h+1} + \sqrt{x+1}\right)}$$

Cancel the h's:

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}}$$

Now take the limit as $h \to 0$:

$$f'(x) = \frac{1}{2\sqrt{x+1}}$$

Final Answer:

$$f'(x) = \boxed{\frac{1}{2\sqrt{x+1}}}$$

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Question 7

Let

$$f(x) = \frac{1}{x}$$

Use the definition of the derivative to compute $f'(x)$. Show all steps clearly without using derivative rules.

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Question 7 - Solution

We are given:

$$f(x) = \frac{1}{x}$$

Use the definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

Combine the terms in the numerator:

$$= \lim_{h \to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h \to 0} \frac{\frac{-h}{x(x+h)}}{h}$$

Simplify:

$$= \lim_{h \to 0} \frac{-1}{x(x+h)}$$

Now take the limit as $h \to 0$:

$$f'(x) = \frac{-1}{x^2}$$

Final Answer:

$$f'(x) = \boxed{-\frac{1}{x^2}}$$

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Question 8

Let

$$f(x) = \sqrt{x}$$

Use the definition of the derivative to compute $f'(x)$. Do not use derivative shortcuts. Show all algebraic simplification and steps clearly.

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Question 8 - Solution

We are given:

$$f(x) = \sqrt{x}$$

Use the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Multiply numerator and denominator by the conjugate:

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

Simplify numerator:

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

Now take the limit as $h \to 0$:

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Final Answer:

$$f'(x) = \boxed{\frac{1}{2\sqrt{x}}}$$

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Question 9

Let

$$f(x) = \sqrt{x^2 + 4x + 5}$$

Task: Use the *definition of the derivative* to find $f'(x)$.

- Use first principles only.
- Do not apply derivative shortcuts.
- Show all algebraic steps.

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Question 9 - Solution

We are given:

$$f(x) = \sqrt{x^2 + 4x + 5}$$

Apply the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{(x+h)^2 + 4(x+h) + 5} - \sqrt{x^2 + 4x + 5}}{h}$$

Expand the expression inside the first square root:

$$(x+h)^2 + 4(x+h) + 5 = x^2 + 2xh + h^2 + 4x + 4h + 5$$

So we have:

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} - \sqrt{x^2 + 4x + 5}}{h}$$

Multiply numerator and denominator by the conjugate:

$$= \lim_{h \to 0} \frac{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} - \sqrt{x^2 + 4x + 5}}{h} \cdot \frac{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5}}{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5}}$$

Use the identity $(a-b)(a+b) = a^2 - b^2$:

$$= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2 + 4x + 4h + 5) - (x^2 + 4x + 5)}{h\left(\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5}\right)}$$

Simplify the numerator:

$$= \lim_{h \to 0} \frac{2xh + h^2 + 4h}{h\left(\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5}\right)}$$

Factor h from the numerator:

$$= \lim_{h \to 0} \frac{h(2x + h + 4)}{h\left(\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5}\right)}$$

Cancel h:

$$= \lim_{h \to 0} \frac{2x + h + 4}{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5}}$$

Now take the limit as $h \to 0$:

$$f'(x) = \frac{2x + 4}{2\sqrt{x^2 + 4x + 5}}$$

Final Answer:

$$f'(x) = \boxed{\frac{2x + 4}{2\sqrt{x^2 + 4x + 5}}}$$

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Question 10

Let

$$f(x) = \sqrt{x+3}$$

Use the definition of the derivative to find $f'(x)$. Do not apply any derivative shortcuts — use first principles only.

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Question 10 - Solution

We are given:

$$f(x) = \sqrt{x+3}$$

Apply the definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h}$$

To evaluate this limit, multiply numerator and denominator by the conjugate of the numerator:

$$= \lim_{h\to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h} \cdot \frac{\sqrt{x+h+3} + \sqrt{x+3}}{\sqrt{x+h+3} + \sqrt{x+3}}$$

Use the identity $(a-b)(a+b) = a^2 - b^2$:

$$= \lim_{h\to 0} \frac{(x+h+3) - (x+3)}{h(\sqrt{x+h+3} + \sqrt{x+3})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+3} + \sqrt{x+3})}$$

Cancel h in numerator and denominator:

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h+3} + \sqrt{x+3}}$$

Now take the limit as $h \to 0$:

$$f'(x) = \frac{1}{2\sqrt{x+3}}$$

Final Answer:

$$f'(x) = \boxed{\frac{1}{2\sqrt{x+3}}}$$

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Question 1

The position of a particle moving along a straight line is given by

$$s(t) = t^3 - 6t^2 + 9t \quad \text{for } t \geq 0$$

(a) Find the velocity and acceleration functions.

(b) At what time(s) is the particle at rest?

(c) Determine when the particle is speeding up and when it is slowing down.

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Question 1 - Solution

We are given the position function:

$$s(t) = t^3 - 6t^2 + 9t$$

(a) Velocity and Acceleration:
Velocity is the first derivative of position:

$$v(t) = s'(t) = 3t^2 - 12t + 9$$

Acceleration is the derivative of velocity:

$$a(t) = v'(t) = 6t - 12$$

(b) Particle at Rest:
The particle is at rest when $v(t) = 0$:

$$3t^2 - 12t + 9 = 0 \Rightarrow t^2 - 4t + 3 = 0 \Rightarrow (t-1)(t-3) = 0 \Rightarrow t = 1, 3$$

So, the particle is at rest at $t = \boxed{1}$ and $t = \boxed{3}$.
(c) Speeding Up / Slowing Down:
Speed increases when velocity and acceleration have the same sign; slows down when signs differ.
Critical points from previous steps:

$$v(t) = 3t^2 - 12t + 9, \quad a(t) = 6t - 12$$

Velocity is 0 at $t = 1, 3$, acceleration is 0 at $t = 2$
Test signs in intervals:
$t < 1$: $v(t) > 0$, $a(t) < 0 \rightarrow$ slowing down
$1 < t < 2$: $v(t) < 0$, $a(t) < 0 \rightarrow$ speeding up
$2 < t < 3$: $v(t) < 0$, $a(t) > 0 \rightarrow$ slowing down
$t > 3$: $v(t) > 0$, $a(t) > 0 \rightarrow$ speeding up
Conclusion:

- Speeding up: $t \in (1, 2) \cup (3, \infty)$
- Slowing down: $t \in (0, 1) \cup (2, 3)$

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Question 2

A rock is thrown vertically upward from the top of a cliff. Its height in meters after t seconds is given by:

$$h(t) = -5t^2 + 20t + 50$$

(a) What is the velocity of the rock after 2 seconds?

(b) When does the rock reach its maximum height?

(c) When does the rock hit the ground?

(d) What is the velocity of the rock at the moment it hits the ground?

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Question 2 - Solution

We are given:

$$h(t) = -5t^2 + 20t + 50$$

(a) Velocity after 2 seconds:
Velocity is the derivative of height:

$$v(t) = h'(t) = -10t + 20$$

So,

$$v(2) = -10(2) + 20 = -20 + 20 = \boxed{0\,\text{m/s}}$$

(b) Maximum Height:
Maximum height occurs when velocity is zero:

$$v(t) = -10t + 20 = 0 \Rightarrow t = 2\,\text{seconds}$$

(c) When does the rock hit the ground?
The rock hits the ground when $h(t) = 0$:

$$-5t^2 + 20t + 50 = 0 \Rightarrow t^2 - 4t - 10 = 0 \Rightarrow t = \frac{4 \pm \sqrt{16 + 40}}{2} = \frac{4 \pm \sqrt{56}}{2} = \frac{4 \pm 2\sqrt{14}}{2} = 2 \pm \sqrt{14}$$

Since time must be positive, we choose the positive root:

$$t = 2 + \sqrt{14} \approx \boxed{5.74\,\text{seconds}}$$

(d) Velocity at impact:
Plug into the velocity function:

$$v(2 + \sqrt{14}) = -10(2 + \sqrt{14}) + 20 = -20 - 10\sqrt{14} + 20 = -10\sqrt{14}$$

So the velocity at impact is:

$$\boxed{-10\sqrt{14}\,\text{m/s}} \quad \text{(downward)}$$

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Question 3

The position of a particle moving along a straight line is given by:

$$s(t) = 3t^3 - 15t^2 + 18t \quad \text{(in meters), for } t \geq 0.$$

(a) Find the velocity and acceleration functions of the particle.

(b) At what times is the particle at rest?

(c) Determine the intervals when the particle is moving forward.

(d) When is the acceleration zero, and what does it signify?

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Question 3 - Solution

Given:

$$s(t) = 3t^3 - 15t^2 + 18t$$

(a) Velocity and Acceleration:
Velocity is the derivative of position:

$$v(t) = s'(t) = 9t^2 - 30t + 18$$

Acceleration is the derivative of velocity:

$$a(t) = v'(t) = 18t - 30$$

(b) Particle at Rest:
Set velocity to zero:

$$9t^2 - 30t + 18 = 0 \Rightarrow t = \frac{30 \pm \sqrt{(-30)^2 - 4(9)(18)}}{2(9)} = \frac{30 \pm \sqrt{900 - 648}}{18} = \frac{30 \pm \sqrt{252}}{18}$$

$$\sqrt{252} = 6\sqrt{7}, \quad t = \frac{30 \pm 6\sqrt{7}}{18} = \frac{5 \pm \sqrt{7}}{3}$$

So the particle is at rest at:

$$\boxed{t = \frac{5 \pm \sqrt{7}}{3}}$$

(c) Particle Moving Forward:
Forward motion means $v(t) > 0$. Use sign chart for:

$$v(t) = 9t^2 - 30t + 18 = 9(t - t_1)(t - t_2), \quad \text{where } t_1 = \frac{5 - \sqrt{7}}{3},\ t_2 = \frac{5 + \sqrt{7}}{3}$$

This quadratic opens upward (coefficient of t^2 is positive), so:

$$v(t) > 0 \quad \text{on } (0, t_1) \cup (t_2, \infty)$$

So the particle moves forward on:

$$\boxed{\left(0, \frac{5 - \sqrt{7}}{3}\right) \cup \left(\frac{5 + \sqrt{7}}{3}, \infty\right)}$$

(d) Acceleration Zero:

$$a(t) = 18t - 30 = 0 \Rightarrow t = \frac{30}{18} = \frac{5}{3}$$

This is when the particle changes concavity or when velocity changes most rapidly.
Conclusion:

$$\boxed{a(t) = 0 \text{ when } t = \frac{5}{3} \text{ seconds}}$$

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Question 4

The cost (in dollars) to produce x items is given by the function:

$$C(x) = 50x + 300\sqrt{x}$$

(a) Find the marginal cost function.

(b) Interpret the meaning of $C'(100)$.

(c) Approximate the cost of producing the 101st item using the derivative.

(d) Compute the actual cost of producing the 101st item and compare with your approximation.

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Question 4 - Solution

Marginal cost. For $x > 0$,

$$\boxed{C'(x) = 50 + 150/\sqrt{x}}.$$

Thus $C'(100) = 65$ dollars per item. This is the instantaneous cost rate and predicts a cost of approximately $\boxed{\$65}$ for one additional item.

Actual additional cost.

$$C(101) - C(100) = 50 + 300(\sqrt{101} - 10) \approx \boxed{\$64.96}.$$

The derivative estimate exceeds the actual cost by about \$0.04. This agrees with $C''(x) = -75x^{-3/2} < 0$: the marginal cost decreases over the next item.

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Question 5

The population of a certain city (in thousands) $P(t)$, as a function of time t (in years since 2000), is modeled by the function:

$$P(t) = 500 + 40t - 0.5t^2$$

(a) Find the rate of change of the population with respect to time.

(b) What is the population growth rate in the year 2010?

(c) At what year is the population growth rate zero?

(d) Interpret your result from (c) in the context of the population.

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Question 5 - Solution

We are given:

$$P(t) = 500 + 40t - 0.5t^2$$

(a) Rate of Change of Population:
The rate of change of the population is given by the derivative:

$$P'(t) = \frac{d}{dt}(500 + 40t - 0.5t^2) = 40 - t$$

$$\boxed{P'(t) = 40 - t}$$

(b) Growth Rate in 2010:
Since t is years since 2000, 2010 corresponds to $t = 10$.

$$P'(10) = 40 - 10 = 30$$

Interpretation: In 2010, the population was increasing at a rate of 30 thousand people per year.

$$\boxed{P'(10) = 30 \quad \text{(thousand people/year)}}$$

(c) When is the Population Growth Rate Zero?
Set $P'(t) = 0$:

$$40 - t = 0 \Rightarrow t = 40$$

$$\boxed{t = 40}$$

(d) Interpretation:
Since $t = 40$ corresponds to the year $2000 + 40 = 2040$, the population stops growing in 2040.
Conclusion: In 2040, the population reaches its maximum and stops increasing. After that, it begins to decline.

$$\boxed{\text{The population stops growing in 2040.}}$$

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Question 6

The temperature T (in °C) in a chemical reaction chamber is modeled by the function

$$T(t) = 100 - 20e^{-0.5t}$$

where t is the time in minutes since the reaction started.

(a) Find the rate at which the temperature is changing with respect to time.

(b) What is the rate of temperature change at $t = 0$?

(c) What is the long-term behavior of the rate of temperature change? Explain.

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Question 6 - Solution

We are given:

$$T(t) = 100 - 20e^{-0.5t}$$

(a) Rate of Change of Temperature:
Differentiate $T(t)$ with respect to t:

$$T'(t) = \frac{d}{dt}(100 - 20e^{-0.5t}) = -20 \cdot \frac{d}{dt}(e^{-0.5t}) = -20 \cdot (-0.5e^{-0.5t}) = 10e^{-0.5t}$$

$$\boxed{T'(t) = 10e^{-0.5t}}$$

This represents the rate of temperature change at time t.

(b) Rate of Change at $t = 0$:

$$T'(0) = 10e^{-0.5 \cdot 0} = 10e^0 = 10$$

$$\boxed{T'(0) = 10 \quad (\text{°C per minute})}$$

Interpretation: At the beginning of the reaction, the temperature is increasing at 10°C per minute.

(c) Long-Term Behavior:
As $t \to \infty$, $e^{-0.5t} \to 0$, so:

$$T'(t) = 10e^{-0.5t} \to 0$$

Conclusion: The rate of temperature change decreases over time and approaches zero. This means the temperature stabilizes and stops changing as time progresses.

$$\boxed{\lim_{t \to \infty} T'(t) = 0}$$

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Question 7

The position of a particle moving along a straight line is given by the function

$$s(t) = t^3 - 6t^2 + 9t$$

where $s(t)$ is in meters and $t \geq 0$ is in seconds.

(a) Find the velocity function $v(t)$.

(b) At what time(s) is the particle at rest?

(c) Determine the direction of motion (moving forward or backward) on the intervals determined by part (b).

(d) Is the particle speeding up or slowing down at $t = 1$? Justify using acceleration.

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Question 7 - Solution

Velocity and rest times.

$$v(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3), \qquad a(t) = 6t - 12.$$

Hence the particle is at rest at $\boxed{t = 1, 3}$.

For $t \geq 0$, it moves forward on $[0, 1) \cup (3, \infty)$ and backward on $(1, 3)$.

Behavior at $t = 1$. Here $v(1) = 0$ and $a(1) = -6$. Just before 1, velocity and acceleration have opposite signs, so speed decreases. Just after 1, both are negative, so speed increases.

The speed $|v(t)|$ has left derivative -6 and right derivative 6 at 1; it is not differentiable there.

$$\boxed{\text{At } t = 1 \text{ it is at rest and reverses direction.}}$$

It slows down immediately before that instant and speeds up immediately afterward.

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Question 8

The height of a ball thrown vertically upward is modeled by the function:

$$h(t) = -16t^2 + 64t + 80$$

where $h(t)$ is the height in feet after t seconds.

(a) Determine the velocity function of the ball.

(b) When does the ball reach its maximum height?

(c) What is the maximum height the ball reaches?

(d) What is the acceleration of the ball at any time t?

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Question 8 - Solution

We are given:

$$h(t) = -16t^2 + 64t + 80$$

(a) Velocity Function:
The velocity is the first derivative of height:

$$v(t) = h'(t) = -32t + 64$$

$$\boxed{v(t) = -32t + 64}$$

(b) Time of Maximum Height:
The ball reaches its maximum height when the velocity is zero:

$$-32t + 64 = 0 \Rightarrow t = 2$$

$$\boxed{t = 2 \text{ seconds}}$$

(c) Maximum Height:
Plug $t = 2$ into the original height function:

$$h(2) = -16(2)^2 + 64(2) + 80 = -64 + 128 + 80 = 144$$

$$\boxed{144 \text{ feet}}$$

(d) Acceleration Function:
Acceleration is the derivative of velocity:

$$a(t) = v'(t) = -32$$

$$\boxed{a(t) = -32 \text{ ft/s}^2 \text{ (constant acceleration due to gravity)}}$$

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Question 9

The position $s(t)$ of a particle moving along a straight line is given by:

$$s(t) = \frac{3t^2 + 2}{t + 1}, \quad t > 0$$

(a) Find the instantaneous velocity of the particle at $t = 1$.

(b) Interpret the sign of the velocity at $t = 1$.

(c) Find all critical points of $s(t)$ and determine whether the particle changes direction.

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Question 9 - Solution

We are given:

$$s(t) = \frac{3t^2 + 2}{t + 1}, \quad t > 0$$

(a) Instantaneous Velocity at $t = 1$:
Velocity is the derivative $s'(t)$. Using the quotient rule:

$$s'(t) = \frac{(t+1)(6t) - (3t^2+2)(1)}{(t+1)^2}$$

$$= \frac{6t^2 + 6t - 3t^2 - 2}{(t+1)^2} = \frac{3t^2 + 6t - 2}{(t+1)^2}$$

Evaluate at $t = 1$:

$$s'(1) = \frac{3 + 6 - 2}{4} = \frac{7}{4}$$

$$\boxed{s'(1) = \frac{7}{4}}$$

(b) Interpretation of the Sign of Velocity:
Since $s'(1) = \frac{7}{4} > 0$, the particle is moving in the positive direction at $t = 1$.

$$\boxed{\text{The particle is moving to the right (increasing position) at } t = 1}$$

(c) Critical Points and Direction Change:
Critical points occur where $s'(t) = 0$ or where $s'(t)$ is undefined.
The derivative is undefined at $t = -1$, which is outside the domain $t > 0$ and therefore ignored.
Set the numerator equal to zero:

$$3t^2 + 6t - 2 = 0$$

$$t = \frac{-6 \pm \sqrt{36 + 24}}{6} = \frac{-6 \pm \sqrt{60}}{6} = \frac{-3 \pm \sqrt{15}}{3}$$

This gives two solutions:

$$t = \frac{-3 - \sqrt{15}}{3} \quad \text{and} \quad t = \frac{-3 + \sqrt{15}}{3}$$

Since time must satisfy $t > 0$, the negative solution is rejected. The only valid critical time is:

$$\boxed{t = \frac{-3 + \sqrt{15}}{3}}$$

To determine whether the particle changes direction, test the sign of $s'(t)$ on either side of this value (within $t > 0$). The derivative changes from negative to positive, indicating a direction change.

$$\boxed{\text{The particle changes direction at } t = \frac{-3 + \sqrt{15}}{3}}$$

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Question 10

Suppose that the function $C(p)$ models the total cost (in dollars) of producing p items, and is given by:

$$C(p) = 200 + 8p + 0.05p^2$$

(a) Find and interpret $C'(p)$.

(b) Find $C'(50)$ and interpret its meaning in context.

(c) Determine whether the marginal cost is increasing or decreasing as p increases.

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Question 10 - Solution

We are given:

$$C(p) = 200 + 8p + 0.05p^2$$

(a) Find and interpret $C'(p)$:
Differentiate:

$$C'(p) = \frac{d}{dp}(200 + 8p + 0.05p^2) = 0 + 8 + 0.1p = 8 + 0.1p$$

Interpretation: $C'(p)$ is the marginal cost — the approximate cost of producing one additional item when p items have already been produced.

(b) Compute and interpret $C'(50)$:

$$C'(50) = 8 + 0.1(50) = 8 + 5 = \boxed{13}$$

Interpretation: When 50 items have been produced, the cost of producing one more item is approximately \$13.

(c) Is marginal cost increasing or decreasing?
Since:

$$C'(p) = 8 + 0.1p$$

and the derivative of this is:

$$C''(p) = 0.1 > 0$$

The marginal cost is increasing as p increases.

$$\boxed{\text{Marginal cost increases with production level.}}$$

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Question 1

Let

$$f(x) = \frac{\sin(x)}{x^2 + 1} + x\cos(x).$$

(a) Find the derivative $f'(x)$ using standard differentiation rules.

(b) Determine the slope of the tangent line to the graph of $f(x)$ at $x = 0$.

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Question 1 - Solution

We are given:

$$f(x) = \frac{\sin(x)}{x^2+1} + x\cos(x)$$

(a) Find $f'(x)$:
Differentiate term by term.
First term (quotient rule):

$$\frac{d}{dx}\left(\frac{\sin(x)}{x^2+1}\right) = \frac{(x^2+1)\cos(x) - \sin(x)(2x)}{(x^2+1)^2}$$

Second term (product rule):

$$\frac{d}{dx}(x\cos(x)) = \cos(x) - x\sin(x)$$

Putting both together:

$$f'(x) = \frac{(x^2+1)\cos(x) - 2x\sin(x)}{(x^2+1)^2} + \cos(x) - x\sin(x)$$

(b) Find the slope at $x = 0$:
Evaluate $f'(0)$.

$$\begin{aligned} f'(0) &= \frac{(0^2+1)\cos(0) - 2(0)\sin(0)}{(0^2+1)^2} + \cos(0) - 0\cdot\sin(0) \\ &= \frac{1\cdot 1 - 0}{1^2} + 1 - 0 \\ &= 1 + 1 = \boxed{2} \end{aligned}$$

So, the slope of the tangent line to the curve at $x = 0$ is $\boxed{2}$.

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Question 2

Let

$$f(x) = \sqrt{x^2 + 3x} + \frac{1}{\sqrt{x^2 + 3x}}.$$

(a) Compute $f'(x)$ using differentiation rules.

(b) Determine all points where the derivative does not exist.

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Question 2 - Solution

Domain. Set $u = x^2 + 3x = x(x+3)$. The reciprocal square root requires $u > 0$, so

$$D = (-\infty, -3) \cup (0, \infty).$$

Differentiate.

$$f = u^{1/2} + u^{-1/2}, \qquad \boxed{f'(x) = \frac{(2x+3)(x^2+3x-1)}{2(x^2+3x)^{3/2}}}, \quad x \in D.$$

The derivative exists at every point of the function's domain. On $[-3, 0]$, the function itself is undefined; these are not nondifferentiable points of its graph.

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Question 3

Let

$$f(x) = \frac{e^x}{\cos x} + \ln(x^2 + 1).$$

(a) Find $f'(x)$.

(b) Determine the domain of $f'(x)$.

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Question 3 - Solution

We are given:

$$f(x) = \frac{e^x}{\cos x} + \ln(x^2 + 1)$$

(a) Find $f'(x)$:
Differentiate each term separately.
For the first term $\frac{e^x}{\cos x}$, use the quotient rule:

$$\frac{d}{dx}\left(\frac{e^x}{\cos x}\right) = \frac{(\cos x)(e^x) - e^x(-\sin x)}{(\cos x)^2} = \frac{e^x(\cos x + \sin x)}{\cos^2 x}$$

For the second term $\ln(x^2 + 1)$, use the chain rule:

$$\frac{d}{dx}\ln(x^2 + 1) = \frac{2x}{x^2 + 1}$$

Combine both:

$$f'(x) = \frac{e^x(\cos x + \sin x)}{\cos^2 x} + \frac{2x}{x^2 + 1}$$

(b) Domain of $f'(x)$:
We must exclude values that cause division by zero.
$\cos x = 0$ when $x = \frac{\pi}{2} + n\pi$, where $n \in \mathbb{Z}$, $x^2 + 1 > 0$ for all real x, so no restrictions there
Domain: All real x such that $\cos x \neq 0$

$$\boxed{f'(x) = \frac{e^x(\cos x + \sin x)}{\cos^2 x} + \frac{2x}{x^2 + 1}, \quad x \in \mathbb{R} \setminus \left\{\frac{\pi}{2} + n\pi \mid n \in \mathbb{Z}\right\}}$$

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Question 4

Let

$$f(x) = \frac{\arctan x}{1 + x^2} + x^5 e^x.$$

(a) Compute $f'(x)$.

(b) Write the equation for critical numbers and approximate one in $(-5, -4)$ to six decimal places.

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Question 4 - Solution

(a) Differentiate

The quotient and product rules give

$$\boxed{f'(x) = \frac{1 - 2x \arctan x}{(1 + x^2)^2} + e^x x^4 (x + 5).}$$

(b) Critical numbers

The derivative exists for every real x, since $1 + x^2 > 0$.

Thus critical numbers solve $f'(x) = 0$. Bisection on $(-5, -4)$ gives

$$\boxed{x \approx -4.995520.}$$

The derivative has opposite signs at the ends of this interval, so continuity guarantees a root there.

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Question 5

Let

$$f(x) = \frac{\ln(x)}{x^3}$$

(a) Find the derivative $f'(x)$ using the quotient rule.

(b) Determine any local extrema of $f(x)$ for $x > 0$.

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Question 5 - Solution

We are given:

$$f(x) = \frac{\ln(x)}{x^3}$$

(a) Find $f'(x)$:
Use the quotient rule:

$$f'(x) = \frac{x^3 \cdot \frac{1}{x} - \ln(x) \cdot 3x^2}{x^6} = \frac{x^2 - 3x^2 \ln(x)}{x^6}$$

Factor the numerator:

$$f'(x) = \frac{x^2(1 - 3\ln(x))}{x^6} = \frac{1 - 3\ln(x)}{x^4}$$

So,

$$f'(x) = \boxed{\frac{1 - 3\ln(x)}{x^4}}$$

(b) Local Extrema:
Critical points occur where $f'(x) = 0$:

$$\frac{1 - 3\ln(x)}{x^4} = 0 \Rightarrow 1 - 3\ln(x) = 0 \Rightarrow \ln(x) = \frac{1}{3} \Rightarrow x = e^{1/3}$$

Now determine if this is a maximum or minimum by checking the sign of $f'(x)$ around $x = e^{1/3}$:
For $x < e^{1/3}$, say $x = 1$: $f'(1) = \frac{1-3\ln(1)}{1^4} = 1 > 0$
For $x > e^{1/3}$, say $x = 2$: $f'(2) = \frac{1-3\ln(2)}{2^4} \approx \frac{1-3(0.693)}{16} \approx \frac{-1.079}{16} < 0$
So $f(x)$ has a local maximum at $x = e^{1/3}$.

Conclusion:
$f'(x) = \frac{1 - 3\ln(x)}{x^4}$, Local maximum at $x = \boxed{e^{1/3}}$

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Question 6

Let

$$f(x) = \sqrt{x^2 + 4x + 5}$$

(a) Differentiate $f(x)$ using the chain rule.

(b) Find all values of x where the tangent line to $f(x)$ is horizontal.

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Question 6 - Solution

We are given:

$$f(x) = \sqrt{x^2 + 4x + 5} = (x^2 + 4x + 5)^{1/2}$$

(a) Differentiate $f(x)$:
Using the chain rule:

$$f'(x) = \frac{1}{2}(x^2 + 4x + 5)^{-1/2} \cdot (2x + 4)$$

Simplify:

$$f'(x) = \frac{2x + 4}{2\sqrt{x^2 + 4x + 5}}$$

So,

$$f'(x) = \boxed{\frac{2x + 4}{2\sqrt{x^2 + 4x + 5}}}$$

(b) Horizontal Tangents:
The tangent is horizontal when $f'(x) = 0$. That occurs when the numerator is zero:

$$2x + 4 = 0 \Rightarrow x = -2$$

Check that the denominator is defined at $x = -2$:

$$\sqrt{(-2)^2 + 4(-2) + 5} = \sqrt{4 - 8 + 5} = \sqrt{1} = 1$$

So it's defined, and the tangent is horizontal at $x = -2$.
Conclusion:

- $f'(x) = \dfrac{2x + 4}{2\sqrt{x^2 + 4x + 5}}$
- Horizontal tangent at $x = \boxed{-2}$

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Question 7

Let

$$f(x) = \frac{e^x \sin x}{x^2 + 1}$$

(a) Differentiate $f(x)$ using the quotient rule.

(b) Evaluate $f'(0)$.

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Question 7 - Solution

We are given:

$$f(x) = \frac{e^x \sin x}{x^2 + 1}$$

(a) Use the Quotient Rule:
The quotient rule is:

$$f(x) = \frac{u(x)}{v(x)} \Rightarrow f'(x) = \frac{u'v - uv'}{v^2}$$

Let:

$$u(x) = e^x \sin x, \quad v(x) = x^2 + 1$$

Differentiate $u(x)$ using the product rule:

$$u'(x) = \frac{d}{dx}[e^x \sin x] = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$$

Differentiate $v(x)$:

$$v'(x) = \frac{d}{dx}[x^2 + 1] = 2x$$

Now apply the quotient rule:

$$f'(x) = \frac{e^x(\sin x + \cos x)(x^2 + 1) - e^x \sin x \cdot 2x}{(x^2 + 1)^2}$$

Factor out e^x:

$$f'(x) = \frac{e^x \left[(\sin x + \cos x)(x^2 + 1) - 2x \sin x\right]}{(x^2 + 1)^2}$$

(b) Evaluate $f'(0)$:
Substitute $x = 0$:
Numerator:

$$e^0 \left[(\sin 0 + \cos 0)(0^2 + 1) - 2(0) \sin 0\right] = 1 \cdot (0 + 1)(1) = 1$$

Denominator:

$$(0^2 + 1)^2 = 1^2 = 1$$

So:

$$f'(0) = \boxed{1}$$

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Question 8

Let

$$f(x) = \ln\left(\sqrt{x^2 + 3x + 2}\right)$$

(a) Simplify the function before differentiating.

(b) Find $f'(x)$.

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Question 8 - Solution

(a) Simplify the Function:

We start with:

$$f(x) = \ln\left(\sqrt{x^2+3x+2}\right)$$

Since

$$\sqrt{x^2+3x+2} = (x^2+3x+2)^{1/2},$$

we can write:

$$f(x) = \ln\left((x^2+3x+2)^{1/2}\right)$$

Using the logarithm rule

$$\ln(A^r) = r\ln(A),$$

we get:

$$f(x) = \frac{1}{2}\ln(x^2+3x+2)$$

So the simplified function is:

$$\boxed{f(x) = \frac{1}{2}\ln(x^2+3x+2)}$$

(b) Find $f'(x)$:

Now differentiate:

$$f(x) = \frac{1}{2}\ln(x^2+3x+2)$$

Using

$$\frac{d}{dx}\ln(u) = \frac{u'}{u},$$

where

$$u = x^2+3x+2$$

and

$$u' = 2x+3,$$

we get:

$$f'(x) = \frac{1}{2}\cdot\frac{2x+3}{x^2+3x+2}$$

Therefore,

$$\boxed{f'(x) = \frac{2x+3}{2(x^2+3x+2)}}$$

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Question 9

Let

$$f(x) = \tan^2(x) + \sec^2(x)$$

(a) Differentiate $f(x)$ using the standard differentiation formulas for trigonometric functions.

(b) Simplify your answer.

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Question 9 - Solution

(a) Differentiate $f(x)$:

We are given:

$$f(x) = \tan^2(x) + \sec^2(x)$$

We apply the chain rule and known derivatives:
$\frac{d}{dx}[\tan^2(x)] = 2\tan(x)\sec^2(x)$ (using chain rule)
$\frac{d}{dx}[\sec^2(x)] = 2\sec^2(x)\tan(x)$ (using chain rule)

So:

$$f'(x) = 2\tan(x)\sec^2(x) + 2\sec^2(x)\tan(x)$$

(b) Simplify:

Both terms are the same:

$$f'(x) = 4\tan(x)\sec^2(x)$$

Final Answer:

$$\boxed{f'(x) = 4\tan(x)\sec^2(x)}$$

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Question 10

Let

$$f(x) = \frac{\sin(x)}{1 + \cos(x)}$$

(a) Differentiate $f(x)$ using the quotient rule.

(b) Simplify the result as much as possible.

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Question 10 - Solution

We are given:

$$f(x) = \frac{\sin(x)}{1 + \cos(x)}$$

Let:

$$u = \sin(x), \quad v = 1 + \cos(x)$$

Then:

$$f'(x) = \frac{v \cdot u' - u \cdot v'}{v^2}$$

Compute the derivatives:

$$u' = \cos(x), \quad v' = -\sin(x)$$

Apply the quotient rule:

$$f'(x) = \frac{(1 + \cos(x))\cos(x) - \sin(x)(-\sin(x))}{(1 + \cos(x))^2}$$

Simplify numerator:

$$(1 + \cos(x))\cos(x) + \sin^2(x) = \cos(x) + \cos^2(x) + \sin^2(x)$$

Use the identity $\sin^2(x) + \cos^2(x) = 1$:

$$= \cos(x) + 1$$

So:

$$f'(x) = \frac{\cos(x) + 1}{(1 + \cos(x))^2}$$

Cancel common factor:

$$f'(x) = \frac{1}{1 + \cos(x)}$$

Final Answer:

$$\boxed{f'(x) = \frac{1}{1 + \cos(x)}}$$

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Question 1

Let

$$f(x) = x^2 \cdot \ln(x)$$

(a) Differentiate $f(x)$ using the product rule.

(b) Find the equation of the tangent line to the graph of $f(x)$ at $x = 1$.

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Question 1 - Solution

We are given:

$$f(x) = x^2 \cdot \ln(x)$$

Let:

$$u = x^2, \quad v = \ln(x) \Rightarrow u' = 2x, \quad v' = \frac{1}{x}$$

Apply the product rule:

$$f'(x) = u'v + uv' = 2x\ln(x) + x^2 \cdot \frac{1}{x} = 2x\ln(x) + x$$

Final derivative:

$$\boxed{f'(x) = 2x\ln(x) + x}$$

Now evaluate the tangent line at $x = 1$:

$$f(1) = 1^2 \cdot \ln(1) = 0$$

$$f'(1) = 2(1)\ln(1) + 1 = 0 + 1 = 1$$

Point on the curve: $(1, 0)$ Slope of tangent: 1
Use point-slope form:

$$y - 0 = 1(x - 1) \Rightarrow y = x - 1$$

Equation of the tangent line:

$$\boxed{y = x - 1}$$

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Question 2

Let

$$f(x) = \frac{x^2 \sin(x)}{x+1}$$

(a) Differentiate $f(x)$ using the quotient rule.

(b) Simplify your answer as much as possible.

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Question 2 - Solution

We are given:

$$f(x) = \frac{x^2 \sin(x)}{x+1}$$

Let

$$u(x) = x^2 \sin(x), \quad v(x) = x + 1$$

Then using the **quotient rule**:

$$f'(x) = \frac{v \cdot u' - u \cdot v'}{v^2}$$

We compute:

$$u'(x) = \frac{d}{dx}[x^2 \sin(x)] = 2x \sin(x) + x^2 \cos(x) \quad \text{(product rule)}$$

$$v'(x) = \frac{d}{dx}[x + 1] = 1$$

Now plug into the quotient rule:

$$f'(x) = \frac{(x+1)(2x \sin(x) + x^2 \cos(x)) - x^2 \sin(x) \cdot 1}{(x+1)^2}$$

Now expand and simplify the numerator:

$$= \frac{(x+1)(2x \sin(x) + x^2 \cos(x)) - x^2 \sin(x)}{(x+1)^2}$$

Distribute:

$$= \frac{2x(x+1)\sin(x) + x^2(x+1)\cos(x) - x^2 \sin(x)}{(x+1)^2}$$

Final simplified expression:

$$\boxed{f'(x) = \frac{2x(x+1)\sin(x) - x^2 \sin(x) + x^2(x+1)\cos(x)}{(x+1)^2}}$$

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Question 3

Let

$$f(x) = \frac{\ln(x)}{x\cos(x)}$$

(a) Differentiate $f(x)$ using the quotient and product rules.

(b) Simplify your answer as much as possible.

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Question 3 - Solution

We are given:

$$f(x) = \frac{\ln(x)}{x\cos(x)}$$

Let

$$u(x) = \ln(x), \quad v(x) = x\cos(x)$$

Then by the **quotient rule**:

$$f'(x) = \frac{v \cdot u' - u \cdot v'}{v^2}$$

Compute each derivative:

$$u'(x) = \frac{1}{x}$$

For $v(x) = x\cos(x)$, use the **product rule**:

$$v'(x) = \frac{d}{dx}[x\cos(x)] = \cos(x) - x\sin(x)$$

Now substitute into the quotient rule:

$$f'(x) = \frac{x\cos(x) \cdot \frac{1}{x} - \ln(x)(\cos(x) - x\sin(x))}{(x\cos(x))^2}$$

Simplify numerator:

$$f'(x) = \frac{\cos(x) - \ln(x)\cos(x) + \ln(x)x\sin(x)}{x^2\cos^2(x)}$$

Final Answer:

$$\boxed{f'(x) = \frac{\cos(x)\big(1 - \ln(x)\big) + \ln(x)x\sin(x)}{x^2\cos^2(x)}}$$

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Question 4

Let

$$f(x) = (x^2 + 3x) \cdot e^x$$

(a) Use the product rule to differentiate $f(x)$.

(b) Factor and simplify your final expression.

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Question 4 - Solution

We are given:

$$f(x) = (x^2 + 3x) \cdot e^x$$

Let:

$$u(x) = x^2 + 3x, \quad v(x) = e^x$$

By the product rule:

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$

Differentiate each part:

$$u'(x) = 2x + 3, \quad v'(x) = e^x$$

Now plug into the formula:

$$f'(x) = (2x + 3)e^x + (x^2 + 3x)e^x$$

Factor out e^x:

$$f'(x) = e^x \left[(2x + 3) + (x^2 + 3x)\right]$$

Simplify inside the brackets:

$$(2x + 3) + (x^2 + 3x) = x^2 + 5x + 3$$

Final Answer:

$$\boxed{f'(x) = e^x(x^2 + 5x + 3)}$$

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Question 5

Let

$$f(x) = x^2 e^x \cos x$$

(a) Compute $f'(x)$ using appropriate differentiation rules.

(b) Factor the result where possible.

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Question 5 - Solution

We are given:

$$f(x) = x^2 e^x \cos x$$

This is a product of three functions: - $u(x) = x^2$ - $v(x) = e^x$ - $w(x) = \cos x$
Use the **product rule** for three functions:

$$f'(x) = u'(x)v(x)w(x) + u(x)v'(x)w(x) + u(x)v(x)w'(x)$$

Compute derivatives: - $u'(x) = 2x$ - $v'(x) = e^x$ - $w'(x) = -\sin x$
Now substitute:

$$f'(x) = 2xe^x \cos x + x^2 e^x \cos x - x^2 e^x \sin x$$

Factor where possible:

$$f'(x) = e^x \left(\cos x(2x + x^2) - x^2 \sin x\right)$$

Final Answer:

$$\boxed{f'(x) = e^x \left[\cos x(x^2 + 2x) - x^2 \sin x\right]}$$

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Question 6

Let

$$f(x) = \frac{x^2 e^{2x} \sin x}{1 + x^3}$$

(a) Compute $f'(x)$.

(b) Simplify and factor your answer as much as possible.

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Question 6 - Solution

We are given:

$$f(x) = \frac{x^2 e^{2x} \sin x}{1 + x^3}$$

Let

$$u(x) = x^2 e^{2x} \sin x, \quad v(x) = 1 + x^3$$

Using the quotient rule:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Step 1: Differentiate $u(x)$

The function $u(x)$ is a product of three functions:

$$u(x) = x^2 \cdot e^{2x} \cdot \sin x$$

Apply the product rule:

$$u'(x) = (x^2)' e^{2x} \sin x + x^2 (e^{2x})' \sin x + x^2 e^{2x} (\sin x)'$$

Compute derivatives:

$$(x^2)' = 2x, \quad (e^{2x})' = 2e^{2x}, \quad (\sin x)' = \cos x$$

So:

$$u'(x) = 2x e^{2x} \sin x + 2x^2 e^{2x} \sin x + x^2 e^{2x} \cos x$$

Factor:

$$u'(x) = e^{2x} \left[\sin x (2x + 2x^2) + x^2 \cos x \right]$$

Step 2: Differentiate $v(x)$

$$v'(x) = 3x^2$$

Step 3: Apply the Quotient Rule

$$f'(x) = \frac{e^{2x} \left[\sin x (2x + 2x^2) + x^2 \cos x \right] (1 + x^3) - x^2 e^{2x} \sin x (3x^2)}{(1 + x^3)^2}$$

Factor common terms in the numerator:

$$f'(x) = \frac{e^{2x} x \left[(1 + x^3)(2 \sin x + 2x \sin x + x \cos x) - 3x^3 \sin x \right]}{(1 + x^3)^2}$$

Final Answer:

$$\boxed{f'(x) = \frac{e^{2x} x \left[(1 + x^3)(2 \sin x + 2x \sin x + x \cos x) - 3x^3 \sin x \right]}{(1 + x^3)^2}}$$

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Question 7

Let

$$f(x) = \frac{x \sin x}{x^2 + 1}$$

(a) Compute $f'(x)$ using the quotient rule.

(b) Simplify the derivative fully.

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Question 7 - Solution

We are given:

$$f(x) = \frac{x \sin x}{x^2 + 1}$$

Let:

$$u(x) = x \sin x, \quad v(x) = x^2 + 1$$

We apply the **quotient rule**:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

First, compute $u'(x)$ using the product rule:

$$u'(x) = \sin x + x \cos x$$

Next, compute $v'(x)$:

$$v'(x) = 2x$$

Now apply the quotient rule:

$$f'(x) = \frac{(\sin x + x \cos x)(x^2 + 1) - x \sin x(2x)}{(x^2 + 1)^2}$$

Expand the numerator:

$$(\sin x)(x^2 + 1) + x \cos x(x^2 + 1) - 2x^2 \sin x$$

Distribute:

$$x^2 \sin x + \sin x + x^3 \cos x + x \cos x - 2x^2 \sin x$$

Combine like terms:

$$-x^2 \sin x + \sin x + x^3 \cos x + x \cos x$$

Factor where possible:

$$\sin x(1 - x^2) + x \cos x(x^2 + 1)$$

Final Answer:

$$\boxed{f'(x) = \frac{\sin x(1 - x^2) + x \cos x(x^2 + 1)}{(x^2 + 1)^2}}$$

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Question 8

Let

$$f(x) = \frac{e^x \cos x}{\ln(x)}$$

(a) Compute $f'(x)$ using the quotient rule.

(b) State the domain of $f'(x)$.

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Question 8 - Solution

We are given:

$$f(x) = \frac{e^x \cos x}{\ln(x)}$$

Let: - $u(x) = e^x \cos x$ - $v(x) = \ln(x)$
We'll use the **quotient rule**:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Step 1: Differentiate $u(x) = e^x \cos x$ using the **product rule**:

$$u'(x) = e^x \cos x + e^x(-\sin x) = e^x(\cos x - \sin x)$$

Step 2: Differentiate $v(x) = \ln(x)$:

$$v'(x) = \frac{1}{x}$$

Now plug into the quotient rule:

$$f'(x) = \frac{e^x(\cos x - \sin x) \cdot \ln x - e^x \cos x \cdot \frac{1}{x}}{(\ln x)^2}$$

Factor e^x from the numerator:

$$f'(x) = \frac{e^x \left[(\cos x - \sin x) \ln x - \frac{\cos x}{x}\right]}{(\ln x)^2}$$

Final Answer:

$$\boxed{f'(x) = \frac{e^x \left[(\cos x - \sin x) \ln x - \frac{\cos x}{x}\right]}{(\ln x)^2}}$$

Domain of $f'(x)$:
$\ln x$ is defined only for $x > 0$
Also, $\ln x \neq 0 \Rightarrow x \neq 1$
And denominator has $\ln(x)^2$, so $x = 1$ is excluded
The term $\frac{\cos x}{x}$ is defined for $x > 0$
Domain:

$$\boxed{(0, 1) \cup (1, \infty)}$$

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Question 9

Let

$$f(x) = (x^2 + 1)\tan(x)$$

(a) Differentiate $f(x)$ using the product rule.

(b) Determine all x for which $f'(x)$ is undefined in the interval $(-\pi, \pi)$.

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Question 9 - Solution

We are given:

$$f(x) = (x^2 + 1)\tan(x)$$

Let: $u(x) = x^2 + 1$ $v(x) = \tan(x)$
We'll use the product rule:

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$

Step 1: Differentiate $u(x)$:

$$u'(x) = 2x$$

Step 2: Differentiate $v(x) = \tan(x)$:

$$v'(x) = \sec^2(x)$$

Now apply the product rule:

$$f'(x) = 2x\tan(x) + (x^2 + 1)\sec^2(x)$$

Final Answer:

$$\boxed{f'(x) = 2x\tan(x) + (x^2 + 1)\sec^2(x)}$$

(b) Where is $f'(x)$ undefined on $(-\pi, \pi)$?
$\tan(x)$ and $\sec^2(x)$ are undefined where $\cos(x) = 0$
$\cos(x) = 0$ at $x = -\frac{\pi}{2}, \frac{\pi}{2}$ in the given interval
So, $f'(x)$ is undefined at:

$$\boxed{x = -\frac{\pi}{2},\ \frac{\pi}{2}}$$

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Question 10

Let

$$f(x) = \frac{x^2 e^x \sin(x)}{x^2 + 1}$$

(a) Differentiate $f(x)$ using appropriate differentiation rules.

(b) Simplify and factor your answer as much as possible.

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Question 10 - Solution

We are given:

$$f(x) = \frac{x^2 e^x \sin(x)}{x^2 + 1}$$

This is a quotient of two functions:

$$u(x) = x^2 e^x \sin(x), \quad v(x) = x^2 + 1$$

We apply the **quotient rule**:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Step 1: Differentiate the numerator

The numerator is a product of three functions:

$$u(x) = x^2 \cdot e^x \cdot \sin(x)$$

Using the product rule for three functions:

$$u'(x) = (x^2)' e^x \sin(x) + x^2 (e^x)' \sin(x) + x^2 e^x (\sin(x))'$$

Compute derivatives:

$$(x^2)' = 2x, \quad (e^x)' = e^x, \quad (\sin x)' = \cos x$$

Substitute:

$$u'(x) = 2x e^x \sin(x) + x^2 e^x \sin(x) + x^2 e^x \cos(x)$$

Factor:

$$u'(x) = e^x \left[\sin(x)(2x + x^2) + x^2 \cos(x)\right]$$

Step 2: Differentiate the denominator

$$v'(x) = 2x$$

Step 3: Apply the quotient rule

$$f'(x) = \frac{e^x \left[\sin(x)(2x + x^2) + x^2 \cos(x)\right](x^2 + 1) - x^2 e^x \sin(x)(2x)}{(x^2 + 1)^2}$$

Step 4: Simplify

Factor out e^x:

$$f'(x) = \frac{e^x \left\{(x^2 + 1)\left[\sin(x)(2x + x^2) + x^2 \cos(x)\right] - 2x^3 \sin(x)\right\}}{(x^2 + 1)^2}$$

This is a fully factored and simplified form.

Final Answer:

$$\boxed{f'(x) = \frac{e^x \left[(x^2 + 1)\left(\sin(x)(2x + x^2) + x^2 \cos(x)\right) - 2x^3 \sin(x)\right]}{(x^2 + 1)^2}}$$

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Question 1

Let

$$f(x) = \sin(x)\cos(x) + \tan(x)$$

(a) Find $f'(x)$, the derivative of $f(x)$, using standard derivative rules.

(b) Simplify the result as much as possible.

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Question 1 - Solution

We are given:

$$f(x) = \sin(x)\cos(x) + \tan(x)$$

(a) Differentiate each term:
We apply the product rule to the first term $\sin(x)\cos(x)$, and recall the derivative of $\tan(x)$ is $\sec^2(x)$:

$$\frac{d}{dx}[\sin(x)\cos(x)] = \cos(x)\cos(x) + \sin(x)(-\sin(x)) = \cos^2(x) - \sin^2(x)$$

So the derivative of the entire function is:

$$f'(x) = \cos^2(x) - \sin^2(x) + \sec^2(x)$$

(b) Simplify the result:
Note that:

$$\cos^2(x) - \sin^2(x) = \cos(2x)$$

Thus:

$$\boxed{f'(x) = \cos(2x) + \sec^2(x)}$$

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Question 2

Let

$$f(x) = x^2 \sec(x) + x \cot(x)$$

(a) Compute the derivative $f'(x)$.

(b) Simplify your result as much as possible.

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Question 2 - Solution

We are given:

$$f(x) = x^2 \sec(x) + x \cot(x)$$

(a) Differentiate each term:
We apply the product rule to both terms:

First Term: $x^2 \sec(x)$
Let $u = x^2$, $v = \sec(x)$, then:

$$\frac{d}{dx}[x^2 \sec(x)] = \frac{d}{dx}[u \cdot v] = u'v + uv' = 2x \sec(x) + x^2 \sec(x) \tan(x)$$

Second Term: $x \cot(x)$
Let $u = x$, $v = \cot(x)$, then:

$$\frac{d}{dx}[x \cot(x)] = u'v + uv' = 1 \cdot \cot(x) + x(-\csc^2(x)) = \cot(x) - x \csc^2(x)$$

(b) Combine all terms:

$$f'(x) = 2x \sec(x) + x^2 \sec(x) \tan(x) + \cot(x) - x \csc^2(x)$$

Final Answer:

$$\boxed{f'(x) = 2x \sec(x) + x^2 \sec(x) \tan(x) + \cot(x) - x \csc^2(x)}$$

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Question 3

Let

$$f(x) = \sin(x)\cos(x) + \tan(x)\sec(x)$$

(a) Find $f'(x)$.

(b) Simplify the expression.

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Question 3 - Solution

We are given:

$$f(x) = \sin(x)\cos(x) + \tan(x)\sec(x)$$

(a) Differentiate each term:

Term 1: $\sin(x)\cos(x)$
Using the product rule:

$$\frac{d}{dx}[\sin(x)\cos(x)] = \cos(x)\cos(x) + \sin(x)(-\sin(x)) = \cos^2(x) - \sin^2(x)$$

Term 2: $\tan(x)\sec(x)$
Using the product rule:

$$\frac{d}{dx}[\tan(x)\sec(x)] = \sec^2(x)\sec(x) + \tan(x)\sec(x)\tan(x) = \sec^3(x) + \sec(x)\tan^2(x)$$

(b) Combine and simplify:

$$f'(x) = \cos^2(x) - \sin^2(x) + \sec^3(x) + \sec(x)\tan^2(x)$$

Use trigonometric identities:

$$\cos^2(x) - \sin^2(x) = \cos(2x)$$

$$\tan^2(x) = \sec^2(x) - 1$$

$$\sec(x)\tan^2(x) = \sec^3(x) - \sec(x)$$

Substitute and combine like terms:

$$f'(x) = \cos(2x) + 2\sec^3(x) - \sec(x)$$

Final Answer:

$$\boxed{f'(x) = \cos(2x) + 2\sec^3(x) - \sec(x)}$$

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Question 4

Let

$$f(x) = \cot(x)\csc(x) + \cos^2(x)$$

(a) Find the derivative $f'(x)$.

(b) Simplify the expression as much as possible.

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Question 4 - Solution

We are given:

$$f(x) = \cot(x)\csc(x) + \cos^2(x)$$

(a) Differentiate each term separately.

Term 1: $\cot(x)\csc(x)$

Using the product rule:

$$\frac{d}{dx}[\cot(x)\csc(x)] = \cot(x)(-\cot(x)\csc(x)) + \csc(x)(-\csc^2(x))$$

$$= -\cot^2(x)\csc(x) - \csc^3(x)$$

Term 2: $\cos^2(x)$

Using the chain rule:

$$\frac{d}{dx}[\cos^2(x)] = 2\cos(x)(-\sin(x)) = -2\sin(x)\cos(x)$$

(b) Combine and simplify:

$$f'(x) = -\cot^2(x)\csc(x) - \csc^3(x) - 2\sin(x)\cos(x)$$

Use the identities:

$$\cot^2(x) = \csc^2(x) - 1 \qquad 2\sin(x)\cos(x) = \sin(2x)$$

$$-\cot^2(x)\csc(x) = -(\csc^2(x) - 1)\csc(x) = -\csc^3(x) + \csc(x)$$

Substitute and combine like terms:

$$f'(x) = \csc(x) - 2\csc^3(x) - \sin(2x)$$

Final Answer:

$$\boxed{f'(x) = \csc(x) - 2\csc^3(x) - \sin(2x)}$$

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Question 5

Let

$$f(x) = x\tan(x) - \sec(x)$$

(a) Find the derivative $f'(x)$.

(b) Identify any values of x where $f'(x)$ is undefined.

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Question 5 - Solution

We are given:

$$f(x) = x\tan(x) - \sec(x)$$

(a) Differentiate each term.
Term 1: $x\tan(x)$ Use the product rule:

$$\frac{d}{dx}[x\tan(x)] = x \cdot \sec^2(x) + \tan(x)$$

Term 2: $-\sec(x)$

$$\frac{d}{dx}[-\sec(x)] = -\sec(x)\tan(x)$$

Combine the terms:

$$f'(x) = x\sec^2(x) + \tan(x) - \sec(x)\tan(x)$$

Final Answer:

$$\boxed{f'(x) = x\sec^2(x) + \tan(x) - \sec(x)\tan(x)}$$

(b) Values where $f'(x)$ is undefined:
Note that $\tan(x)$ and $\sec(x)$ are undefined at:

$$x = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}$$

So, the derivative $f'(x)$ is undefined at:

$$\boxed{x = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}}$$

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Question 6

Let

$$f(x) = \frac{\sin(x)}{1 + \cos(x)}$$

(a) Find the derivative $f'(x)$.

(b) Simplify the derivative as much as possible.

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Question 6 - Solution

We are given:

$$f(x) = \frac{\sin(x)}{1 + \cos(x)}$$

(a) Use the Quotient Rule:
The quotient rule is:

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$$

Let:

$$g(x) = \sin(x), \quad h(x) = 1 + \cos(x)$$
$$g'(x) = \cos(x), \quad h'(x) = -\sin(x)$$

Then:

$$f'(x) = \frac{\cos(x)(1 + \cos(x)) - \sin(x)(-\sin(x))}{(1 + \cos(x))^2}$$

$$= \frac{\cos(x)(1 + \cos(x)) + \sin^2(x)}{(1 + \cos(x))^2}$$

(b) Simplify the numerator:
First expand:

$$\cos(x)(1 + \cos(x)) = \cos(x) + \cos^2(x)$$

So the full numerator is:

$$\cos(x) + \cos^2(x) + \sin^2(x)$$

Use the identity:

$$\cos^2(x) + \sin^2(x) = 1$$

So the numerator becomes:

$$\cos(x) + 1$$

Therefore,

$$f'(x) = \frac{\cos(x) + 1}{(1 + \cos(x))^2}$$

Simplify:

$$f'(x) = \frac{1}{1 + \cos(x)}$$

Final Answer:

$$\boxed{f'(x) = \frac{1}{1 + \cos(x)}}$$

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Question 7

Let

$$f(x) = \tan(x) \cdot \sec(x)$$

(a) Find the derivative $f'(x)$.

(b) Simplify the result using trigonometric identities.

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Question 7 - Solution

We are given:

$$f(x) = \tan(x) \cdot \sec(x)$$

(a) Apply the Product Rule:
The product rule states:

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$

Let:

$$u(x) = \tan(x), \quad v(x) = \sec(x)$$
$$u'(x) = \sec^2(x), \quad v'(x) = \sec(x)\tan(x)$$

Then:

$$f'(x) = \sec^2(x) \cdot \sec(x) + \tan(x) \cdot \sec(x) \cdot \tan(x)$$
$$= \sec^3(x) + \sec(x)\tan^2(x)$$

(b) Factor the result:
Both terms have a factor of $\sec(x)$, so factor it out:

$$f'(x) = \sec(x)\left(\sec^2(x) + \tan^2(x)\right)$$

Final Answer:

$$\boxed{f'(x) = \sec(x)\left(\sec^2(x) + \tan^2(x)\right)}$$

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Question 8

Let

$$f(x) = \frac{\sin(x)\cos(x)}{1+\sin^2(x)}$$

(a) Differentiate $f(x)$.

(b) Simplify the result as much as possible.

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Question 8 - Solution

We are given:

$$f(x) = \frac{\sin(x)\cos(x)}{1+\sin^2(x)}$$

Step 1: Rewrite using a double-angle identity.
Recall:

$$\sin(x)\cos(x) = \tfrac{1}{2}\sin(2x)$$

So we rewrite the function:

$$f(x) = \frac{\frac{1}{2}\sin(2x)}{1+\sin^2(x)}$$

Step 2: Apply the quotient rule.
Let:

$$u(x) = \tfrac{1}{2}\sin(2x), \qquad v(x) = 1+\sin^2(x)$$

Differentiate each function:

$$u'(x) = \cos(2x)$$
$$v'(x) = 2\sin(x)\cos(x) = \sin(2x)$$

The quotient rule states:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Substitute:

$$f'(x) = \frac{\cos(2x)(1+\sin^2(x)) - \frac{1}{2}\sin(2x)\sin(2x)}{(1+\sin^2(x))^2}$$

Step 3: Simplify carefully.
Rewrite the squared sine:

$$\sin^2(2x) = 1-\cos^2(2x)$$

So the numerator becomes:

$$\cos(2x)(1+\sin^2(x)) - \tfrac{1}{2}(1-\cos^2(2x))$$

Expand:

$$= \cos(2x) + \cos(2x)\sin^2(x) - \tfrac{1}{2} + \tfrac{1}{2}\cos^2(2x)$$

Now use:

$$\sin^2(x) = \tfrac{1-\cos(2x)}{2}$$

Substitute:

$$\cos(2x)\sin^2(x) = \tfrac{1}{2}\cos(2x) - \tfrac{1}{2}\cos^2(2x)$$

Combine terms:

$$= \tfrac{3}{2}\cos(2x) - \tfrac{1}{2}$$

Final Simplified Result:

$$f'(x) = \frac{3\cos(2x)-1}{2(1+\sin^2(x))^2}$$

Final Answer:

$$\boxed{f'(x) = \frac{3\cos(2x)-1}{2(1+\sin^2(x))^2}}$$

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Question 9

Let

$$f(x) = x\tan(x)$$

(a) Use the product rule to find the derivative of $f(x)$.

(b) Find the equation of the tangent line to $f(x)$ at $x = \frac{\pi}{4}$.

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Question 9 - Solution

We are given:

$$f(x) = x\tan(x)$$

(a) Differentiate Using the Product Rule:
Let:

$$u(x) = x, \quad v(x) = \tan(x)$$

Then:

$$u'(x) = 1, \quad v'(x) = \sec^2(x)$$

Product rule:

$$f'(x) = u'(x)v(x) + u(x)v'(x) = 1 \cdot \tan(x) + x \cdot \sec^2(x) = \tan(x) + x\sec^2(x)$$

$$\boxed{f'(x) = \tan(x) + x\sec^2(x)}$$

(b) Tangent Line at $x = \frac{\pi}{4}$:
Evaluate:

$$f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} \cdot \tan\left(\frac{\pi}{4}\right) = \frac{\pi}{4} \cdot 1 = \frac{\pi}{4}$$

$$f'\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) + \frac{\pi}{4} \cdot \sec^2\left(\frac{\pi}{4}\right) = 1 + \frac{\pi}{4} \cdot 2 = 1 + \frac{\pi}{2}$$

Use point-slope form:

$$y - f\left(\tfrac{\pi}{4}\right) = f'\left(\tfrac{\pi}{4}\right)\left(x - \tfrac{\pi}{4}\right) \Rightarrow y - \frac{\pi}{4} = \left(1 + \frac{\pi}{2}\right)\left(x - \frac{\pi}{4}\right)$$

Final Answer: The tangent line is:

$$\boxed{y = \left(1 + \frac{\pi}{2}\right)\left(x - \frac{\pi}{4}\right) + \frac{\pi}{4}}$$

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Question 10

Consider the function:

$$f(x) = \sin(x)\cos(x)$$

(a) Differentiate $f(x)$ using the product rule.

(b) Simplify the derivative using trigonometric identities.

(c) Evaluate $f'(x)$ at $x = \frac{\pi}{3}$.

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Question 10 - Solution

We are given:

$$f(x) = \sin(x)\cos(x)$$

(a) Differentiate Using the Product Rule:
Let:

$$u(x) = \sin(x), \quad v(x) = \cos(x)$$

Then:

$$u'(x) = \cos(x), \quad v'(x) = -\sin(x)$$

Using the product rule:

$$f'(x) = u'(x)v(x) + u(x)v'(x) = \cos(x)\cos(x) + \sin(x)(-\sin(x)) = \cos^2(x) - \sin^2(x)$$

$$\boxed{f'(x) = \cos^2(x) - \sin^2(x)}$$

(b) Simplify Using Trig Identity:
We use the identity:

$$\cos^2(x) - \sin^2(x) = \cos(2x)$$

So:

$$\boxed{f'(x) = \cos(2x)}$$

(c) Evaluate at $x = \frac{\pi}{3}$:

$$f'\left(\frac{\pi}{3}\right) = \cos\left(2 \cdot \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$$\boxed{f'\left(\frac{\pi}{3}\right) = -\frac{1}{2}}$$

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Question 1

Let

$$f(x) = \ln\left(x^2 + 1\right) \cdot e^{3x}$$

(a) Differentiate $f(x)$ using the product rule and the chain rule.

(b) Simplify your answer.

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Question 1 - Solution

We are given:

$$f(x) = \ln(x^2 + 1) \cdot e^{3x}$$

(a) Differentiate Using Product Rule:
Let:

$$u(x) = \ln(x^2 + 1), \quad v(x) = e^{3x}$$

Then:

$$u'(x) = \frac{1}{x^2 + 1} \cdot 2x = \frac{2x}{x^2 + 1}, \quad v'(x) = 3e^{3x}$$

Now apply the product rule:

$$f'(x) = u'(x)v(x) + u(x)v'(x) = \frac{2x}{x^2 + 1} \cdot e^{3x} + \ln(x^2 + 1) \cdot 3e^{3x}$$

(b) Final Simplified Answer:
Factor out e^{3x}:

$$f'(x) = e^{3x} \left(\frac{2x}{x^2 + 1} + 3\ln(x^2 + 1) \right)$$

$$\boxed{f'(x) = e^{3x} \left(\frac{2x}{x^2 + 1} + 3\ln(x^2 + 1) \right)}$$

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Question 2

Consider the function:

$$f(x) = \frac{e^x}{\ln(x)}$$

(a) Find the derivative $f'(x)$.

(b) State the domain of $f(x)$ and $f'(x)$.

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Question 2 - Solution

We are given:

$$f(x) = \frac{e^x}{\ln(x)}$$

(a) Differentiate $f(x)$:
This is a quotient, so we apply the quotient rule:

$$f'(x) = \frac{\left(\ln(x) \cdot \frac{d}{dx}[e^x]\right) - \left(e^x \cdot \frac{d}{dx}[\ln(x)]\right)}{(\ln(x))^2}$$

Differentiate:

$$\frac{d}{dx}[e^x] = e^x, \quad \frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

Now substitute:

$$f'(x) = \frac{\ln(x) \cdot e^x - e^x \cdot \frac{1}{x}}{(\ln(x))^2} = \frac{e^x\left(\ln(x) - \frac{1}{x}\right)}{(\ln(x))^2}$$

$$\boxed{f'(x) = \frac{e^x\left(\ln(x) - \frac{1}{x}\right)}{(\ln(x))^2}}$$

(b) Domain of $f(x)$:
$\ln(x)$ is only defined for $x > 0$
But we also need $\ln(x) \neq 0 \Rightarrow x \neq 1$
So domain of $f(x)$ is:

$$\boxed{(0,1) \cup (1,\infty)}$$

The derivative involves the same terms (no new restrictions), so:

$$\text{Domain of } f'(x) = \boxed{(0,1) \cup (1,\infty)}$$

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Question 3

Let $f(x) = x^x$ for $x > 0$.

(a) Use logarithmic differentiation to find $f'(x)$.

(b) Evaluate $f'(x)$ at $x = 2$.

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Question 3 - Solution

We are given:

$$f(x) = x^x, \quad x > 0$$

(a) Logarithmic Differentiation
Take natural log on both sides:

$$\ln(f(x)) = \ln(x^x) = x \ln(x)$$

Differentiate both sides implicitly:

$$\frac{1}{f(x)} \cdot f'(x) = \frac{d}{dx}[x \ln(x)]$$

Differentiate the right-hand side using product rule:

$$\frac{d}{dx}[x \ln(x)] = \ln(x) + 1$$

So:

$$\frac{1}{f(x)} \cdot f'(x) = \ln(x) + 1 \Rightarrow f'(x) = f(x) \cdot (\ln(x) + 1)$$

Recall $f(x) = x^x$, so:

$$\boxed{f'(x) = x^x(\ln(x) + 1)}$$

(b) Evaluate at $x = 2$:

$$f'(2) = 2^2(\ln(2) + 1) = 4(\ln(2) + 1)$$

Exact Answer:

$$\boxed{f'(2) = 4(\ln(2) + 1)}$$

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Question 4

Let the function $f(x) = e^{x \ln(\sin x)}$, for $0 < x < \pi$.

(a) Simplify the expression $f(x)$.

(b) Find the derivative $f'(x)$.

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Question 4 - Solution

We are given:

$$f(x) = e^{x \ln(\sin x)}, \quad 0 < x < \pi$$

(a) Simplify the expression:
Recall the identity:

$$e^{x \ln(\sin x)} = (e^{\ln(\sin x)})^x = (\sin x)^x$$

So:

$$f(x) = (\sin x)^x$$

(b) Differentiate $f(x) = (\sin x)^x$ using logarithmic differentiation:
Take natural log on both sides:

$$\ln(f(x)) = x \ln(\sin x)$$

Differentiate both sides:

$$\frac{1}{f(x)} \cdot f'(x) = \frac{d}{dx}[x \ln(\sin x)]$$

Apply product rule on the right-hand side:

$$\frac{d}{dx}[x \ln(\sin x)] = \ln(\sin x) + x \cdot \frac{\cos x}{\sin x} = \ln(\sin x) + x \cot x$$

Multiply both sides by $f(x) = (\sin x)^x$:

$$f'(x) = (\sin x)^x \cdot (\ln(\sin x) + x \cot x)$$

Final Answer:

$$\boxed{f'(x) = (\sin x)^x \left(\ln(\sin x) + x \cot x\right)}$$

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Question 5

Let the function $f(x) = \ln\left(\frac{x^2\sqrt{e^x+1}}{\ln x}\right)$, for $x > 1$.

(a) Simplify the expression using logarithmic properties.

(b) Find the derivative $f'(x)$.

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Question 5 - Solution

We are given:

$$f(x) = \ln\left(\frac{x^2\sqrt{e^x+1}}{\ln x}\right), \quad x > 1$$

(a) Simplify the expression:
Use logarithmic identities:

$$\ln\left(\frac{A \cdot B}{C}\right) = \ln(A) + \ln(B) - \ln(C)$$

Apply to this expression:

$$f(x) = \ln(x^2) + \ln\left((e^x+1)^{1/2}\right) - \ln(\ln x)$$

Now simplify powers:

$$f(x) = 2\ln x + \frac{1}{2}\ln(e^x+1) - \ln(\ln x)$$

(b) Differentiate $f(x)$:
Differentiate term-by-term:

$\frac{d}{dx}[2\ln x] = \frac{2}{x}$
$\frac{d}{dx}\left[\frac{1}{2}\ln(e^x+1)\right] = \frac{1}{2}\cdot\frac{e^x}{e^x+1}$
$\frac{d}{dx}[\ln(\ln x)] = \frac{1}{x\ln x}$

Putting it all together:

$$f'(x) = \frac{2}{x} + \frac{1}{2}\cdot\frac{e^x}{e^x+1} - \frac{1}{x\ln x}$$

Final Answer:

$$\boxed{f'(x) = \frac{2}{x} + \frac{e^x}{2(e^x+1)} - \frac{1}{x\ln x}}$$

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Question 6

Let $f(x) = x^x$, defined for $x > 0$.

(a) Find the derivative $f'(x)$.

(b) Evaluate $f'(1)$.

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Question 6 - Solution

We are given:

$$f(x) = x^x, \quad x > 0$$

This is a function where the base and exponent both depend on x. To differentiate it, we take logarithms on both sides.

Step 1: Take natural log of both sides:
Let $y = x^x$. Then,

$$\ln y = \ln(x^x) = x \ln x$$

Step 2: Differentiate both sides implicitly:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(x \ln x) = \ln x + 1$$

Step 3: Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y(\ln x + 1) = x^x(\ln x + 1)$$

Final answer:

$$\boxed{f'(x) = x^x(\ln x + 1)}$$

(b) Evaluate $f'(1)$:

$$f'(1) = 1^1(\ln 1 + 1) = 1(0 + 1) = \boxed{1}$$

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Question 7

Let $f(x) = \ln\left(\frac{e^x+1}{e^x-1}\right)$, for $x > 0$.

(a) Find $f'(x)$.

(b) Simplify your answer as much as possible.

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Question 7 - Solution

We are given:

$$f(x) = \ln\left(\frac{e^x + 1}{e^x - 1}\right)$$

Step 1: Use the derivative of the natural log of a quotient:

$$\frac{d}{dx}\ln\left(\frac{u(x)}{v(x)}\right) = \frac{d}{dx}\left[\ln(u(x)) - \ln(v(x))\right] = \frac{u'(x)}{u(x)} - \frac{v'(x)}{v(x)}$$

Here, $u(x) = e^x + 1$, and $v(x) = e^x - 1$
So,

$$f'(x) = \frac{d}{dx}\left[\ln(e^x + 1) - \ln(e^x - 1)\right] = \frac{e^x}{e^x + 1} - \frac{e^x}{e^x - 1}$$

Step 2: Combine the two terms:

$$f'(x) = \frac{e^x}{e^x + 1} - \frac{e^x}{e^x - 1}$$

Get a common denominator:

$$f'(x) = e^x\left(\frac{1}{e^x + 1} - \frac{1}{e^x - 1}\right) = e^x\left(\frac{(e^x - 1) - (e^x + 1)}{(e^x + 1)(e^x - 1)}\right) = e^x\left(\frac{e^x - 1 - e^x - 1}{e^{2x} - 1}\right)$$

$$f'(x) = e^x\left(\frac{-2}{e^{2x} - 1}\right) = \boxed{\frac{-2e^x}{e^{2x} - 1}}$$

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Question 8

Let $f(x) = x^x$, where $x > 0$.

(a) Find the derivative $f'(x)$.

(b) Simplify your answer as much as possible.

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Question 8 - Solution

We are given:

$$f(x) = x^x \quad \text{for } x > 0$$

This is a function where the variable is both the base and the exponent. To differentiate it, we take the natural logarithm of both sides:

Let

$$y = x^x$$

Take the natural log of both sides:

$$\ln y = \ln(x^x) = x \ln x$$

Differentiate both sides implicitly:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(x \ln x)$$

Now differentiate the right-hand side using the product rule:

$$\frac{d}{dx}(x \ln x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

Now solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y(\ln x + 1)$$

Recall that $y = x^x$, so:

$$f'(x) = \boxed{x^x(\ln x + 1)}$$

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Question 9

Let $a > 0$ be a real constant and define the function

$$f(x) = \ln\left(\frac{e^{ax}+1}{e^{ax}-1}\right).$$

(a) Find the derivative $f'(x)$ in terms of a.

(b) Simplify your answer as much as possible.

(c) Determine the domain of $f(x)$.

(d) Show that $f'(x)$ can be written in the form

$$f'(x) = -\frac{2a}{e^{ax}-e^{-ax}}.$$

(e) Hence, or otherwise, determine whether $f(x)$ is increasing or decreasing on its domain.

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Question 9 - Solution

We are given

$$f(x) = \ln\left(\frac{e^{ax}+1}{e^{ax}-1}\right), \quad a > 0.$$

(a) Differentiation

Rewrite the function using logarithmic identities:

$$f(x) = \ln(e^{ax}+1) - \ln(e^{ax}-1).$$

Differentiate term by term using the chain rule:

$$f'(x) = \frac{ae^{ax}}{e^{ax}+1} - \frac{ae^{ax}}{e^{ax}-1}.$$

(b) Simplification

Factor out ae^{ax}:

$$f'(x) = ae^{ax}\left(\frac{1}{e^{ax}+1} - \frac{1}{e^{ax}-1}\right).$$

Combine the fractions:

$$= ae^{ax}\left(\frac{(e^{ax}-1)-(e^{ax}+1)}{(e^{ax}+1)(e^{ax}-1)}\right) = ae^{ax}\left(\frac{-2}{e^{2ax}-1}\right).$$

Thus,

$$f'(x) = -\frac{2ae^{ax}}{e^{2ax}-1}.$$

(c) Domain of $f(x)$

The logarithm requires:

$$\frac{e^{ax}+1}{e^{ax}-1} > 0, \quad e^{ax}-1 \neq 0.$$

Since $e^{ax} > 0$ for all x, the expression is positive only when

$$e^{ax}-1 > 0 \quad \Rightarrow \quad x > 0.$$

Hence, the domain is:

$$\boxed{(0,\infty)}.$$

(d) Alternative Form

Multiply numerator and denominator of the derivative by e^{-ax}:

$$f'(x) = -\frac{2ae^{ax}}{e^{2ax}-1} = -\frac{2a}{e^{ax}-e^{-ax}}.$$

(e) Monotonicity

For $x > 0$,

$$e^{ax} - e^{-ax} > 0, \quad a > 0.$$

Therefore,

$$f'(x) < 0 \quad \text{for all } x > 0.$$

$$\boxed{\text{The function } f(x) \text{ is strictly decreasing on its domain.}}$$

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Question 10

Let $a > 0$ be a real constant and define

$$f(x) = x^{ax}, \quad x > 0.$$

(a) Use logarithmic differentiation to find $f'(x)$.

(b) Simplify your answer as much as possible.

(c) Find the second derivative $f''(x)$.

(d) Determine all stationary points of $f(x)$.

(e) Hence, determine the intervals on which $f(x)$ is increasing and decreasing.

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Question 10 - Solution

We are given

$$f(x) = x^{ax}, \quad x > 0, \quad a > 0.$$

(a) Logarithmic Differentiation

Let

$$y = x^{ax}.$$

Taking natural logarithms:

$$\ln y = ax \ln x.$$

(b) First Derivative

Differentiate both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = a \ln x + a.$$

Multiply through by y:

$$\frac{dy}{dx} = y(a \ln x + a).$$

Substitute back $y = x^{ax}$:

$$f'(x) = x^{ax}\, a(\ln x + 1).$$

$$\boxed{f'(x) = ax^{ax}(\ln x + 1)}$$

(c) Second Derivative

Differentiate $f'(x)$ using the product rule:

$$f'(x) = ax^{ax}(\ln x + 1).$$

Let

$$u = x^{ax}, \quad v = \ln x + 1.$$

Then:

$$u' = ax^{ax}(\ln x + 1), \quad v' = \frac{1}{x}.$$

Thus,

$$f''(x) = a\left[u'v + uv'\right] = a\left[x^{ax}a(\ln x + 1)^2 + x^{ax}\frac{1}{x}\right].$$

Factor out x^{ax}:

$$f''(x) = ax^{ax}\left[a(\ln x + 1)^2 + \frac{1}{x}\right].$$

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(d) Stationary Points

Stationary points occur when

$$f'(x) = 0.$$

Since $a > 0$ and $x^{ax} > 0$ for $x > 0$, we require:

$$\ln x + 1 = 0.$$

Solving:

$$\ln x = -1 \quad \Rightarrow \quad x = e^{-1}.$$

Thus, the only stationary point occurs at:

$$\boxed{x = \frac{1}{e}}.$$

(e) Increasing and Decreasing Behaviour

From

$$f'(x) = ax^{ax}(\ln x + 1),$$

the sign of $f'(x)$ depends on $\ln x + 1$.

- If $0 < x < \frac{1}{e}$, then $\ln x + 1 < 0 \Rightarrow f'(x) < 0$.
- If $x > \frac{1}{e}$, then $\ln x + 1 > 0 \Rightarrow f'(x) > 0$.

$$\boxed{\begin{array}{l}\text{Decreasing on } (0, e^{-1}),\\ \text{Increasing on } (e^{-1}, \infty).\end{array}}$$

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Question 1

Differentiate the function:

$$f(x) = x \cdot \arcsin(x)$$

Then evaluate $f'(x)$ at $x = \frac{1}{2}$.

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Question 1 - Solution

We are given:

$$f(x) = x \cdot \arcsin(x)$$

Use the product rule:

$$f'(x) = \arcsin(x) + x \cdot \frac{1}{\sqrt{1-x^2}}$$

Answer:

$$f'(x) = \boxed{\arcsin(x) + \frac{x}{\sqrt{1-x^2}}}$$

Now evaluate at $x = \frac{1}{2}$:

$$f'\left(\frac{1}{2}\right) = \arcsin\left(\frac{1}{2}\right) + \frac{\frac{1}{2}}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = \frac{\pi}{6} + \frac{1/2}{\sqrt{3}/2} = \frac{\pi}{6} + \frac{1}{\sqrt{3}}$$

Answer: $f'\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{6} + \frac{1}{\sqrt{3}}}$

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Question 2

Let

$$f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

(a) Show that $f(x) = 2\tan^{-1}(x)$ for all $x \in (-1, 1)$

(b) Use the identity in part (a) to find $f'(x)$.

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Question 2 - Solution

(a) Use an identity for tangent of double angle:

We recall:

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

Let $\theta = \tan^{-1}(x) \Rightarrow \tan(\theta) = x$. Then:

$$\tan(2\theta) = \frac{2x}{1-x^2} \Rightarrow 2\theta = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

So:

$$f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1}(x) \quad \text{for } x \in (-1,1)$$

(b) Differentiate using identity:

From (a), $f(x) = 2\tan^{-1}(x)$

Use the derivative:

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \Rightarrow f'(x) = 2\cdot\frac{1}{1+x^2} = \boxed{\frac{2}{1+x^2}}$$

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Question 3

Let

$$f(x) = \arcsin\left(\frac{2x}{1+x^2}\right), \qquad x > 0.$$

(a) Express f in terms of $\arctan x$, separately for $0 < x \leq 1$ and $x > 1$.

(b) Find $f'(x)$ wherever it exists, and analyze $x = 1$.

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Question 3 - Solution

Put $t = \arctan x$. For $x > 0$, $0 < t < \pi/2$ and

$$\frac{2x}{1+x^2} = \sin(2t).$$

The principal range of arcsin is $[-\pi/2, \pi/2]$. Therefore

$$\boxed{f(x) = \begin{cases} 2\arctan x, & 0 < x \le 1, \\ \pi - 2\arctan x, & x > 1. \end{cases}}$$

Differentiating on the two open intervals gives

$$\boxed{f'(x) = \begin{cases} 2/(1+x^2), & 0 < x < 1, \\ -2/(1+x^2), & x > 1. \end{cases}}$$

At $x = 1$ the left and right derivatives are 1 and -1, respectively, so no derivative exists there.

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Question 4

Let

$$f(x) = \tan^{-1}(\sqrt{1-x^2})$$

(a) Determine the domain of $f(x)$.

(b) Compute $f'(x)$ for $x \in (-1, 1)$.

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Question 4 - Solution

(a) Domain:

Since $f(x) = \tan^{-1}(\sqrt{1-x^2})$, we require the expression under the square root to be non-negative:

$$1 - x^2 \geq 0 \Rightarrow x^2 \leq 1 \Rightarrow -1 \leq x \leq 1$$

But the square root function $\sqrt{1-x^2}$ is only real for $x \in [-1, 1]$, and since it appears inside an inverse tangent (which is defined for all real inputs), the domain of f is:

$$\boxed{[-1, 1]}$$

(b) Compute the derivative:

Let's differentiate using the chain rule:

$$f(x) = \tan^{-1}(u), \quad \text{where } u = \sqrt{1-x^2} \Rightarrow f'(x) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

Now compute $\frac{du}{dx}$ where $u = \sqrt{1-x^2}$:

$$\frac{du}{dx} = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

Next, compute $1 + u^2 = 1 + (1 - x^2) = 2 - x^2$

So:

$$f'(x) = \frac{1}{2-x^2} \cdot \left(\frac{-x}{\sqrt{1-x^2}}\right) = \boxed{\frac{-x}{(2-x^2)\sqrt{1-x^2}}}$$

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Question 5

Let

$$f(x) = \sec^{-1}\left(\frac{2x+3}{x-1}\right)$$

(a) Determine the domain of $f(x)$.

(b) Compute $f'(x)$.

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Question 5 - Solution

(a) Domain:

The inverse secant function is defined for $|u| \geq 1$. Thus, we require

$$\left|\frac{2x+3}{x-1}\right| \geq 1$$

Squaring both sides:

$$\frac{(2x+3)^2}{(x-1)^2} \geq 1 \quad \Rightarrow \quad (2x+3)^2 \geq (x-1)^2$$

Expanding:

$$4x^2 + 12x + 9 \geq x^2 - 2x + 1$$

$$3x^2 + 14x + 8 \geq 0$$

Factor:

$$(3x+2)(x+4) \geq 0$$

Critical points are $x = -4$ and $x = -\frac{2}{3}$. Since the quadratic opens upward, the inequality holds outside the interval between the roots:

$$x \leq -4 \quad \text{or} \quad x \geq -\tfrac{2}{3}$$

We must also exclude $x = 1$, where the denominator is zero.
Therefore, the domain of $f(x)$ is:

$$\boxed{(-\infty, -4] \cup \left[-\tfrac{2}{3}, 1\right) \cup (1, \infty)}$$

(b) Derivative:

Using the derivative formula:

$$\frac{d}{dx}\left(\sec^{-1}(u)\right) = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

Let

$$u = \frac{2x+3}{x-1}$$

Then

$$\frac{du}{dx} = \frac{(x-1)(2) - (2x+3)(1)}{(x-1)^2} = \frac{-5}{(x-1)^2}$$

Substitute into the formula:

$$f'(x) = \frac{1}{\left|\frac{2x+3}{x-1}\right| \sqrt{\left(\frac{2x+3}{x-1}\right)^2 - 1}} \cdot \frac{-5}{(x-1)^2}$$

$$\boxed{f'(x) = \frac{-5}{(x-1)^2 \left|\frac{2x+3}{x-1}\right| \sqrt{\left(\frac{2x+3}{x-1}\right)^2 - 1}}}$$

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Question 6

Let

$$f(x) = \tan^{-1}\left(\sqrt{1-x^2}\right)$$

(a) Find the derivative $f'(x)$.

(b) Determine the domain of $f'(x)$.

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Question 6 - Solution

We are given:

$$f(x) = \tan^{-1}\left(\sqrt{1-x^2}\right)$$

(a) Finding $f'(x)$:

Let $u(x) = \sqrt{1-x^2}$, so:

$$f(x) = \tan^{-1}(u) \Rightarrow f'(x) = \frac{1}{1+u^2} \cdot u'(x)$$

First, compute $u'(x)$:

$$u(x) = (1-x^2)^{1/2} \Rightarrow u'(x) = \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

Now compute:

$$f'(x) = \frac{1}{1+(1-x^2)} \cdot \frac{-x}{\sqrt{1-x^2}} = \frac{1}{2-x^2} \cdot \frac{-x}{\sqrt{1-x^2}}$$

Final Answer:

$$\boxed{f'(x) = \frac{-x}{(2-x^2)\sqrt{1-x^2}}}$$

(b) Domain of $f'(x)$:

We need: $\sqrt{1-x^2}$ to be real $1-x^2 \geq 0 \Rightarrow x \in [-1,1]$, Denominator $\sqrt{1-x^2} \neq 0$ exclude $x = \pm 1$, Also ensure $2-x^2 \neq 0 \Rightarrow x \neq \sqrt{2}, -\sqrt{2}$ (but these are already excluded)

So domain of $f'(x)$ is:

$$\boxed{(-1,1)}$$

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Question 7

Let

$$f(x) = \arcsin\left(\frac{2x}{\sqrt{1+x^2}+1}\right).$$

(a) Determine the real domain.

(b) Decide whether $f(x) = \arctan x$ holds on that domain. Justify your answer.

(c) Compute $f'(x)$ wherever it exists.

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Question 7 - Solution

Domain. Let $s = \sqrt{1+x^2} \geq 1$. The argument of arcsin must satisfy

$$\frac{4x^2}{(s+1)^2} = \frac{4(s-1)}{s+1} \leq 1.$$

This is equivalent to $s \leq 5/3$, hence

$$\boxed{D = [-4/3, 4/3]}.$$

Test the claimed identity. At $x = 4/3$, the arcsine is $\pi/2$, while $\arctan(4/3) < \pi/2$. Thus $f(x) = \arctan x$ is false.

Differentiate. For $|x| < 4/3$, put $u = 2x/(s+1)$. Then

$$u' = \frac{2}{s(s+1)}, \qquad 1 - u^2 = \frac{5-3s}{s+1}.$$

Consequently

$$\boxed{f'(x) = \frac{2}{s\sqrt{s+1}\sqrt{5-3s}}}, \qquad s = \sqrt{1+x^2}.$$

The derivative is defined on $(-4/3, 4/3)$ and grows without bound toward either endpoint.

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Question 8

Let $f(x) = \tan^{-1}\left(\sqrt{1-x^2}\right)$, where $-1 < x < 1$.

(a) Compute $f'(x)$.

(b) Simplify your result as much as possible.

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Question 8 - Solution

We are given:

$$f(x) = \tan^{-1}\left(\sqrt{1-x^2}\right)$$

(a) Differentiate using the chain rule:
Let $u = \sqrt{1-x^2}$, so:

$$f(x) = \tan^{-1}(u) \Rightarrow f'(x) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

Now compute each part:

$$u = (1-x^2)^{1/2}, \quad \frac{du}{dx} = \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

Now plug into the derivative:

$$f'(x) = \frac{1}{1+(1-x^2)} \cdot \frac{-x}{\sqrt{1-x^2}} = \frac{1}{2-x^2} \cdot \frac{-x}{\sqrt{1-x^2}}$$

$$f'(x) = \boxed{\frac{-x}{(2-x^2)\sqrt{1-x^2}}}$$

(b) Final simplified form:

$$\boxed{f'(x) = \frac{-x}{(2-x^2)\sqrt{1-x^2}}}$$

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Question 9

Let $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, where $-1 < x < 1$.

(a) Compute $f'(x)$.

(b) Show that the result simplifies to a rational function.

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Question 9 - Solution

We are given:

$$f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Let $u = \frac{2x}{1+x^2}$, so:

$$f'(x) = \frac{1}{\sqrt{1-u^2}} \cdot u'$$

Step 1: Differentiate $u = \frac{2x}{1+x^2}$:
Using the quotient rule:

$$u' = \frac{(1+x^2)(2) - 2x(2x)}{(1+x^2)^2} = \frac{2(1+x^2) - 4x^2}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}$$

Step 2: Simplify $\sqrt{1-u^2}$:

$$u = \frac{2x}{1+x^2} \Rightarrow u^2 = \frac{4x^2}{(1+x^2)^2} \Rightarrow 1-u^2 = 1 - \frac{4x^2}{(1+x^2)^2} = \frac{(1+x^2)^2 - 4x^2}{(1+x^2)^2}$$

$$= \frac{1+2x^2+x^4-4x^2}{(1+x^2)^2} = \frac{1-2x^2+x^4}{(1+x^2)^2} = \frac{(1-x^2)^2}{(1+x^2)^2} \Rightarrow \sqrt{1-u^2} = \frac{|1-x^2|}{1+x^2}$$

Since $-1 < x < 1$, we have $1 - x^2 > 0$, so:

$$\sqrt{1-u^2} = \frac{1-x^2}{1+x^2}$$

Step 3: Combine all parts:

$$f'(x) = \frac{1}{\sqrt{1-u^2}} \cdot u' = \frac{1}{\frac{1-x^2}{1+x^2}} \cdot \frac{2(1-x^2)}{(1+x^2)^2} = \frac{1+x^2}{1-x^2} \cdot \frac{2(1-x^2)}{(1+x^2)^2}$$

Cancel $1 - x^2$ and one factor of $1 + x^2$:

$$f'(x) = \frac{2}{1+x^2}$$

Final Answer:

$$\boxed{f'(x) = \frac{2}{1+x^2}}$$

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Question 10

Let $f(x) = \tan^{-1}(\sqrt{1-x^2})$ for $-1 < x < 1$.

(a) Find $f'(x)$.

(b) Simplify your answer as much as possible.

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Question 10 - Solution

We are given:

$$f(x) = \tan^{-1}(\sqrt{1-x^2})$$

Let $u = \sqrt{1-x^2}$, then by the chain rule:

$$f'(x) = \frac{1}{1+u^2} \cdot u'$$

Step 1: Compute u':

$$u = (1-x^2)^{1/2} \Rightarrow u' = \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

Step 2: Compute $1+u^2$:

$$1 + u^2 = 1 + (1-x^2) = 2 - x^2$$

Step 3: Combine everything:

$$f'(x) = \frac{1}{1+u^2} \cdot u' = \frac{1}{2-x^2} \cdot \frac{-x}{\sqrt{1-x^2}} = \boxed{\frac{-x}{(2-x^2)\sqrt{1-x^2}}}$$

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Question 1

Differentiate the function:

$$f(x) = \sinh(x) \cdot \cosh(x)$$

Then express your answer in terms of a single hyperbolic function if possible.

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Question 1 - Solution

We are given:

$$f(x) = \sinh(x) \cdot \cosh(x)$$

Use the product rule:

$$f'(x) = \cosh(x) \cdot \cosh(x) + \sinh(x) \cdot \sinh(x) = \cosh^2(x) + \sinh^2(x)$$

Recall the identity:

$$\cosh^2(x) - \sinh^2(x) = 1 \quad \Rightarrow \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1$$

Thus:

$$f'(x) = \boxed{\cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1}$$

Answer: $f'(x) = \boxed{\cosh(2x)}$

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Question 2

Let $f(x) = \frac{\sinh(3x)}{\cosh(x)}$.

(a) Compute $f'(x)$ using differentiation rules.

(b) Simplify the result as much as possible using hyperbolic identities.

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Question 2 - Solution

The quotient rule gives

$$f'(x) = \frac{3\cosh(3x)\cosh x - \sinh(3x)\sinh x}{\cosh^2 x}.$$

Use the product identities

$$\cosh(3x)\cosh x = \tfrac{1}{2}(\cosh 4x + \cosh 2x),$$

$$\sinh(3x)\sinh x = \tfrac{1}{2}(\cosh 4x - \cosh 2x).$$

Thus

$$\boxed{f'(x) = \frac{\cosh 4x + 2\cosh 2x}{\cosh^2 x}}.$$

At zero the derivative is 3, agreeing with the original quotient-rule expression.

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Question 3

Let $f(x) = \tanh(x)\sinh(x)$.

(a) Compute $f'(x)$.

(b) Simplify your result as much as possible using hyperbolic identities.

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Question 3 - Solution

We are given:

$$f(x) = \tanh(x)\sinh(x)$$

This is a product of two functions, so we apply the product rule:

$$f'(x) = \tanh'(x)\sinh(x) + \tanh(x)\sinh'(x)$$

Recall the derivatives:

$$\tanh'(x) = \operatorname{sech}^2(x), \quad \sinh'(x) = \cosh(x)$$

Substitute:

$$f'(x) = \operatorname{sech}^2(x)\sinh(x) + \tanh(x)\cosh(x)$$

Now simplify. Recall:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \Rightarrow \tanh(x)\cosh(x) = \sinh(x)$$

So:

$$f'(x) = \operatorname{sech}^2(x)\sinh(x) + \sinh(x) = \sinh(x)\left(\operatorname{sech}^2(x) + 1\right)$$

Final Answer:

$$\boxed{f'(x) = \sinh(x)\left(\operatorname{sech}^2(x) + 1\right)}$$

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Question 4

Let $f(x) = \frac{\cosh^2(x)-1}{\sinh(x)}$.

(a) Simplify the expression before differentiating.

(b) Compute $f'(x)$.

(c) Identify any points where $f'(x)$ is undefined.

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Question 4 - Solution

(a) Simplification:

We start with:

$$f(x) = \frac{\cosh^2(x) - 1}{\sinh(x)}$$

Use the identity:

$$\cosh^2(x) - 1 = \sinh^2(x)$$

Thus:

$$f(x) = \frac{\sinh^2(x)}{\sinh(x)} = \sinh(x)$$

(b) Differentiate:

$$f'(x) = \frac{d}{dx}\sinh(x) = \cosh(x)$$

(c) Points of Non-Differentiability:

Since $f(x) = \sinh(x)$ is differentiable for all real x, and the simplification is valid as long as the denominator of the original function $\sinh(x) \neq 0$, we must consider the domain of the original function.

The original function is undefined at:

$$\sinh(x) = 0 \Rightarrow x = 0$$

So although $f(x) = \sinh(x)$ is differentiable everywhere, the original form $f(x) = \frac{\cosh^2(x)-1}{\sinh(x)}$ is undefined at $x = 0$, and thus f is not defined or differentiable there in its original form.

Conclusion:

$$\boxed{f'(x) = \cosh(x), \quad f'(x) \text{ undefined at } x = 0}$$

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Question 5

Let $f(x) = \frac{\sinh(x)}{\cosh(x)}$.

(a) Simplify the function if possible.

(b) Find $f'(x)$.

(c) Determine the intervals where $f(x)$ is increasing or decreasing.

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Question 5 - Solution

(a) Simplify the Function:
We have:

$$f(x) = \frac{\sinh(x)}{\cosh(x)}$$

This is the definition of the hyperbolic tangent function:

$$f(x) = \tanh(x)$$

(b) Differentiate $f(x) = \tanh(x)$:
The derivative of $\tanh(x)$ is:

$$f'(x) = \operatorname{sech}^2(x)$$

(c) Increasing/Decreasing Intervals:
Since $f'(x) = \operatorname{sech}^2(x) > 0$ for all real x, the function is strictly increasing on its entire domain.
Conclusion:

$$\boxed{f'(x) = \operatorname{sech}^2(x), \quad f(x) \text{ is increasing on } (-\infty, \infty)}$$

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Question 6

Let $f(x) = x \cdot \cosh(x^2)$.

(a) Find $f'(x)$.

(b) Determine the value of the derivative at $x = 0$.

(c) Is there a local minimum, maximum, or neither at $x = 0$? Justify using the derivative.

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Question 6 - Solution

We are given:

$$f(x) = x \cdot \cosh(x^2)$$

(a) Find $f'(x)$:

This is a product of two functions: Let $u = x$, $v = \cosh(x^2)$

Then:

$$f'(x) = u'v + uv' = 1 \cdot \cosh(x^2) + x \cdot \frac{d}{dx}[\cosh(x^2)]$$

Differentiate $\cosh(x^2)$ using the chain rule:

$$\frac{d}{dx}[\cosh(x^2)] = \sinh(x^2) \cdot 2x$$

So:

$$f'(x) = \cosh(x^2) + x \cdot \big(2x \cdot \sinh(x^2)\big) = \cosh(x^2) + 2x^2 \sinh(x^2)$$

$$\boxed{f'(x) = \cosh(x^2) + 2x^2 \sinh(x^2)}$$

(b) Value at $x = 0$:

$$f'(0) = \cosh(0^2) + 2 \cdot 0^2 \cdot \sinh(0^2) = \cosh(0) = 1$$

$$\boxed{f'(0) = 1}$$

(c) Local Behavior at $x = 0$:

Since $f'(0) = 1 > 0$, the function is increasing at $x = 0$, so:

Conclusion: There is **no local minimum or maximum** at $x = 0$, because the derivative is positive and the function is increasing there.

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Question 7

Let $f(x) = \tanh(x^2) \cdot \operatorname{sech}(x)$.

(a) Differentiate $f(x)$ using the product rule and the chain rule.

(b) Simplify the derivative as much as possible using hyperbolic identities.

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Question 7 - Solution

We are given:

$$f(x) = \tanh(x^2) \cdot \operatorname{sech}(x)$$

(a) Use the product rule:
Let:

$$u = \tanh(x^2), \quad v = \operatorname{sech}(x)$$

Then:

$$f'(x) = u'v + uv'$$

Differentiate each part:
Step 1: $u = \tanh(x^2) \Rightarrow u' = \operatorname{sech}^2(x^2) \cdot 2x$ using chain rule
Step 2: $v = \operatorname{sech}(x) \Rightarrow v' = -\operatorname{sech}(x) \cdot \tanh(x)$
Now substitute into the product rule:

$$f'(x) = \left[\operatorname{sech}^2(x^2) \cdot 2x\right] \cdot \operatorname{sech}(x) + \tanh(x^2) \cdot [-\operatorname{sech}(x) \cdot \tanh(x)]$$

(b) Final expression:

$$\boxed{f'(x) = 2x \operatorname{sech}^2(x^2) \operatorname{sech}(x) - \tanh(x^2) \operatorname{sech}(x) \tanh(x)}$$

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Question 8

Let $f(x) = \frac{\sinh(3x)}{x^2+1}$.

(a) Find the derivative $f'(x)$ using the quotient rule.

(b) Evaluate the derivative at $x = 0$.

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Question 8 - Solution

We are given:

$$f(x) = \frac{\sinh(3x)}{x^2+1}$$

(a) Find $f'(x)$:
This is a quotient, so we use the quotient rule:

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$$

Let:

$$g(x) = \sinh(3x), \quad h(x) = x^2+1$$

Then:

$$g'(x) = 3\cosh(3x), \quad h'(x) = 2x$$

So:

$$f'(x) = \frac{3\cosh(3x)(x^2+1) - \sinh(3x)(2x)}{(x^2+1)^2}$$

$$\boxed{f'(x) = \frac{3(x^2+1)\cosh(3x) - 2x\sinh(3x)}{(x^2+1)^2}}$$

(b) Evaluate at $x = 0$:
First note:

$$\cosh(0) = 1, \quad \sinh(0) = 0$$

Then:

$$f'(0) = \frac{3(0^2+1)\cdot 1 - 0}{(0^2+1)^2} = \frac{3}{1} = 3$$

$$\boxed{f'(0) = 3}$$

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Question 9

Let $f(x) = \tanh(x^2 + 1)$.

(a) Find the derivative $f'(x)$ using the chain rule and hyperbolic identities.

(b) Determine the value of the derivative at $x = 0$.

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Question 9 - Solution

We are given:

$$f(x) = \tanh(x^2 + 1)$$

(a) Find $f'(x)$:
Use the chain rule:

$$f'(x) = \frac{d}{dx}\left[\tanh(u)\right] \cdot \frac{d}{dx}(x^2 + 1)$$

Recall:

$$\frac{d}{dx}[\tanh(u)] = \operatorname{sech}^2(u)$$

So let $u = x^2 + 1$, then:

$$f'(x) = \operatorname{sech}^2(x^2 + 1) \cdot 2x$$

$$\boxed{f'(x) = 2x \cdot \operatorname{sech}^2(x^2 + 1)}$$

(b) Evaluate at $x = 0$:

$$f'(0) = 2 \cdot 0 \cdot \operatorname{sech}^2(0^2 + 1) = 0$$

$$\boxed{f'(0) = 0}$$

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Question 10

Let $f(x) = \frac{\sinh(3x)}{\cosh(x)}$.

(a) Differentiate $f(x)$ using the quotient rule and known derivatives of hyperbolic functions.

(b) Simplify the expression for $f'(x)$ as much as possible.

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Question 10 - Solution

We are given:

$$f(x) = \frac{\sinh(3x)}{\cosh(x)}$$

(a) Apply the quotient rule:
The quotient rule is:

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Let:

$$u = \sinh(3x), \quad v = \cosh(x)$$

Differentiate:

$$u' = 3\cosh(3x), \quad v' = \sinh(x)$$

Apply the rule:

$$f'(x) = \frac{3\cosh(3x)\cosh(x) - \sinh(3x)\sinh(x)}{\cosh^2(x)}$$

(b) Final simplified expression:

$$\boxed{f'(x) = \frac{3\cosh(3x)\cosh(x) - \sinh(3x)\sinh(x)}{\cosh^2(x)}}$$

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Question 1

Differentiate the function:

$$f(x) = \sqrt{1 + e^{3x^2}}$$

Clearly show how the chain rule is used in multiple layers.

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Question 1 - Solution

We are given:

$$f(x) = \sqrt{1 + e^{3x^2}} = (1 + e^{3x^2})^{1/2}$$

Use the chain rule:

$$f'(x) = \frac{1}{2}(1 + e^{3x^2})^{-1/2} \cdot \frac{d}{dx}(1 + e^{3x^2})$$

$$= \frac{1}{2\sqrt{1 + e^{3x^2}}} \cdot \frac{d}{dx}(e^{3x^2})$$

Now apply the chain rule again to e^{3x^2}:

$$\frac{d}{dx}(e^{3x^2}) = e^{3x^2} \cdot \frac{d}{dx}(3x^2) = e^{3x^2} \cdot 6x$$

Putting it all together:

$$f'(x) = \frac{1}{2\sqrt{1 + e^{3x^2}}} \cdot e^{3x^2} \cdot 6x = \boxed{\frac{3x \cdot e^{3x^2}}{\sqrt{1 + e^{3x^2}}}}$$

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Question 2

Let $f(x) = \sin\left(\sqrt{1+x^4}\right)$.

(a) Find $f'(x)$ using the chain rule.

(b) Evaluate $f'(1)$ exactly (do not approximate numerically).

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Question 2 - Solution

We are given:

$$f(x) = \sin\left(\sqrt{1+x^4}\right)$$

Let's apply the chain rule step by step.

Step 1: Let

$$u(x) = \sqrt{1+x^4} = (1+x^4)^{1/2} \quad \text{so that} \quad f(x) = \sin(u(x))$$

Step 2: Apply the chain rule:

$$f'(x) = \cos(u(x)) \cdot u'(x)$$

Step 3: Differentiate $u(x)$:

$$u(x) = (1+x^4)^{1/2} \Rightarrow u'(x) = \frac{1}{2}(1+x^4)^{-1/2} \cdot 4x^3 = \frac{2x^3}{\sqrt{1+x^4}}$$

Final derivative:

$$f'(x) = \cos\left(\sqrt{1+x^4}\right) \cdot \frac{2x^3}{\sqrt{1+x^4}}$$

$$\boxed{f'(x) = \frac{2x^3 \cos\left(\sqrt{1+x^4}\right)}{\sqrt{1+x^4}}}$$

(b) Evaluate at $x = 1$:

$$f'(1) = \frac{2(1)^3 \cdot \cos\left(\sqrt{1+1^4}\right)}{\sqrt{1+1^4}} = \frac{2 \cdot \cos(\sqrt{2})}{\sqrt{2}}$$

$$\boxed{f'(1) = \frac{2\cos(\sqrt{2})}{\sqrt{2}}}$$

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Question 3

Let $y = \ln\left(\tan^3(5x^2+1)\right)$.

(a) Differentiate y with respect to x using the chain rule.

(b) Simplify the derivative as much as possible.

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Question 3 - Solution

We are given:

$$y = \ln\left(\tan^3(5x^2+1)\right)$$

Step 1: Use logarithmic identity:

$$\ln\left(\tan^3(5x^2+1)\right) = 3\ln\left(\tan(5x^2+1)\right)$$

Step 2: Differentiate using chain rule:

$$\frac{dy}{dx} = 3 \cdot \frac{1}{\tan(5x^2+1)} \cdot \sec^2(5x^2+1) \cdot (10x)$$

Step 3: Simplify the expression:

$$\frac{dy}{dx} = \frac{30x\sec^2(5x^2+1)}{\tan(5x^2+1)}$$

$$\boxed{\frac{dy}{dx} = \frac{30x\sec^2(5x^2+1)}{\tan(5x^2+1)}}$$

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Question 4

Let $y = \sqrt{1 + \sin^2(3x)}$.

(a) Find $\dfrac{dy}{dx}$ using the chain rule.

(b) Simplify the result as much as possible.

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Question 4 - Solution

We are given:

$$y = \sqrt{1 + \sin^2(3x)} = (1 + \sin^2(3x))^{1/2}$$

Step 1: Apply the chain rule:
Let: - Outer function: $u^{1/2}$ - Inner function: $u = 1 + \sin^2(3x)$
We compute:

$$\frac{dy}{dx} = \frac{1}{2}(1 + \sin^2(3x))^{-1/2} \cdot \frac{d}{dx}[\sin^2(3x)]$$

Now differentiate $\sin^2(3x)$ using the chain rule:

$$\frac{d}{dx}[\sin^2(3x)] = 2\sin(3x) \cdot \cos(3x) \cdot 3 = 6\sin(3x)\cos(3x)$$

Step 2: Combine terms:

$$\frac{dy}{dx} = \frac{1}{2}(1 + \sin^2(3x))^{-1/2} \cdot 6\sin(3x)\cos(3x)$$

$$\frac{dy}{dx} = \frac{3\sin(3x)\cos(3x)}{\sqrt{1 + \sin^2(3x)}}$$

$$\boxed{\frac{dy}{dx} = \frac{3\sin(3x)\cos(3x)}{\sqrt{1 + \sin^2(3x)}}}$$

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Question 5

Let $f(x) = e^{\tan(x^3+1)}$.

(a) Use the chain rule to find $f'(x)$.

(b) Clearly identify each function used in the composition and the order in which the chain rule is applied.

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Question 5 - Solution

We are given:

$$f(x) = e^{\tan(x^3+1)}$$

This is a composition of multiple functions. Let's break it down:

- Outer function: $u = e^v$
- Middle function: $v = \tan(w)$
- Inner function: $w = x^3 + 1$

Using the chain rule:

$$f'(x) = \frac{d}{dx}\left[e^{\tan(x^3+1)}\right] = e^{\tan(x^3+1)} \cdot \frac{d}{dx}\left[\tan(x^3+1)\right]$$

Now,

$$\frac{d}{dx}\left[\tan(x^3+1)\right] = \sec^2(x^3+1) \cdot \frac{d}{dx}[x^3+1] = \sec^2(x^3+1) \cdot 3x^2$$

Putting it all together:

$$f'(x) = e^{\tan(x^3+1)} \cdot \sec^2(x^3+1) \cdot 3x^2$$

$$\boxed{f'(x) = 3x^2 \cdot \sec^2(x^3+1) \cdot e^{\tan(x^3+1)}}$$

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Question 6

Let $f(x) = \ln\left(\sqrt{1+\sin^2(5x)}\right)$.

(a) Use the chain rule to compute $f'(x)$.

(b) Identify the layers of composition involved and describe the differentiation process step-by-step.

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Question 6 - Solution

We are given:

$$f(x) = \ln\left(\sqrt{1+\sin^2(5x)}\right)$$

Let's simplify before differentiating:

$$f(x) = \ln\left((1+\sin^2(5x))^{1/2}\right) = \frac{1}{2}\ln(1+\sin^2(5x))$$

Now we differentiate:

$$f'(x) = \frac{1}{2}\cdot\frac{1}{1+\sin^2(5x)}\cdot\frac{d}{dx}[\sin^2(5x)]$$

We apply the chain rule to $\sin^2(5x)$:

$$\frac{d}{dx}[\sin^2(5x)] = 2\sin(5x)\cdot\cos(5x)\cdot 5 = 10\sin(5x)\cos(5x)$$

Therefore:

$$f'(x) = \frac{1}{2}\cdot\frac{10\sin(5x)\cos(5x)}{1+\sin^2(5x)}$$

$$\boxed{f'(x) = \frac{5\sin(5x)\cos(5x)}{1+\sin^2(5x)}}$$

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Question 7

Let $y = e^{\tan^3(2x)}$.

(a) Use the chain rule to compute $\frac{dy}{dx}$.

(b) Clearly outline the layers of composition and how each derivative is applied.

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Question 7 - Solution

We are given:

$$y = e^{\tan^3(2x)} = e^{(\tan(2x))^3}$$

Let's identify the layers of composition:

- Outermost: e^u
- Middle: $u = v^3$
- Inner: $v = \tan(2x)$
- Innermost: $2x$

Apply the chain rule step-by-step:

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{(\tan(2x))^3}\right) = e^{(\tan(2x))^3} \cdot \frac{d}{dx}\left((\tan(2x))^3\right)$$

Differentiate the power:

$$\frac{d}{dx}\left((\tan(2x))^3\right) = 3(\tan(2x))^2 \cdot \frac{d}{dx}[\tan(2x)]$$

Differentiate $\tan(2x)$:

$$\frac{d}{dx}[\tan(2x)] = \sec^2(2x) \cdot 2$$

Now put it all together:

$$\frac{dy}{dx} = e^{(\tan(2x))^3} \cdot 3(\tan(2x))^2 \cdot \sec^2(2x) \cdot 2$$

Simplify:

$$\boxed{\frac{dy}{dx} = 6(\tan(2x))^2 \sec^2(2x) \cdot e^{(\tan(2x))^3}}$$

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Question 8

Let $y = \ln\left(\sqrt{1+e^{3x^2}}\right)$.

(a) Use the chain rule to compute $\frac{dy}{dx}$.

(b) Explain how the composition of functions influences the order of derivatives.

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Question 8 - Solution

We are given:

$$y = \ln\left(\sqrt{1 + e^{3x^2}}\right)$$

We rewrite the square root as a power:

$$y = \ln\left((1 + e^{3x^2})^{1/2}\right)$$

Using the log rule:

$$y = \frac{1}{2}\ln(1 + e^{3x^2})$$

Differentiate:

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{1 + e^{3x^2}} \cdot \frac{d}{dx}[e^{3x^2}]$$

Differentiate the exponential:

$$\frac{d}{dx}[e^{3x^2}] = e^{3x^2} \cdot \frac{d}{dx}[3x^2] = e^{3x^2} \cdot 6x$$

Putting it all together:

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{1 + e^{3x^2}} \cdot e^{3x^2} \cdot 6x$$

Simplify:

$$\boxed{\frac{dy}{dx} = \frac{3xe^{3x^2}}{1 + e^{3x^2}}}$$

Explanation: This function has a nested composition: Innermost: $3x^2$, Exponential: e^{3x^2} , Added to 1 , Square root $\rightarrow$ converted to a power , Then natural logarithm

At each step, the derivative is taken from the outside and multiplied by the derivative of the inside, following the chain rule.

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Question 9

Let $f(x) = \sin^2(5x^3 + 1)$.

(a) Use the chain rule to compute $f'(x)$.

(b) Write down each layer of function composition clearly before differentiating.

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Question 9 - Solution

We are given:

$$f(x) = \sin^2(5x^3 + 1)$$

This can be rewritten as:

$$f(x) = [\sin(5x^3 + 1)]^2$$

Layers of function composition:
1. Innermost function: $u(x) = 5x^3 + 1$ 2. Middle function: $v(u) = \sin(u)$ 3. Outer function: $f(v) = v^2$
Differentiate step by step:
Start from the outside:

$$f'(x) = 2 \cdot \sin(5x^3 + 1) \cdot \frac{d}{dx}[\sin(5x^3 + 1)]$$

Now differentiate the inner sine function:

$$\frac{d}{dx}[\sin(5x^3 + 1)] = \cos(5x^3 + 1) \cdot \frac{d}{dx}[5x^3 + 1] = \cos(5x^3 + 1) \cdot 15x^2$$

Putting it all together:

$$f'(x) = 2 \cdot \sin(5x^3 + 1) \cdot \cos(5x^3 + 1) \cdot 15x^2$$

$$\boxed{f'(x) = 30x^2 \cdot \sin(5x^3 + 1) \cdot \cos(5x^3 + 1)}$$

Alternatively, you may express the result using the identity:

$$2 \sin\theta \cos\theta = \sin(2\theta)$$

$$\Rightarrow f'(x) = 15x^2 \cdot \sin(2(5x^3 + 1)) = \boxed{15x^2 \cdot \sin(10x^3 + 2)}$$

Either boxed expression is valid.

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Question 10

Let $y = \ln\left(\sqrt{1+e^{4x}}\right)$.

(a) Use the chain rule to find $\dfrac{dy}{dx}$.

(b) Clearly indicate the nested functions and show each step of differentiation.

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Question 10 - Solution

We are given:

$$y = \ln\left(\sqrt{1+e^{4x}}\right)$$

First, simplify the expression:

$$y = \ln\left[(1+e^{4x})^{1/2}\right] = \frac{1}{2}\ln(1+e^{4x})$$

Now differentiate:

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{1+e^{4x}}\cdot\frac{d}{dx}(1+e^{4x})$$

Differentiate the inner exponential:

$$\frac{d}{dx}(1+e^{4x}) = 4e^{4x}$$

Putting it all together:

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{4e^{4x}}{1+e^{4x}} = \boxed{\frac{2e^{4x}}{1+e^{4x}}}$$

Nested Function Structure:
$u = e^{4x}$, $v = 1+u$, $w = \ln(\sqrt{v}) = \frac{1}{2}\ln(v)$
Differentiation followed this path:

$$\frac{dw}{dv}\cdot\frac{dv}{du}\cdot\frac{du}{dx}$$

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Question 1

Given the equation:

$$x^2 + xy + y^2 = 7$$

(a) Use implicit differentiation to find $\dfrac{dy}{dx}$.

(b) Find the slope of the tangent line at the point $(1, 2)$.

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Question 1 - Solution

We are given:

$$x^2 + xy + y^2 = 7$$

Differentiate both sides with respect to x, using the product rule on xy:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$

$$2x + \left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 0$$

Group $\frac{dy}{dx}$ terms:

$$x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x - y \Rightarrow (x + 2y)\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx} = \boxed{\frac{-2x - y}{x + 2y}}$$

(b) Evaluate at the point $(1, 2)$:

$$\frac{dy}{dx} = \frac{-2(1) - 2}{1 + 2(2)} = \frac{-4}{5}$$

Answer: Slope of the tangent line at $(1, 2)$ is $\boxed{-\frac{4}{5}}$

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Question 2

Let the equation of a curve be defined implicitly by:

$$x^3 + y^3 = 6xy$$

(a) Use implicit differentiation to find $\dfrac{dy}{dx}$.

(b) Find the slope of the tangent line at the point $(3, 3)$.

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Question 2 - Solution

We are given:

$$x^3 + y^3 = 6xy$$

(a) Differentiate both sides with respect to x:
Left side:

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 3x^2 + 3y^2\frac{dy}{dx}$$

Right side (Product Rule):

$$\frac{d}{dx}(6xy) = 6x\frac{dy}{dx} + 6y$$

Now equate both sides:

$$3x^2 + 3y^2\frac{dy}{dx} = 6x\frac{dy}{dx} + 6y$$

Bring like terms together:

$$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

Factor out $\frac{dy}{dx}$:

$$\left(3y^2 - 6x\right)\frac{dy}{dx} = 6y - 3x^2$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$

Simplify:

$$\boxed{\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}}$$

(b) Evaluate the derivative at $(3, 3)$:
Substitute $x = 3$, $y = 3$:

$$\frac{dy}{dx} = \frac{2(3) - 3^2}{3^2 - 2(3)} = \frac{6 - 9}{9 - 6} = \frac{-3}{3} = \boxed{-1}$$

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Question 3

Consider the equation:

$$x^2 \sin(y) + y^2 \cos(x) = 1$$

(a) Use implicit differentiation to find $\dfrac{dy}{dx}$.

(b) Find the slope of the tangent line to the curve at the point $(0, 1)$.

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Question 3 - Solution

We are given:

$$x^2 \sin(y) + y^2 \cos(x) = 1$$

Differentiate both sides implicitly with respect to x:

$$\frac{d}{dx}\left[x^2 \sin(y)\right] + \frac{d}{dx}\left[y^2 \cos(x)\right] = 0$$

Using the product rule and chain rule:

$$2x \sin(y) + x^2 \cos(y)\frac{dy}{dx} + 2y\frac{dy}{dx}\cos(x) - y^2 \sin(x) = 0$$

Move the non-$\frac{dy}{dx}$ terms to the other side:

$$x^2 \cos(y)\frac{dy}{dx} + 2y\cos(x)\frac{dy}{dx} = y^2 \sin(x) - 2x \sin(y)$$

Factor out $\frac{dy}{dx}$:

$$\left(x^2 \cos(y) + 2y\cos(x)\right)\frac{dy}{dx} = y^2 \sin(x) - 2x \sin(y)$$

Solve for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = \frac{y^2 \sin(x) - 2x \sin(y)}{x^2 \cos(y) + 2y\cos(x)}}$$

At $(0, 1)$:

$$\frac{dy}{dx} = \frac{(1)^2 \sin(0) - 2(0)\sin(1)}{(0)^2 \cos(1) + 2(1)\cos(0)} = \frac{0}{2} = \boxed{0}$$

Therefore, the slope of the tangent line at $(0, 1)$ is:

$$\boxed{0}$$

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Question 4

Consider the equation:

$$\sin(xy) = x + y$$

(a) Use implicit differentiation to find $\dfrac{dy}{dx}$.

(b) Find the slope of the tangent line to the curve at the point $(0, 0)$.

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Question 4 - Solution

(b) Tangent slope

At $(0, 0)$, the original equation reads $\sin 0 = 0 + 0$, so the point belongs to the curve.

(a) Implicit derivative

Implicit differentiation gives

$$\cos(xy)(y + xy') = 1 + y'.$$

Collecting derivative terms,

$$(x\cos(xy) - 1)y' = 1 - y\cos(xy).$$

Thus, wherever the denominator is nonzero,

$$\boxed{y' = \frac{1 - y\cos(xy)}{x\cos(xy) - 1}}.$$

At the specified point the denominator is -1, and the slope is $\boxed{-1}$.

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Question 5

Consider the equation:

$$x^2y + e^{xy} = y^3 + 4x$$

(a) Use implicit differentiation to find $\dfrac{dy}{dx}$.

(b) Find the slope of the tangent line to the curve at the point $(0, 1)$.

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Question 5 - Solution

We are given:

$$x^2y + e^{xy} = y^3 + 4x$$

Differentiate both sides implicitly with respect to x:

$$\frac{d}{dx}[x^2y] + \frac{d}{dx}[e^{xy}] = \frac{d}{dx}[y^3] + \frac{d}{dx}[4x]$$

Using the product rule and chain rule:

$$x^2\frac{dy}{dx} + 2xy + e^{xy}\left(x\frac{dy}{dx} + y\right) = 3y^2\frac{dy}{dx} + 4$$

Expand:

$$x^2\frac{dy}{dx} + 2xy + xe^{xy}\frac{dy}{dx} + ye^{xy} = 3y^2\frac{dy}{dx} + 4$$

Move the $\frac{dy}{dx}$ terms to one side:

$$x^2\frac{dy}{dx} + xe^{xy}\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 4 - 2xy - ye^{xy}$$

Factor:

$$\left(x^2 + xe^{xy} - 3y^2\right)\frac{dy}{dx} = 4 - 2xy - ye^{xy}$$

Therefore,

$$\boxed{\frac{dy}{dx} = \frac{4 - 2xy - ye^{xy}}{x^2 + xe^{xy} - 3y^2}}$$

At $(0, 1)$, we have $xy = 0$, so $e^{xy} = e^0 = 1$. Thus,

$$\frac{dy}{dx} = \frac{4 - 2(0)(1) - (1)(1)}{0^2 + 0(1) - 3(1)^2} = \frac{3}{-3} = \boxed{-1}$$

Therefore, the slope of the tangent line at $(0, 1)$ is:

$$\boxed{-1}$$

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Question 6

Consider the equation defined implicitly:

$$x^2 + y^2 + \ln(y) = 10$$

(a) Use implicit differentiation to find $\dfrac{dy}{dx}$.

(b) Find the slope of the tangent line at the point $(3, 1)$.

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Question 6 - Solution

The logarithm requires $y > 0$. Differentiating gives

$$2x + 2yy' + y'/y = 0, \qquad \boxed{y' = \frac{-2x}{2y + 1/y}}.$$

At $(3, 1)$, $3^2 + 1^2 + \ln 1 = 10$, so the point is on the curve.
The denominator is 3, giving $\boxed{y'(3) = -2}$.

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Question 7

Given the curve defined implicitly by the equation:

$$\sin(xy) + x^2 = y$$

(a) Use implicit differentiation to find $\frac{dy}{dx}$.

(b) Find the value of $\frac{dy}{dx}$ at the point $(0, 0)$.

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Question 7 - Solution

We are given:

$$\sin(xy) + x^2 = y$$

(a) Differentiate both sides implicitly with respect to x:
Use the chain rule on $\sin(xy)$, noting that xy is a product of functions:

$$\frac{d}{dx}[\sin(xy)] + \frac{d}{dx}[x^2] = \frac{d}{dx}[y]$$

Differentiate:

$$\cos(xy) \cdot \left(y + x \cdot \frac{dy}{dx}\right) + 2x = \frac{dy}{dx}$$

Now solve for $\frac{dy}{dx}$. Expand the left-hand side:

$$\cos(xy)y + \cos(xy)x \cdot \frac{dy}{dx} + 2x = \frac{dy}{dx}$$

Bring all $\frac{dy}{dx}$ terms to one side:

$$\cos(xy)x \cdot \frac{dy}{dx} - \frac{dy}{dx} = -\cos(xy)y - 2x$$

Factor $\frac{dy}{dx}$:

$$(\cos(xy)x - 1) \cdot \frac{dy}{dx} = -\cos(xy)y - 2x$$

Solve:

$$\boxed{\frac{dy}{dx} = \frac{-\cos(xy)y - 2x}{\cos(xy)x - 1}}$$

(b) Evaluate at the point $(0, 0)$:
Substitute $x = 0$, $y = 0$ into the derivative:

$$\cos(0) = 1$$

$$\frac{dy}{dx} = \frac{-1(0) - 2(0)}{1(0) - 1} = \frac{0}{-1} = 0$$

$$\boxed{\left.\frac{dy}{dx}\right|_{(0,0)} = 0}$$

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Question 8

Consider the equation:

$$\ln(x+y) + x^2 y = y^2 + 3x + \ln 2 - 3$$

(a) Use implicit differentiation to find $\dfrac{dy}{dx}$.

(b) Find the slope of the tangent line to the curve at the point $(1, 1)$.

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Question 8 - Solution

At $(1,1)$ both sides of the corrected equation equal $\ln 2 + 1$.

(a) Implicit derivative

Differentiating (with $x + y > 0$) gives

$$\frac{1 + y'}{x + y} + 2xy + x^2 y' = 2yy' + 3.$$

Therefore

$$\boxed{y' = \frac{3 - 2xy - 1/(x+y)}{1/(x+y) + x^2 - 2y}}.$$

(b) Tangent slope

At $(1,1)$ the numerator is $1/2$ and the denominator is $-1/2$, hence the slope is $\boxed{-1}$.

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Question 9

Consider the equation:

$$x\sin(y) + y\cos(x) = 1$$

(a) Use implicit differentiation to find $\dfrac{dy}{dx}$.

(b) Find the slope of the tangent line to the curve at the point $(0, 1)$.

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Question 9 - Solution

We are given:

$$x\sin(y) + y\cos(x) = 1$$

Differentiate both sides implicitly with respect to x:

$$\frac{d}{dx}[x\sin(y)] + \frac{d}{dx}[y\cos(x)] = 0$$

Using the product rule:

$$\sin(y) + x\cos(y)\frac{dy}{dx} + \cos(x)\frac{dy}{dx} - y\sin(x) = 0$$

Move the non-$\frac{dy}{dx}$ terms to the other side:

$$x\cos(y)\frac{dy}{dx} + \cos(x)\frac{dy}{dx} = y\sin(x) - \sin(y)$$

Factor out $\frac{dy}{dx}$:

$$(x\cos(y) + \cos(x))\frac{dy}{dx} = y\sin(x) - \sin(y)$$

Solve for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = \frac{y\sin(x) - \sin(y)}{x\cos(y) + \cos(x)}}$$

At $(0, 1)$:

$$\frac{dy}{dx} = \frac{(1)\sin(0) - \sin(1)}{(0)\cos(1) + \cos(0)}$$

$$\frac{dy}{dx} = \frac{0 - \sin(1)}{0 + 1} = \boxed{-\sin(1)}$$

Therefore, the slope of the tangent line at $(0, 1)$ is:

$$\boxed{-\sin(1)}$$

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Question 10

Consider the real relation

$$x^2 + y^2 = \sin(xy).$$

(a) Derive the formal relation obtained by implicit differentiation. State when division would be valid.

(b) Determine the real solution set and decide whether a tangent slope exists at $(0, 0)$.

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Question 10 - Solution

Formal differentiation. If a differentiable branch existed, it would obey

$$2x + 2yy' = \cos(xy)(y + xy'),$$

so $(2y - x\cos(xy))y' = y\cos(xy) - 2x$. Division would require a nonzero coefficient.

Analyze the real point set first. For every real x, y,

$$\sin(xy) \le |xy| \le \frac{x^2 + y^2}{2}.$$

Thus the given equation implies

$$0 \le x^2 + y^2 \le \frac{x^2 + y^2}{2},$$

which forces $x = y = 0$. The real solution set is the singleton $\boxed{\{(0, 0)\}}$.

There is no curve branch through this isolated point and hence no tangent slope dy/dx there. The formal expression $0/0$ alone would not establish this conclusion.

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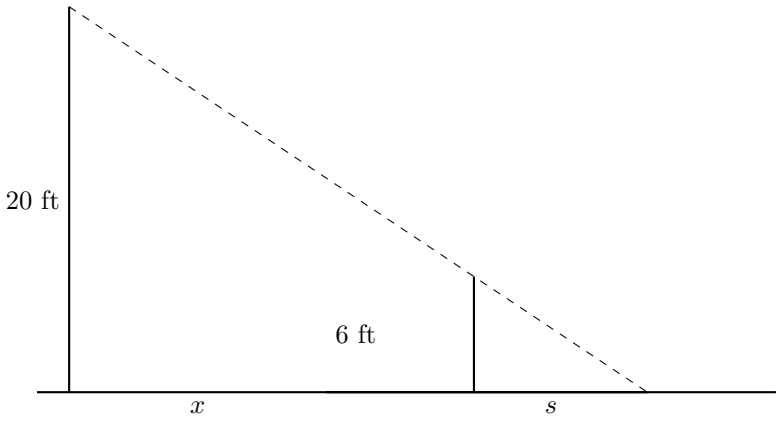
Question 1

A streetlight is mounted at the top of a 20-foot pole. A person 6 feet tall walks directly away from the pole at a constant rate of 4 ft/sec.

(a) How fast is the tip of the person's shadow moving along the ground when the person is 10 feet from the pole?

(b) How fast is the length of the person's shadow changing at that instant?

(c) Explain why the tip of the shadow moves faster than the person.



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Question 1 - Solution

Let:

$$x(t) = \text{distance from the pole to the person}$$

$$s(t) = \text{length of the shadow}$$

The distance from the pole to the tip of the shadow is:

$$x + s$$

Step 1: Set up the relationship

Using similar triangles:

$$\frac{20}{x+s} = \frac{6}{s}$$

Cross-multiply:

$$20s = 6(x+s)$$

Simplify:

$$20s = 6x + 6s \Rightarrow 14s = 6x \Rightarrow s = \frac{3}{7}x$$

Step 2: Differentiate with respect to time

Differentiate both sides:

$$\frac{ds}{dt} = \frac{3}{7}\frac{dx}{dt}$$

Given:

$$\frac{dx}{dt} = 4$$

So:

$$\frac{ds}{dt} = \frac{12}{7}$$

(a) Speed of the tip of the shadow

$$\frac{d}{dt}(x+s) = \frac{dx}{dt} + \frac{ds}{dt} = 4 + \frac{12}{7} = \frac{40}{7}$$

Answer (a):

$$\boxed{\frac{40}{7} \text{ ft/sec}}$$

(b) Rate of change of the shadow length

$$\boxed{\frac{ds}{dt} = \frac{12}{7} \text{ ft/sec}}$$

(c) Explanation

The shadow tip moves faster than the person because its motion includes both the person's movement and the changing length of the shadow.

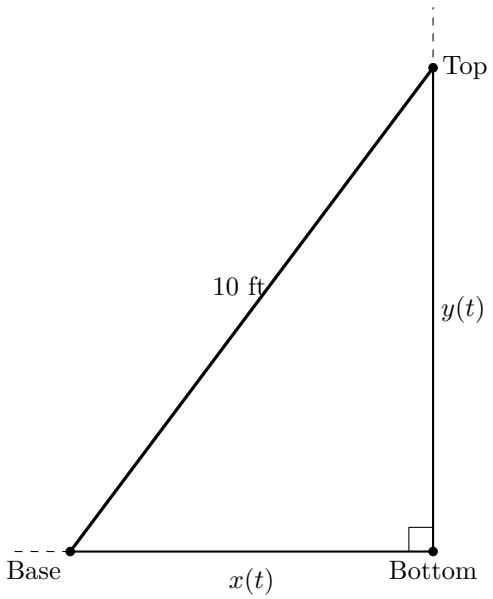
Conclusion: The shadow tip moves faster than the person.

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Question 2

A ladder 10 feet long is leaning against a vertical wall. The bottom of the ladder is sliding away from the wall at a rate of 2 ft/sec.

(a) How fast is the top of the ladder sliding down the wall when the bottom is 6 feet from the wall?



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Question 2 - Solution

Let: $x(t)$: distance from the wall to the bottom of the ladder (horizontal leg), $y(t)$: height of the ladder on the wall (vertical leg), Ladder: 10 ft, so by Pythagoras:

$$x^2 + y^2 = 10^2 = 100$$

Differentiate both sides w.r.t. time t:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

Given: $x = 6$, Ladder is 10 ft, so $y = \sqrt{100 - 36} = 8$, $\frac{dx}{dt} = 2$

$$\frac{dy}{dt} = -\frac{6}{8} \cdot 2 = -\frac{12}{8} = -\frac{3}{2}$$

Answer:

$$\boxed{\frac{dy}{dt} = -\frac{3}{2} \text{ ft/sec}}$$

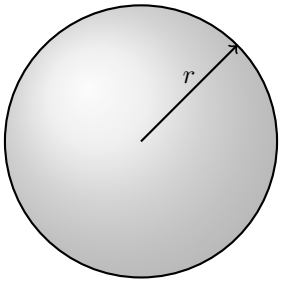
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Question 3

Problem:

A spherical balloon is being inflated such that its radius increases at a constant rate of $\frac{dr}{dt} = 3$ cm/min.

(a) At what rate is the volume of the balloon increasing when the radius is 10 cm?
(b) Include units and interpret the meaning of your result.



Spherical balloon

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Question 3 - Solution

We are given:

$$\frac{dr}{dt} = 3 \text{ cm/min}, \quad r = 10 \text{ cm}$$

The volume of a sphere is given by:

$$V = \frac{4}{3}\pi r^3$$

Differentiate both sides with respect to t:

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2 \frac{dr}{dt}$$

Substitute the known values:

$$\frac{dV}{dt} = 4\pi(10)^2(3) = 4\pi(100)(3) = 1200\pi$$

$$\frac{dV}{dt} \approx 3769.91 \text{ cm}^3/\text{min}$$

Answer: The volume is increasing at a rate of $\boxed{1200\pi \approx 3769.91 \text{ cm}^3/\text{min}}$ when the radius is 10 cm.

Interpretation: When the balloon's radius is 10 cm and growing at 3 cm per minute, the volume is expanding at approximately 3769.91 cubic centimeters per minute. This tells us how rapidly the space inside the balloon is increasing at that instant.

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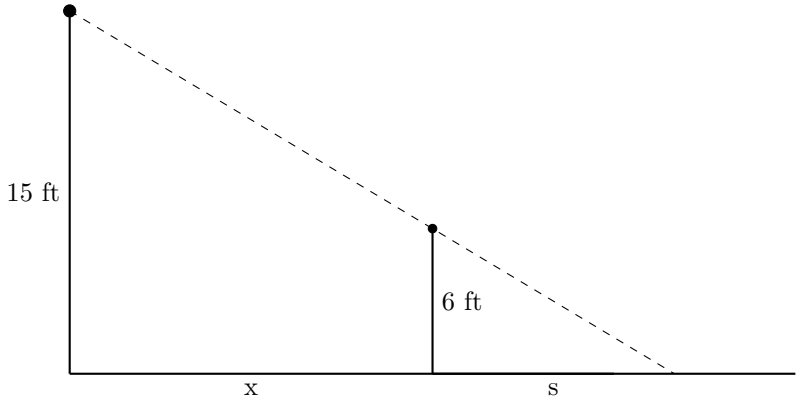
Question 4

Problem:

A 6-ft tall person walks away from a 15-ft tall streetlight at a rate of 3 ft/s.

(a) How fast is the tip of the shadow moving when the person is 30 ft from the base of the light?

(b) At the same moment, how fast is the length of the shadow increasing?



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Question 4 - Solution

Let: - $x(t)$ be the distance from the person to the light pole - $s(t)$ be the length of the shadow - The tip of the shadow is at a distance $x + s$ from the pole

Using similar triangles:

$$\frac{6}{s} = \frac{15}{x+s} \Rightarrow 6(x+s) = 15s \Rightarrow 6x + 6s = 15s \Rightarrow 6x = 9s \Rightarrow s = \frac{2}{3}x$$

Differentiate with respect to t:

$$\frac{ds}{dt} = \frac{2}{3}\frac{dx}{dt}$$

Given:

$$\frac{dx}{dt} = 3 \text{ ft/s} \Rightarrow \frac{ds}{dt} = \frac{2}{3}(3) = 2 \text{ ft/s}$$

Now, the tip of the shadow moves at:

$$\frac{d}{dt}(x+s) = \frac{dx}{dt} + \frac{ds}{dt} = 3 + 2 = \boxed{5 \text{ ft/s}}$$

Answers:

(a) The tip of the shadow is moving at $\boxed{5 \text{ ft/s}}$

(b) The shadow is lengthening at $\boxed{2 \text{ ft/s}}$

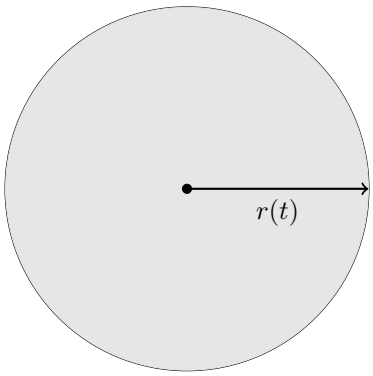
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Question 5

A circular ripple is formed on the surface of a still pond when a pebble is dropped into it. The radius of the ripple increases at a constant rate of $0.05\,\text{m/s}$.

(a) At what rate is the area enclosed by the ripple increasing when the radius is $0.3\,\text{m}$?

(b) At what rate is the circumference of the ripple increasing at that moment?



Expanding ripple with changing radius

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Question 5 - Solution

Let $r(t)$ be the radius of the ripple at time t, in meters.

(a) Rate of Change of Area

The area of a circle is:

$$A = \pi r^2$$

Differentiate both sides with respect to time t:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

We're given:

$$\frac{dr}{dt} = 0.05\,\text{m/s}, \quad r = 0.3\,\text{m}$$

Substitute:

$$\frac{dA}{dt} = 2\pi(0.3)(0.05) = 0.03\pi \approx \boxed{0.0942\,\text{m}^2/\text{s}}$$

(b) Rate of Change of Circumference

Circumference is given by:

$$C = 2\pi r$$

Differentiate:

$$\frac{dC}{dt} = 2\pi \frac{dr}{dt}$$

Substitute:

$$\frac{dC}{dt} = 2\pi(0.05) = \boxed{0.1\pi \approx 0.3142\,\text{m/s}}$$

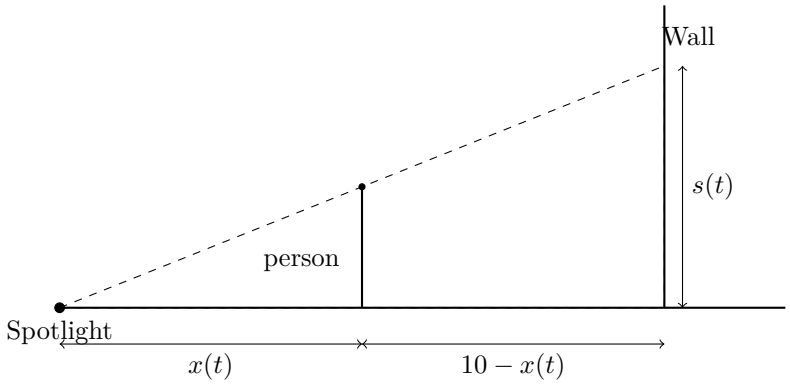
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Question 6

A spotlight is located on the ground 10 m away from a vertical wall. A person 2 m tall walks directly away from the wall at a speed of 1.2 m/s.

(a) How fast is the length of the shadow on the wall changing when the person is 5 m from the wall?

(b) At that moment, how fast is the tip of the shadow moving up the wall?



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Question 6 - Solution

Let:

$$x(t) = \text{horizontal distance from the spotlight to the person (m)}$$

$$s(t) = \text{height of the shadow on the wall (m)}$$

From similar triangles:

$$\frac{s(t)}{10} = \frac{2}{x(t)} \Rightarrow s(t) = \frac{20}{x(t)}$$

Differentiate both sides:

$$\frac{ds}{dt} = -\frac{20}{x^2} \cdot \frac{dx}{dt}$$

Given:

$$x = 5 \text{ m}, \quad \frac{dx}{dt} = -1.2 \text{ m/s}$$

$$\frac{ds}{dt} = -\frac{20}{25}(-1.2) = 0.96 \text{ m/s}$$

(a) Rate of change of shadow height:

$$\boxed{\frac{ds}{dt} = 0.96\,\text{m/s}}$$

(b) Speed of the tip of the shadow:
The tip moves upward at the same rate:

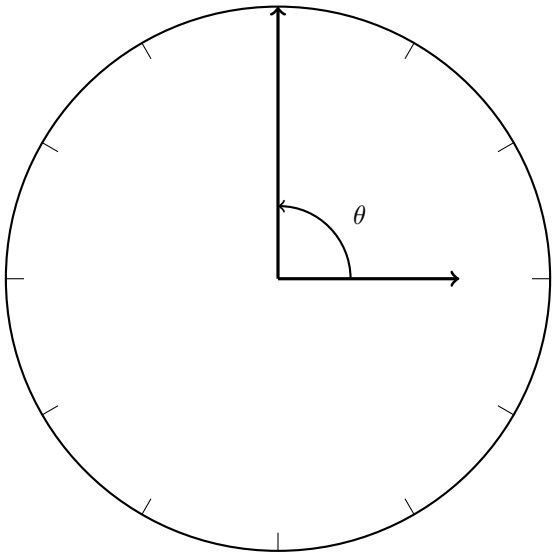
$$\boxed{0.96\,\text{m/s upward}}$$

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Question 7

The minute hand of a clock completes one full revolution every 60 minutes, and the hour hand completes one full revolution every 12 hours.

Question: At exactly 3:00, how fast is the *smaller angle* between the hour and minute hands changing?



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Solution — Method 1 (Formal Related Rates)

Let $\theta(t)$ be the signed angle (in degrees) from the hour hand to the minute hand.

$$\omega_m = 360/60 = 6^\circ/\text{min}, \quad \omega_h = 360/(12 * 60) = 0.5^\circ/\text{min}$$

At 3:00, the hour hand is at 90° and the minute hand is at 0°.

$$\theta(t) = (6t) - (90 + 0.5t)$$

Differentiate:

$$\frac{d\theta}{dt} = 6 - 0.5 = 5.5$$

Since the smaller angle is initially decreasing,

$$\boxed{\frac{d\theta}{dt} = -5.5^\circ/\text{min}}$$

Question 7 - Solution)

The angle between the hands changes at the difference of their angular speeds.

$$\text{Minute hand speed} = 6^\circ/\text{min}$$

$$\text{Hour hand speed} = 0.5^\circ/\text{min}$$

So the angle closes at:

$$6 - 0.5 = 5.5^\circ/\text{min}$$

Since the hands are moving toward each other at 3:00, the angle is decreasing.

$$\boxed{5.5^\circ/\text{min (decreasing)}}$$

Final Answer

$$\boxed{\frac{d\theta}{dt} = -5.5^\circ/\text{min}}$$

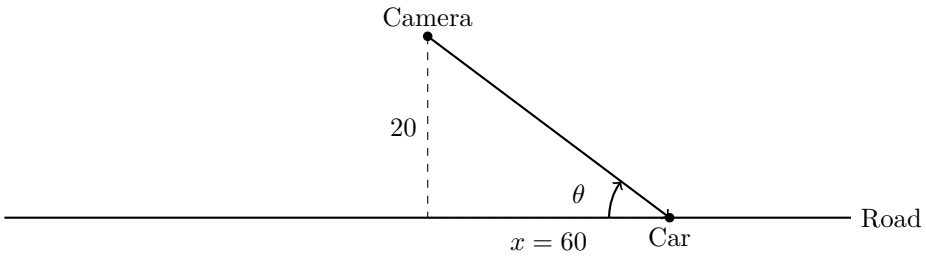
The negative sign indicates the angle between the hands is decreasing.

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Question 8

A camera is placed 20 meters from a straight road. A car is driving along the road away from the point closest to the camera at a speed of 30 m/s.

How fast is the angle between the camera's line of sight and the road changing when the car is 60 meters from the point on the road closest to the camera?



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Question 8 - Solution

Let:

$x(t)$: horizontal distance from car to closest point on road to camera , $\theta(t)$: angle between the camera's line of sight and the road , The camera is 20 m from the road (fixed)

We have:

$$\tan(\theta) = \frac{20}{x}$$

Differentiate both sides with respect to time:

$$\sec^2(\theta) \cdot \frac{d\theta}{dt} = \frac{-20}{x^2} \cdot \frac{dx}{dt}$$

At the instant:
$x = 60$, $\frac{dx}{dt} = 30$ m/s (moving away) , $\tan(\theta) = \frac{20}{60} = \frac{1}{3} \Rightarrow \theta \approx 18.43°$
Use identity:

$$\sec^2(\theta) = 1 + \tan^2(\theta) = 1 + \left(\frac{1}{3}\right)^2 = \frac{10}{9}$$

Now plug in:

$$\frac{10}{9} \cdot \frac{d\theta}{dt} = \frac{-20}{60^2} \cdot 30 = \frac{-20 \cdot 30}{3600} = \frac{-600}{3600} = -\frac{1}{6}$$

So:

$$\frac{d\theta}{dt} = \boxed{-\frac{9}{60} = -0.15 \text{ radians per second}}$$

Conclusion: The angle between the camera's line of sight and the road is decreasing at a rate of:

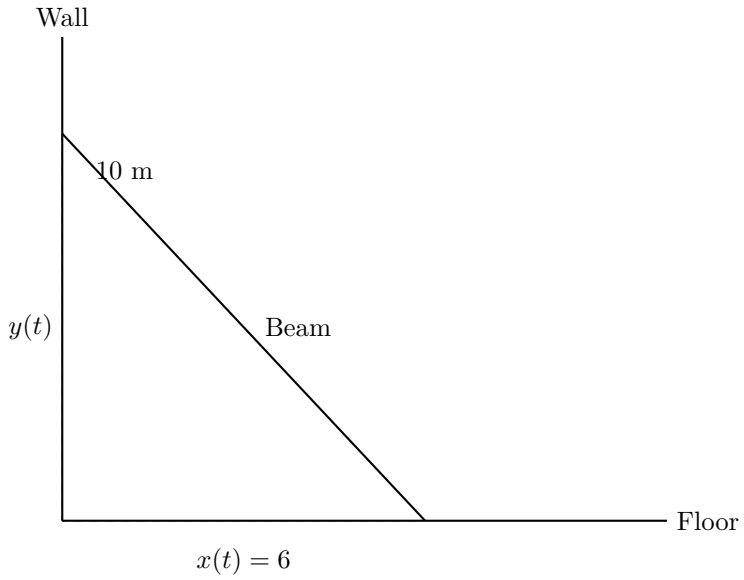
$$\boxed{0.15 \text{ radians/second}}$$

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Question 9

A 10-meter-long beam is leaning against a vertical wall and a horizontal floor, forming a right triangle. The bottom of the beam is sliding away from the wall along the floor at a rate of 1.5 m/s.

How fast is the top of the beam sliding down the wall when the bottom is 6 meters away from the wall?



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Question 9 - Solution

Let:

$x(t)$: distance from the base of the wall to the bottom of the beam
$y(t)$: height of the beam on the wall
The beam is 10 m long $\Rightarrow x^2 + y^2 = 100$
Differentiate both sides with respect to time:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Given: $x = 6$, $\frac{dx}{dt} = 1.5$
Find y from Pythagoras:

$$y = \sqrt{100 - 6^2} = \sqrt{64} = 8$$

Now solve for $\frac{dy}{dt}$:

$$6(1.5) + 8\frac{dy}{dt} = 0 \Rightarrow 9 + 8\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{9}{8}$$

Conclusion: The top of the beam is sliding down at a rate of:

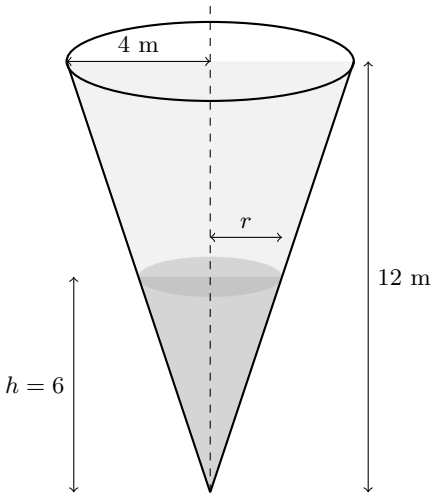
$$\boxed{\frac{dy}{dt} = -\frac{9}{8} = -1.125 \text{ m/s}}$$

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Question 10

Water is poured into an inverted conical tank at a rate of $3\,\text{m}^3/\text{min}$. The tank is 12 meters deep and has a top radius of 4 meters.

How fast is the water level rising when the water is 6 meters deep?



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Question 10 - Solution

Let:

V: volume of water
r: radius of water surface
h: depth of water

Given:

$$\frac{dV}{dt} = 3\,\text{m}^3/\text{min}$$

Tank dimensions: height = 12 m, top radius = 4 m
Volume of cone: $V = \frac{1}{3}\pi r^2 h$

Use similar triangles:

$$\frac{r}{h} = \frac{4}{12} = \frac{1}{3} \Rightarrow r = \frac{h}{3}$$

Substitute into volume formula:

$$V = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h = \frac{\pi}{27}h^3$$

Differentiate:

$$\frac{dV}{dt} = \frac{\pi}{9}h^2 \cdot \frac{dh}{dt}$$

Substitute known values:

$$3 = \frac{\pi}{9}(6)^2 \cdot \frac{dh}{dt} = \frac{\pi}{9} \cdot 36 \cdot \frac{dh}{dt} \Rightarrow 3 = 4\pi \cdot \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{3}{4\pi}$$

Final Answer:

$$\boxed{\frac{dh}{dt} = \frac{3}{4\pi}\,\text{m/min} \approx 0.239\,\text{m/min}}$$

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Question 1

Let:

$$f(x) = x^4 - 4x^3 + 6x^2$$

(a) Find the first and second derivatives of $f(x)$.
(b) Determine the intervals on which $f(x)$ is concave up and concave down.
(c) Find all points of inflection.

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Question 1 - Solution

We are given:

$$f(x) = x^4 - 4x^3 + 6x^2$$

(a) First derivative:

$$f'(x) = 4x^3 - 12x^2 + 12x$$

Second derivative:

$$f''(x) = 12x^2 - 24x + 12$$

Factor:

$$f''(x) = 12(x^2 - 2x + 1) = 12(x - 1)^2$$

Answer:

$$f'(x) = 4x^3 - 12x^2 + 12x, \quad f''(x) = 12(x - 1)^2$$

(b) Analyze sign of $f''(x)$:
Since $f''(x) = 12(x - 1)^2 \geq 0$ for all x, the second derivative is never negative.
$f''(x) > 0$ for $x \neq 1 \rightarrow$ concave up
$f''(1) = 0 \rightarrow$ possible inflection point
But since the sign does not change around $x = 1$, it is not an inflection point.
Answer: Concave up: $(-\infty, \infty)$ Concave down: None

(c) No change in concavity at $x = 1$, so:
Answer: $\boxed{\text{No points of inflection}}$

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Question 2

Let $f(x) = x^2 e^{-x}$.

(a) Compute the first four derivatives of $f(x)$.

(b) Identify a pattern in the derivatives.

(c) Use the pattern to write a general formula for $f^{(n)}(x)$ for $n \geq 0$.

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Question 2 - Solution

We are given $f(x) = x^2 e^{-x}$. Let's compute derivatives step-by-step.

(a) First, compute the derivatives:
First derivative:

$$f'(x) = \frac{d}{dx}(x^2 e^{-x}) = 2xe^{-x} + x^2(-e^{-x}) = (2x - x^2)e^{-x}$$

Second derivative:

$$f''(x) = \frac{d}{dx}[(2x - x^2)e^{-x}] = [2 - 2x]e^{-x} + (2x - x^2)(-e^{-x}) = [2 - 2x - 2x + x^2]e^{-x} = (x^2 - 4x + 2)e^{-x}$$

Third derivative:

$$f^{(3)}(x) = \frac{d}{dx}[(x^2 - 4x + 2)e^{-x}] = (2x - 4)e^{-x} + (x^2 - 4x + 2)(-e^{-x}) = (-x^2 + 6x - 6)e^{-x}$$

Fourth derivative:

$$f^{(4)}(x) = \frac{d}{dx}[(-x^2 + 6x - 6)e^{-x}] = (-2x + 6)e^{-x} + (-x^2 + 6x - 6)(-e^{-x}) = (x^2 - 8x + 12)e^{-x}$$

(b) Pattern:
Each derivative is a polynomial times e^{-x}. The polynomials alternate in sign and follow a recursive-like behavior:

$$\begin{aligned} f(x) &= x^2 e^{-x} \\ f'(x) &= (2x - x^2)e^{-x} \\ f''(x) &= (x^2 - 4x + 2)e^{-x} \\ f^{(3)}(x) &= (-x^2 + 6x - 6)e^{-x} \\ f^{(4)}(x) &= (x^2 - 8x + 12)e^{-x} \end{aligned}$$

(c) General formula (structure):
Let:

$$f^{(n)}(x) = P_n(x)e^{-x}$$

Where $P_n(x)$ is a polynomial of degree 2, alternating in sign:

$$P_n(x) = (-1)^n(x^2 - 2nx + n(n-1))$$

Final boxed result:

$$\boxed{f^{(n)}(x) = (-1)^n(x^2 - 2nx + n(n-1))e^{-x}}$$

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Question 3

Let $f(x) = \ln(x^2 + 1)$.

(a) Compute the first three derivatives of $f(x)$.

(b) Determine the value of $f^{(3)}(0)$.

(c) Based on the derivatives at $x = 0$, write the third-degree Taylor polynomial centered at $x = 0$.

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Question 3 - Solution

We are given:

$$f(x) = \ln(x^2 + 1)$$

(a) First three derivatives:
First derivative:

$$f'(x) = \frac{d}{dx} \ln(x^2 + 1) = \frac{2x}{x^2 + 1}$$

Second derivative:

$$f''(x) = \frac{d}{dx} \left(\frac{2x}{x^2 + 1} \right) = \frac{2(x^2 + 1) - 2x(2x)}{(x^2 + 1)^2} = \frac{2(x^2 + 1 - 2x^2)}{(x^2 + 1)^2} = \frac{2(1 - x^2)}{(x^2 + 1)^2}$$

Third derivative:
We apply the quotient rule to:

$$f''(x) = \frac{2(1 - x^2)}{(x^2 + 1)^2}$$

Let's compute:

$$f^{(3)}(x) = \frac{d}{dx} \left(\frac{2(1 - x^2)}{(x^2 + 1)^2} \right)$$

Numerator derivative:

$$\frac{d}{dx}[2(1 - x^2)] = -4x$$

Denominator derivative:

$$\frac{d}{dx}[(x^2 + 1)^2] = 2(x^2 + 1)(2x) = 4x(x^2 + 1)$$

Now apply quotient rule:

$$f^{(3)}(x) = \frac{(-4x)(x^2 + 1)^2 - 2(1 - x^2)(4x)(x^2 + 1)}{(x^2 + 1)^4}$$

Factor out $4x(x^2 + 1)$:

$$f^{(3)}(x) = \frac{-4x(x^2 + 1) \left[(x^2 + 1) + 2(1 - x^2) \right]}{(x^2 + 1)^4}$$

Simplify:

$$(x^2 + 1) + 2(1 - x^2) = x^2 + 1 + 2 - 2x^2 = 3 - x^2$$

So:

$$f^{(3)}(x) = \frac{-4x(x^2 + 1)(3 - x^2)}{(x^2 + 1)^4} = \frac{-4x(3 - x^2)}{(x^2 + 1)^3}$$

(b) Evaluate at $x = 0$:

$$f(0) = \ln(1) = 0, \quad f'(0) = 0, \quad f''(0) = 2, \quad f^{(3)}(0) = \frac{-4(0)(3 - 0)}{1^3} = 0$$

(c) Taylor polynomial of degree 3 at $x = 0$:

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 = 0 + 0 + \frac{2}{2}x^2 + 0 = x^2$$

$$\boxed{T_3(x) = x^2}$$

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Question 4

Let $f(x) = e^x \cos(x)$.

(a) Compute the first three derivatives: $f'(x)$, $f''(x)$, and $f^{(3)}(x)$.

(b) Find the value of $f''(0)$.

(c) Determine the third-degree Taylor polynomial of $f(x)$ centered at $x = 0$.

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Question 4 - Solution

We are given:

$$f(x) = e^x \cos(x)$$

(a) First three derivatives:
Use product rule: $f(x) = u(x)v(x)$, where $u = e^x$, $v = \cos(x)$
First derivative:

$$f'(x) = u'v + uv' = e^x \cos(x) - e^x \sin(x) = e^x(\cos(x) - \sin(x))$$

Second derivative:

$$f''(x) = \frac{d}{dx}[e^x(\cos(x) - \sin(x))] = e^x(\cos(x) - \sin(x)) + e^x(-\sin(x) - \cos(x))$$

$$= e^x[(\cos(x) - \sin(x)) + (-\sin(x) - \cos(x))] = e^x(-2\sin(x))$$

Third derivative:

$$f^{(3)}(x) = \frac{d}{dx}[e^x(-2\sin(x))] = e^x(-2\sin(x)) + e^x(-2\cos(x)) = -2e^x(\sin(x) + \cos(x))$$

(b) Evaluate $f''(0)$:

$$f''(0) = e^0(-2\sin(0)) = 1 \cdot 0 = \boxed{0}$$

(c) Taylor polynomial of degree 3 at $x = 0$:

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3$$

Compute values:

$$f(0) = e^0 \cos(0) = 1, \quad f'(0) = e^0(\cos(0) - \sin(0)) = 1, \quad f''(0) = 0$$

$$f^{(3)}(0) = -2e^0(\sin(0) + \cos(0)) = -2(0 + 1) = -2$$

So:

$$T_3(x) = 1 + x + 0 + \frac{-2}{6}x^3 = 1 + x - \frac{1}{3}x^3$$

$$\boxed{T_3(x) = 1 + x - \frac{1}{3}x^3}$$

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Question 5

Let $f(x) = \sin(x^2)$.

(a) Find $f'(x)$, $f''(x)$, and $f^{(3)}(x)$.

(b) Compute $f''(0)$.

(c) Use your results to write the second-degree Taylor polynomial of $f(x)$ centered at $x = 0$.

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Question 5 - Solution

We are given:

$$f(x) = \sin(x^2)$$

(a) First three derivatives:
First derivative:

$$f'(x) = \frac{d}{dx}\sin(x^2) = \cos(x^2) \cdot 2x = 2x\cos(x^2)$$

Second derivative: Use product rule:

$$f''(x) = \frac{d}{dx}\left(2x\cos(x^2)\right) = 2\cos(x^2) + 2x \cdot (-\sin(x^2)) \cdot 2x = 2\cos(x^2) - 4x^2\sin(x^2)$$

Third derivative: Differentiate $f''(x)$:

$$f^{(3)}(x) = \frac{d}{dx}\left(2\cos(x^2) - 4x^2\sin(x^2)\right)$$

First term:

$$\frac{d}{dx}[2\cos(x^2)] = -2\sin(x^2) \cdot 2x = -4x\sin(x^2)$$

Second term (product rule):

$$\frac{d}{dx}[4x^2\sin(x^2)] = 8x\sin(x^2) + 4x^2 \cdot \cos(x^2) \cdot 2x = 8x\sin(x^2) + 8x^3\cos(x^2)$$

So:

$$f^{(3)}(x) = -4x\sin(x^2) - \left(8x\sin(x^2) + 8x^3\cos(x^2)\right) = -12x\sin(x^2) - 8x^3\cos(x^2)$$

(b) Compute $f''(0)$:

$$f''(x) = 2\cos(x^2) - 4x^2\sin(x^2) \Rightarrow f''(0) = 2\cos(0) - 0 = 2$$

(c) Taylor Polynomial $T_2(x)$:
We use:

$$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

$$f(0) = \sin(0) = 0, \quad f'(0) = 2(0)\cos(0) = 0, \quad f''(0) = 2$$

So:

$$T_2(x) = 0 + 0 + \frac{2}{2}x^2 = x^2$$

$$\boxed{T_2(x) = x^2}$$

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Question 6

Let $f(x) = \frac{1}{1-x}$.

(a) Find the first four derivatives $f'(x), f''(x), f^{(3)}(x), f^{(4)}(x)$.

(b) Find a general formula for $f^{(n)}(x)$.

(c) Evaluate $f^{(n)}(0)$ and write the Taylor series for $f(x)$ centered at $x = 0$.

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Question 6 - Solution

We are given:

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

(a) Compute derivatives:
Using the chain rule,

$$\frac{d}{dx}(1-x)^{-1} = -1(1-x)^{-2} \cdot (-1) = (1-x)^{-2}$$

$$f'(x) = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{d}{dx}(1-x)^{-2} = -2(1-x)^{-3} \cdot (-1) = \frac{2}{(1-x)^3}$$

$$f^{(3)}(x) = \frac{d}{dx}\left(\frac{2}{(1-x)^3}\right) = 2 \cdot 3(1-x)^{-4} = \frac{6}{(1-x)^4}$$

$$f^{(4)}(x) = \frac{d}{dx}\left(\frac{6}{(1-x)^4}\right) = 6 \cdot 4(1-x)^{-5} = \frac{24}{(1-x)^5}$$

(b) General formula:
Each derivative introduces a factorial factor and increases the power of the denominator by 1:

$$\boxed{f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}}$$

(c) Evaluate at $x = 0$:

$$f^{(n)}(0) = \frac{n!}{1^{n+1}} = n!$$

Thus the Taylor series centered at $x = 0$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} x^n$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1}$$

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Question 7

Let $f(x) = x^2 e^x$.

(a) Find the first three derivatives of $f(x)$.

(b) Find a general expression for $f^{(n)}(x)$.

(c) Evaluate $f^{(3)}(0)$.

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Question 7 - Solution

We are given:

$$f(x) = x^2 e^x$$

(a) Compute Derivatives:
We use the product rule: If $u = x^2$, $v = e^x$, then $f = uv$

$$f'(x) = (x^2)'e^x + x^2(e^x) = 2xe^x + x^2e^x = e^x(2x + x^2)$$

$$f''(x) = \frac{d}{dx}[e^x(2x + x^2)] = e^x(2 + 2x + 2x + x^2) = e^x(x^2 + 4x + 2)$$

$$f^{(3)}(x) = \frac{d}{dx}[e^x(x^2 + 4x + 2)] = e^x(x^2 + 4x + 2 + 2x + 4 + 0) = e^x(x^2 + 6x + 6)$$

(b) General Formula:
We notice a pattern in each derivative:

$$f^{(n)}(x) = e^x \cdot P_n(x)$$

where $P_n(x)$ is a degree-2 polynomial (since the original function had x^2).
This suggests:

$$\boxed{f^{(n)}(x) = e^x \cdot (x^2 + 2nx + n(n-1))}$$

(This can be proven by induction.)

(c) Evaluate $f^{(3)}(0)$:
From (a):

$$f^{(3)}(0) = e^0(0^2 + 6 \cdot 0 + 6) = 1 \cdot 6 = \boxed{6}$$

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Question 8

Let $f(x) = \frac{1}{x}$.

(a) Find the first four derivatives of $f(x)$.

(b) Based on the pattern, write a general formula for the n-th derivative $f^{(n)}(x)$.

(c) Evaluate $f^{(6)}(x)$.

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Question 8 - Solution

We are given:

$$f(x) = \frac{1}{x} = x^{-1}$$

(a) Compute derivatives:

$$f'(x) = \frac{d}{dx}(x^{-1}) = -x^{-2}$$
$$f''(x) = \frac{d}{dx}(-x^{-2}) = 2x^{-3}$$
$$f^{(3)}(x) = \frac{d}{dx}(2x^{-3}) = -6x^{-4}$$
$$f^{(4)}(x) = \frac{d}{dx}(-6x^{-4}) = 24x^{-5}$$

(b) General Pattern:
Observe the signs and coefficients:

$$f^{(n)}(x) = (-1)^n \cdot n! \cdot x^{-(n+1)}$$

Answer: $\boxed{f^{(n)}(x) = (-1)^n \cdot n! \cdot x^{-(n+1)}}$

(c) Evaluate $f^{(6)}(x)$:
Use the formula:

$$f^{(6)}(x) = (-1)^6 \cdot 6! \cdot x^{-7} = 720x^{-7}$$

Answer: $\boxed{f^{(6)}(x) = \frac{720}{x^7}}$

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Question 9

Let $f(x) = \cos(2x)$.

(a) Find the first four derivatives of $f(x)$.

(b) Find a general formula for the n-th derivative $f^{(n)}(x)$.

(c) Evaluate $f^{(15)}(x)$.

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Question 9 - Solution

Successive differentiation yields

$$f' = -2\sin 2x, \quad f'' = -4\cos 2x, \quad f^{(3)} = 8\sin 2x, \quad f^{(4)} = 16\cos 2x.$$

Each derivative multiplies by 2 and advances the phase by $\pi/2$; induction therefore gives

$$\boxed{f^{(n)}(x) = 2^n \cos(2x + n\pi/2)}.$$

Since $15\pi/2$ is congruent to $3\pi/2$ modulo 2π,

$$\boxed{f^{(15)}(x) = 32768\cos(2x + 3\pi/2) = 32768\sin 2x}.$$

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Question 10

Let

$$f(x) = \ln(1 + x^2).$$

(a) Show that all odd-order derivatives of $f(x)$ evaluated at $x = 0$ are zero.

(b) Find a closed-form expression for the $2n$-th derivative of $f(x)$ evaluated at $x = 0$, where $n \in \mathbb{N}$.

(c) Using your result from part (b), write the Maclaurin series for $f(x)$ in summation notation.

(d) Determine the radius of convergence of the Maclaurin series and justify your answer.

(e) Use the Maclaurin series to approximate $\ln(1.1)$ to within an error of 10^{-4}, and explain how you control the error.

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Question 10 - Solution

We are given

$$f(x) = \ln(1 + x^2).$$

(a) Odd-order derivatives at $x = 0$

Observe that $f(x)$ is an **even function**, since

$$f(-x) = \ln(1 + (-x)^2) = \ln(1 + x^2) = f(x).$$

The derivative of an even function is odd, and the derivative of an odd function is even. Therefore:

- $f'(x)$ is odd,
- $f''(x)$ is even,
- $f^{(3)}(x)$ is odd,
- and so on.

All odd functions evaluate to zero at $x = 0$. Hence,

$$\boxed{f^{(2k+1)}(0) = 0 \quad \text{for all } k \geq 0.}$$

(b) Closed form for $f^{(2n)}(0)$

We begin with the known Taylor expansion:

$$\ln(1 + u) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{u^n}{n}, \quad |u| < 1.$$

Substitute $u = x^2$:

$$\ln(1 + x^2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}.$$

Comparing this with the Maclaurin series definition:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

we see that only even powers appear. Matching coefficients of x^{2n}, we obtain:

$$\frac{f^{(2n)}(0)}{(2n)!} = (-1)^{n+1} \frac{1}{n}.$$

Solving for $f^{(2n)}(0)$:

$$\boxed{f^{(2n)}(0) = (-1)^{n+1} \frac{(2n)!}{n}.}$$

(c) Maclaurin series

Using the result above, the Maclaurin series for $f(x)$ is:

$$\boxed{\ln(1 + x^2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}, \quad |x| < 1.}$$

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(d) Radius of convergence

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n}$$

is a power series in x^2.

Using the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{x^{2(n+1)}/(n+1)}{x^{2n}/n}\right| = |x|^2.$$

The series converges when:

$$|x|^2 < 1 \quad \Rightarrow \quad |x| < 1.$$

Thus, the radius of convergence is:

$$\boxed{R = 1.}$$

(e) Approximating $\ln(1.1)$

We write:

$$\ln(1.1) = \ln(1+0.1) = \ln(1+x^2) \quad \text{with } x^2 = 0.1.$$

Using the alternating series:

$$\ln(1.1) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.1)^n}{n}.$$

Since this is an alternating series with decreasing terms, the error after N terms is bounded by the magnitude of the next term:

$$|R_N| \le \frac{(0.1)^{N+1}}{N+1}.$$

We test values:

$$\frac{(0.1)^3}{3} \approx 0.00033 > 10^{-4}, \quad \frac{(0.1)^4}{4} = 0.000025 < 10^{-4}.$$

Thus, using the first four terms guarantees the desired accuracy.

$$\boxed{\ln(1.1) \approx 0.1 - \frac{0.1^2}{2} + \frac{0.1^3}{3} - \frac{0.1^4}{4}}$$

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Question 1

Use logarithmic differentiation to find the derivative of:

$$f(x) = (x^2 + 1)^x$$

Simplify your answer as much as possible.

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Question 1 - Solution

We are given:

$$f(x) = (x^2 + 1)^x$$

Take the natural log of both sides:

$$\ln f(x) = \ln\left((x^2 + 1)^x\right) = x \ln(x^2 + 1)$$

Differentiate both sides implicitly:

$$\frac{1}{f(x)} \cdot f'(x) = \ln(x^2 + 1) + x \cdot \frac{1}{x^2 + 1} \cdot 2x = \ln(x^2 + 1) + \frac{2x^2}{x^2 + 1}$$

Multiply both sides by $f(x) = (x^2 + 1)^x$:

$$f'(x) = (x^2 + 1)^x \left[\ln(x^2 + 1) + \frac{2x^2}{x^2 + 1}\right]$$

Final Answer:

$$f'(x) = \boxed{(x^2 + 1)^x \left[\ln(x^2 + 1) + \frac{2x^2}{x^2 + 1}\right]}$$

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Question 2

Let the function be:

$$f(x) = \frac{(x^2+1)^5 \cdot \sqrt{x+3}}{(x-1)^4}$$

(a) Use logarithmic differentiation to find $f'(x)$.

(b) Simplify your final expression as much as possible.

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Question 2 - Solution

The real domain is $x \geq -3,\ x \neq 1$. Where the function is nonzero, use absolute values in logarithmic differentiation:

$$\ln|f(x)| = 5\ln(x^2+1) + \tfrac{1}{2}\ln(x+3) - 4\ln|x-1|$$

Differentiating gives

$$\boxed{f'(x) = f(x)\left(\frac{10x}{x^2+1} + \frac{1}{2(x+3)} - \frac{4}{x-1}\right).}$$

This formula holds for $x > -3,\ x \neq 1$. At $x = -3$ there is no finite derivative because of the square-root factor.

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Question 3

Let the function be:

$$f(x) = \frac{(3x+1)^4 \cdot \ln(x)}{(x^2-9)^{3/2}}$$

(a) Use logarithmic differentiation to find $f'(x)$.

(b) Clearly indicate how you apply the logarithmic properties before differentiating.

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Question 3 - Solution

We are given:

$$f(x) = \frac{(3x+1)^4 \cdot \ln(x)}{(x^2-9)^{3/2}}$$

Step 1: Take the natural log of both sides:

$$\ln f(x) = \ln\left(\frac{(3x+1)^4 \cdot \ln(x)}{(x^2-9)^{3/2}}\right)$$

Apply logarithmic identities:

$$\ln f(x) = 4\ln(3x+1) + \ln(\ln x) - \frac{3}{2}\ln(x^2-9)$$

Step 2: Differentiate both sides implicitly:

$$\frac{f'(x)}{f(x)} = \frac{4 \cdot 3}{3x+1} + \frac{1}{\ln x} \cdot \frac{1}{x} - \frac{3}{2} \cdot \frac{2x}{x^2-9}$$

Simplify:

$$\frac{f'(x)}{f(x)} = \frac{12}{3x+1} + \frac{1}{x\ln x} - \frac{3x}{x^2-9}$$

Step 3: Multiply both sides by $f(x)$ to isolate $f'(x)$:

$$f'(x) = f(x)\left(\frac{12}{3x+1} + \frac{1}{x\ln x} - \frac{3x}{x^2-9}\right)$$

Substitute back the original $f(x)$:

$$f'(x) = \frac{(3x+1)^4 \cdot \ln(x)}{(x^2-9)^{3/2}}\left(\frac{12}{3x+1} + \frac{1}{x\ln x} - \frac{3x}{x^2-9}\right)$$

$$\boxed{f'(x) = \frac{(3x+1)^4 \cdot \ln(x)}{(x^2-9)^{3/2}}\left(\frac{12}{3x+1} + \frac{1}{x\ln x} - \frac{3x}{x^2-9}\right)}$$

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Question 4

For $x > 1$ (so the base is positive), let the function be:

$$f(x) = \left(\frac{x^2 \cdot \sqrt{x+1}}{(x^2+4)^3 \cdot \ln x} \right)^{\sin x}$$

(a) Use logarithmic differentiation to find $f'(x)$.

(b) Simplify your answer as much as possible.

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Question 4 - Solution

For $x > 1$, define the positive base

$$B(x) = \frac{x^2\sqrt{x+1}}{(x^2+4)^3 \ln x}, \qquad f(x) = B(x)^{\sin x}.$$

Set

$$A(x) = \ln B(x) = 2\ln x + \tfrac{1}{2}\ln(x+1) - 3\ln(x^2+4) - \ln(\ln x).$$

Then $\ln f(x) = \sin x\, A(x)$ and

$$A'(x) = \frac{2}{x} + \frac{1}{2(x+1)} - \frac{6x}{x^2+4} - \frac{1}{x\ln x}.$$

Logarithmic differentiation yields

$$\boxed{f'(x) = B(x)^{\sin x}\big(\cos x\, A(x) + \sin x\, A'(x)\big), \quad x > 1.}$$

Here B, A, A' are the explicit expressions above.

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Question 5

Let the function be:

$$f(x) = \frac{(x^2+1)^5 \cdot \sqrt[3]{\ln(x)}}{e^{x^3} \cdot (x-4)^2}$$

(a) Use logarithmic differentiation to find $f'(x)$.

(b) Simplify your answer as much as possible.

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Question 5 - Solution

The real domain is $x > 0,\ x \neq 4$. Where the function is nonzero, use absolute values in logarithmic differentiation:

$$\ln|f(x)| = 5\ln(x^2+1) + \tfrac{1}{3}\ln|\ln x| - x^3 - 2\ln|x-4|$$

Differentiating gives

$$\boxed{f'(x) = f(x)\left(\frac{10x}{x^2+1} + \frac{1}{3x\ln x} - 3x^2 - \frac{2}{x-4}\right).}$$

This formula holds for $x > 0,\ x \neq 1, 4$. At $x = 1$, $f(1) = 0$, but the cube-root factor gives no finite derivative.

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Question 6

Let the function be:

$$f(x) = \frac{(x^2 + 3x + 5)^4 \cdot \sqrt{x^2 - 1}}{(x + 2)^5 \cdot (3x - 1)^{1/3}}$$

(a) Use logarithmic differentiation to find $f'(x)$.

(b) Simplify the result as much as possible.

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Question 6 - Solution

The real domain is $|x| \geq 1,\ x \neq -2$. Where the function is nonzero, use absolute values in logarithmic differentiation:

$$\ln|f(x)| = 4\ln(x^2+3x+5) + \tfrac{1}{2}\ln(x^2-1) - 5\ln|x+2| - \tfrac{1}{3}\ln|3x-1|$$

Differentiating gives

$$\boxed{f'(x) = f(x)\left(\frac{4(2x+3)}{x^2+3x+5} + \frac{x}{x^2-1} - \frac{5}{x+2} - \frac{1}{3x-1}\right).}$$

This formula holds for $|x| > 1,\ x \neq -2$. At $x = \pm 1$ there is no finite derivative. The cube root is the real cube root.

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Question 7

Let the function be:

$$y = \left(\frac{(2x^2+1)^5}{\sqrt{5x-3} \cdot (x^2+4)^3} \right)^2$$

(a) Use logarithmic differentiation to find $\frac{dy}{dx}$.

(b) Simplify the result as much as possible.

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Question 7 - Solution

We are given:

$$y = \left(\frac{(2x^2+1)^5}{\sqrt{5x-3} \cdot (x^2+4)^3} \right)^2$$

Take the natural logarithm of both sides:

$$\ln y = \ln \left[\left(\frac{(2x^2+1)^5}{(5x-3)^{1/2} \cdot (x^2+4)^3} \right)^2 \right]$$

Bring the exponent outside:

$$\ln y = 2 \cdot \ln \left(\frac{(2x^2+1)^5}{(5x-3)^{1/2} \cdot (x^2+4)^3} \right)$$

Use log rules:

$$\ln y = 2 \left[5 \ln(2x^2+1) - \frac{1}{2} \ln(5x-3) - 3 \ln(x^2+4) \right]$$

Differentiate both sides:

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 \left[\frac{5 \cdot 4x}{2x^2+1} - \frac{1}{2} \cdot \frac{5}{5x-3} - \frac{3 \cdot 2x}{x^2+4} \right]$$

Simplify:

$$\frac{dy}{dx} = y \cdot \left(\frac{40x}{2x^2+1} - \frac{5}{5x-3} - \frac{12x}{x^2+4} \right)$$

Substitute y back:

$$\frac{dy}{dx} = \left(\frac{(2x^2+1)^5}{\sqrt{5x-3} \cdot (x^2+4)^3} \right)^2 \cdot \left(\frac{40x}{2x^2+1} - \frac{5}{5x-3} - \frac{12x}{x^2+4} \right)$$

Final Answer:

$$\boxed{\frac{dy}{dx} = \left(\frac{(2x^2+1)^5}{\sqrt{5x-3} \cdot (x^2+4)^3} \right)^2 \cdot \left(\frac{40x}{2x^2+1} - \frac{5}{5x-3} - \frac{12x}{x^2+4} \right)}$$

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Question 8

Let the function be:

$$y = (x^2 + 3x + 1)^4 \cdot \sqrt{x^2 - 1} \cdot \frac{1}{(2x + 5)^3}$$

(a) Use logarithmic differentiation to find $\frac{dy}{dx}$.

(b) Simplify your answer as much as possible.

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Question 8 - Solution

Put $A = x^2 + 3x + 1$. The real domain is $|x| \geq 1$, $x \neq -5/2$.

For $A \neq 0$ and $|x| > 1$, logarithmic differentiation gives

$$\ln|y| = 4\ln|A| + \tfrac{1}{2}\ln(x^2-1) - 3\ln|2x+5|,$$

$$\frac{y'}{y} = \frac{4(2x+3)}{A} + \frac{x}{x^2-1} - \frac{6}{2x+5}.$$

Cancel before evaluating at zeros of A:

$$\boxed{y' = \frac{A^3\left[4(2x+3)(x^2-1)(2x+5) + Ax(2x+5) - 6A(x^2-1)\right]}{\sqrt{x^2-1}(2x+5)^4}.}$$

This formula holds for $|x| > 1$, $x \neq -5/2$, including $x = (-3-\sqrt{5})/2$, where $y' = 0$. The other root of A is outside the domain.

There is no finite derivative at $x = \pm 1$.

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Question 9

Let

$$y = \frac{\sqrt{(x^2+4)^3(x-1)^2}}{(x^2-1)^4}$$

(a) Use logarithmic differentiation to find $\frac{dy}{dx}$.

(b) Simplify your final answer fully.

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Question 9 - Solution

For $x \neq \pm 1$, the square root simplifies to

$$y = \frac{(x^2+4)^{3/2}|x-1|}{(x^2-1)^4} > 0.$$

Hence

$$\ln y = \tfrac{3}{2}\ln(x^2+4) + \ln|x-1| - 4\ln|x^2-1|.$$

Differentiating and multiplying by y gives

$$\boxed{y' = \frac{(x^2+4)^{3/2}|x-1|}{(x^2-1)^4}\left(\frac{3x}{x^2+4} + \frac{1}{x-1} - \frac{8x}{x^2-1}\right), \quad x \neq \pm 1.}$$

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Question 10

Let

$$y = \frac{(x^2+1)^5 \cdot \sqrt[3]{x^2-4}}{\sqrt{2x+1} \cdot (x-3)^2}$$

(a) Use logarithmic differentiation to find $\frac{dy}{dx}$.

(b) Simplify your answer fully.

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Question 10 - Solution

The real domain is $x > -\frac{1}{2},\ x \neq 3$. Where the function is nonzero, use absolute values in logarithmic differentiation:

$$\ln|y| = 5\ln(x^2+1) + \tfrac{1}{3}\ln|x^2-4| - \tfrac{1}{2}\ln(2x+1) - 2\ln|x-3|$$

Differentiating gives

$$\boxed{y' = y\left(\frac{10x}{x^2+1} + \frac{2x}{3(x^2-4)} - \frac{1}{2x+1} - \frac{2}{x-3}\right).}$$

This formula holds for $x > -\frac{1}{2},\ x \neq 2, 3$. At $x = 2$, $y = 0$, but there is no finite derivative because of the cube-root factor.

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Question 1

The position of an object moving along a straight line is given by:

$$s(t) = 3t^3 - 5t^2 + 2t \quad \text{(in meters, where } t \text{ is in seconds)}$$

(a) Find the velocity function $v(t)$.
(b) Find the instantaneous velocity at $t = 2$ seconds.
(c) Find the average velocity between $t = 1$ and $t = 2$.

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Question 1 - Solution

We are given:

$$s(t) = 3t^3 - 5t^2 + 2t$$

(a) Velocity is the derivative of position:

$$v(t) = s'(t) = 9t^2 - 10t + 2$$

(b) Instantaneous velocity at $t = 2$:

$$v(2) = 9(4) - 10(2) + 2 = 36 - 20 + 2 = 18$$

Answer: $\boxed{v(2) = 18 \text{ m/s}}$

(c) Average velocity from $t = 1$ to $t = 2$:

$$\frac{s(2) - s(1)}{2 - 1} = s(2) - s(1)$$

Compute $s(2)$:

$$s(2) = 3(8) - 5(4) + 2(2) = 24 - 20 + 4 = 8$$

Compute $s(1)$:

$$s(1) = 3(1) - 5(1) + 2(1) = 3 - 5 + 2 = 0$$

$$\text{Average velocity} = \frac{8 - 0}{1} = 8$$

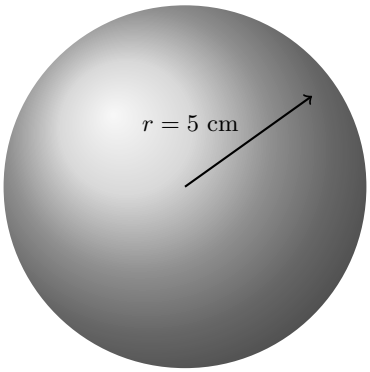
Answer: $\boxed{8 \text{ m/s}}$

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Question 2

Problem:

A spherical balloon is being inflated so that its volume increases at a rate of 100 cm^3/sec. How fast is the radius increasing when the radius is 5 cm?



Inflating sphere

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Question 2 - Solution

We are given the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

Differentiate both sides with respect to time t:

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

We are told that:

$$\frac{dV}{dt} = 100, \quad r = 5$$

Substitute into the equation:

$$100 = 4\pi(5)^2 \cdot \frac{dr}{dt} \Rightarrow 100 = 100\pi \cdot \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{\pi}$$

Answer:

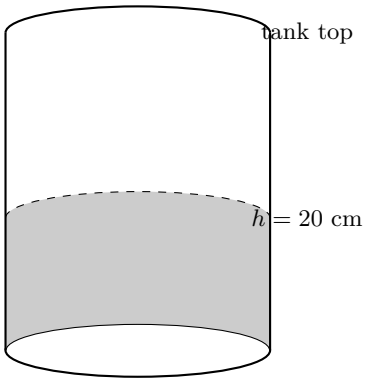
$$\boxed{\frac{dr}{dt} = \frac{1}{\pi} \text{ cm/sec}}$$

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Question 3

Problem:

Water is being poured into a cylindrical tank at a rate of $500\,\text{cm}^3/\text{s}$. The tank has a radius of $10\,\text{cm}$. How fast is the water level rising when the water is $20\,\text{cm}$ deep?



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Question 3 - Solution

The volume of a cylinder is:

$$V = \pi r^2 h$$

Given: $\frac{dV}{dt} = 500$, $r = 10$, $h = 20$
We differentiate with respect to time:

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} \Rightarrow 500 = \pi(10)^2 \frac{dh}{dt} \Rightarrow 500 = 100\pi \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{500}{100\pi} = \frac{5}{\pi}$$

Answer:

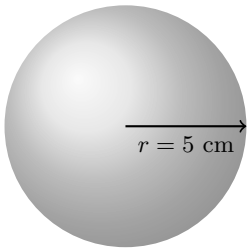
$$\boxed{\frac{dh}{dt} = \frac{5}{\pi} \text{ cm/sec}}$$

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Question 4

Problem:

A spherical balloon is being inflated so that its volume increases at a rate of $100\,\text{cm}^3/\text{s}$. How fast is the radius increasing when the radius is $5\,\text{cm}$?



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Question 4 - Solution

We use the formula for the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

Differentiate both sides with respect to t:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Substitute:

$$\frac{dV}{dt} = 100, \quad r = 5$$

$$100 = 4\pi(5)^2 \frac{dr}{dt} \Rightarrow 100 = 100\pi \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{100}{100\pi} = \frac{1}{\pi}$$

Answer:

$$\boxed{\frac{dr}{dt} = \frac{1}{\pi} \text{ cm/sec}}$$

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Question 5

Lighthouse Problem:

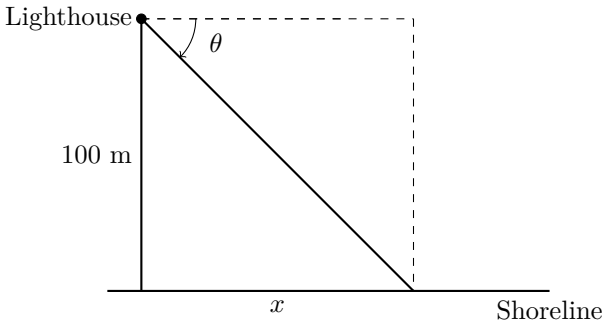
A lighthouse is located at the top of a cliff that is 100 meters high. The shoreline is straight and lies in a vertical plane. The beam of light rotates in this plane at a constant rate of

$$\frac{d\theta}{dt} = 0.05 \text{ rad/s}.$$

Let x be the distance along the shoreline from the base of the cliff to the point where the light hits the shore.

How fast is the spot of light moving along the shoreline when

$$\theta = \frac{\pi}{4}\,?$$



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Question 5 - Solution

Let:

- θ be the angle between the light beam and the horizontal shoreline.
- x be the distance along the shoreline to the spot of light.

From the right triangle formed by the beam, the cliff, and the shoreline:

$$\tan(\theta) = \frac{x}{100} \quad \Rightarrow \quad x = 100\tan(\theta)$$

Differentiate both sides with respect to time t:

$$\frac{dx}{dt} = 100\sec^2(\theta)\,\frac{d\theta}{dt}$$

Substitute the given values:

$$\theta = \frac{\pi}{4}, \quad \frac{d\theta}{dt} = 0.05 \text{ rad/s}, \quad \sec^2\left(\frac{\pi}{4}\right) = 2$$

$$\frac{dx}{dt} = 100(2)(0.05) = 10 \text{ m/s}$$

Answer:

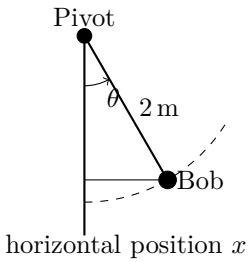
$$\boxed{\frac{dx}{dt} = 10 \text{ m/s}}$$

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Question 6

Problem:

A pendulum of length 2 meters swings back and forth in a vertical plane. At a certain instant, the angle θ between the pendulum and the vertical is $\frac{\pi}{6}$, and the angle is increasing at a rate of $0.4\,\text{rad/s}$. How fast is the pendulum bob moving horizontally at that instant?



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Question 6 - Solution

Let:

- θ be the angle between the pendulum and vertical
- x be the horizontal displacement of the bob from the vertical

Using trigonometry:

$$x = L\sin(\theta) = 2\sin(\theta)$$

Differentiate both sides with respect to time t:

$$\frac{dx}{dt} = 2\cos(\theta) \cdot \frac{d\theta}{dt}$$

Substitute:

$$\theta = \frac{\pi}{6}, \quad \frac{d\theta}{dt} = 0.4\,\text{rad/s}, \quad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\frac{dx}{dt} = 2 \cdot \frac{\sqrt{3}}{2} \cdot 0.4 = \sqrt{3} \cdot 0.4 \approx 0.6928\,\text{m/s}$$

Answer:

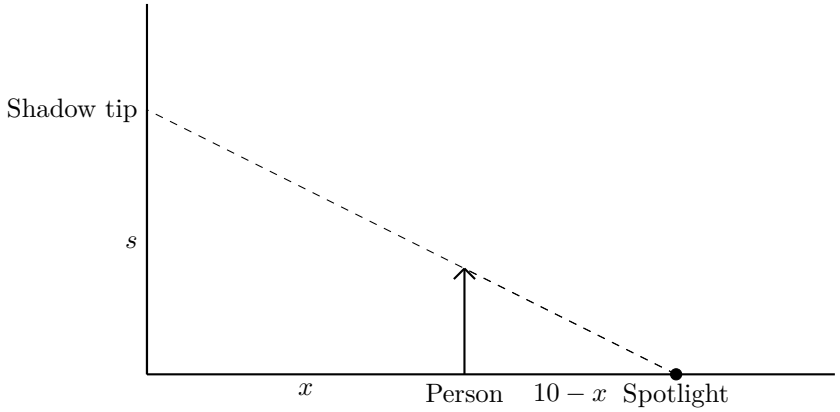
$$\boxed{\frac{dx}{dt} \approx 0.693\,\text{m/s}}$$

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Question 7

Problem:

A spotlight is located on the ground 10 meters from a vertical wall. A person 2 meters tall walks away from the wall at a speed of 1.5 m/s. How fast is the length of their shadow on the wall increasing when the person is 6 meters from the wall?



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Question 7 - Solution

Let:

- x: distance from person to wall (in m)
- s: length of the shadow on the wall (in m)
- Person's height: 2 m
- Spotlight is 10 m from the wall

Using similar triangles (person is between wall and spotlight, so spotlight–person distance is $10 - x$):

$$\frac{2}{10-x} = \frac{s}{10} \quad \Rightarrow \quad s = \frac{20}{10-x}$$

Differentiate with respect to t:

$$\frac{ds}{dt} = 20 \cdot \frac{d}{dt}\left((10-x)^{-1}\right) = 20(10-x)^{-2} \cdot \frac{dx}{dt}$$

Substitute $x = 6$ and $\frac{dx}{dt} = 1.5$:

$$\frac{ds}{dt} = \frac{20(1.5)}{(10-6)^2} = \frac{30}{16} = \frac{15}{8}$$

Answer:

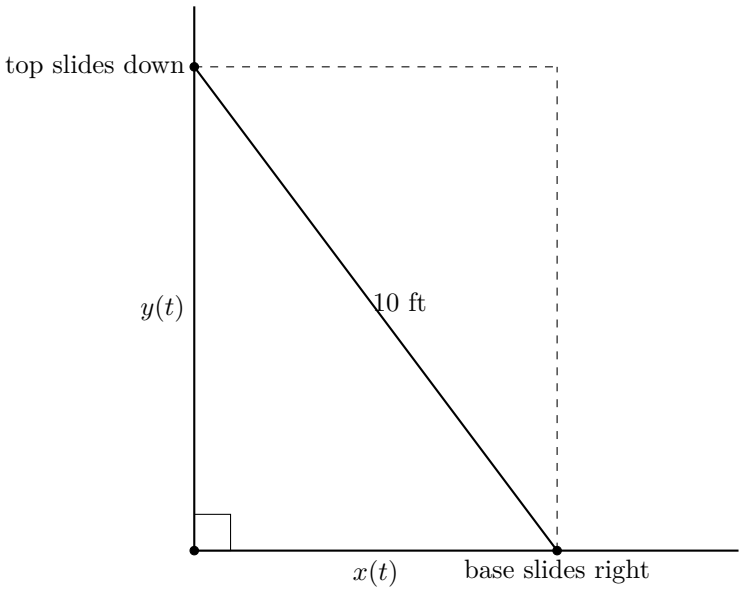
$$\boxed{\frac{ds}{dt} = \frac{15}{8} \text{ m/s}}$$ (The shadow is increasing at this rate.)

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Question 8

Problem:

A 10-foot ladder is leaning against a vertical wall. The bottom of the ladder is sliding away from the wall at a rate of 2 ft/s. How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 feet from the wall?



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Question 8 - Solution

Let:

- $x(t)$: distance from the wall to the base of the ladder
- $y(t)$: height of the top of the ladder on the wall
- Given: $\frac{dx}{dt} = 2\,\text{ft/s}, \quad x = 6$

Using the Pythagorean theorem:

$$x^2 + y^2 = 10^2 = 100$$

Differentiate both sides with respect to time t:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

We know:

$$x = 6, \quad \frac{dx}{dt} = 2$$

Solve for y:

$$36 + y^2 = 100 \Rightarrow y^2 = 64 \Rightarrow y = 8$$

Substitute into the differentiated equation:

$$6(2) + 8\frac{dy}{dt} = 0 \Rightarrow 12 + 8\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{12}{8} = -\frac{3}{2}$$

Answer:

$$\boxed{\frac{dy}{dt} = -\frac{3}{2}\ \text{ft/s}}$$

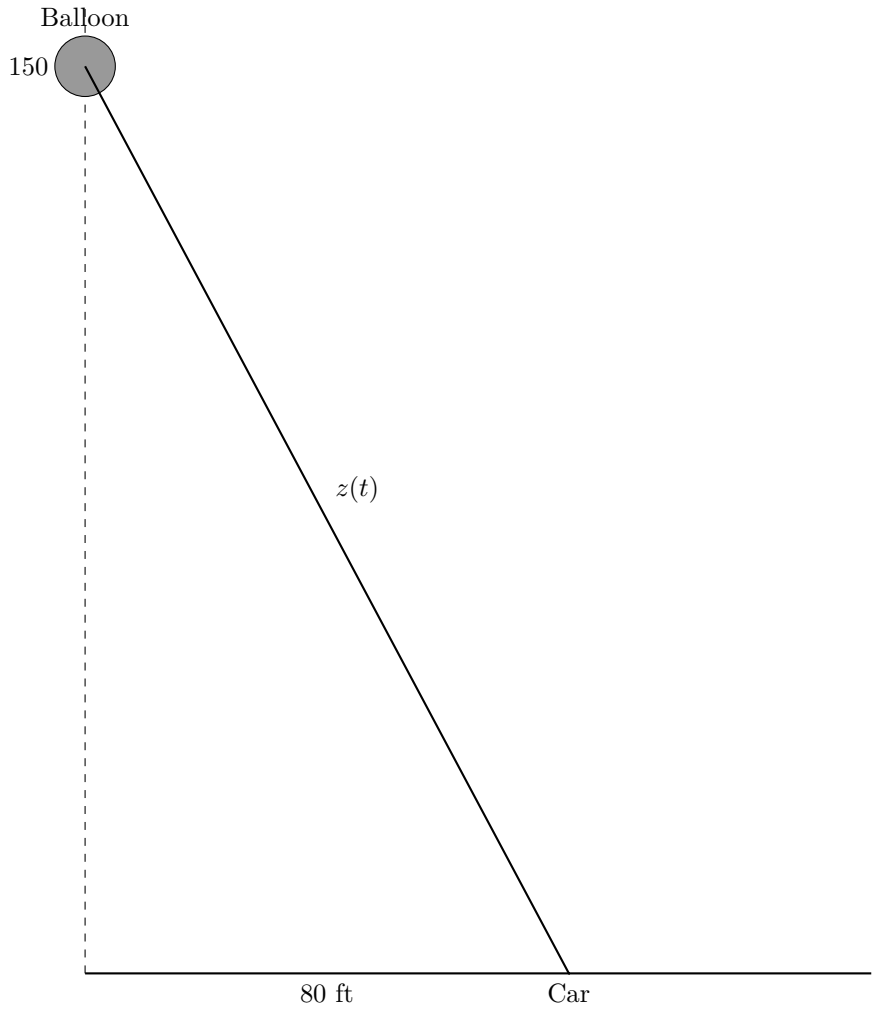
The negative sign indicates that the top of the ladder is sliding downward.

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Question 9

Problem:

A balloon is rising vertically at a constant rate of 5 ft/s, and at the same time, a car drives directly away from the point on the ground beneath the balloon at 60 ft/s. How fast is the distance between the car and the balloon increasing when the car is 80 feet from the balloon's launch point and the balloon is 150 feet high?



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Question 9 - Solution

Let:

- $x(t)$: distance from car to balloon launch point (horizontal) — $\frac{dx}{dt} = 60$
- $y(t)$: height of balloon — $\frac{dy}{dt} = 5$
- $z(t)$: distance between car and balloon (hypotenuse)

We apply the Pythagorean theorem:

$$z^2 = x^2 + y^2$$

Differentiate both sides:

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

Plug in values:

$$x = 80, \quad y = 150 \Rightarrow z = \sqrt{80^2 + 150^2} = \sqrt{6400 + 22500} = \sqrt{28900} = 170$$

Now plug in:

$$170 \cdot \frac{dz}{dt} = 80 \cdot 60 + 150 \cdot 5 \Rightarrow 170 \cdot \frac{dz}{dt} = 4800 + 750 = 5550 \Rightarrow \frac{dz}{dt} = \frac{5550}{170} = 32.65\,\text{ft/s}$$

Answer:

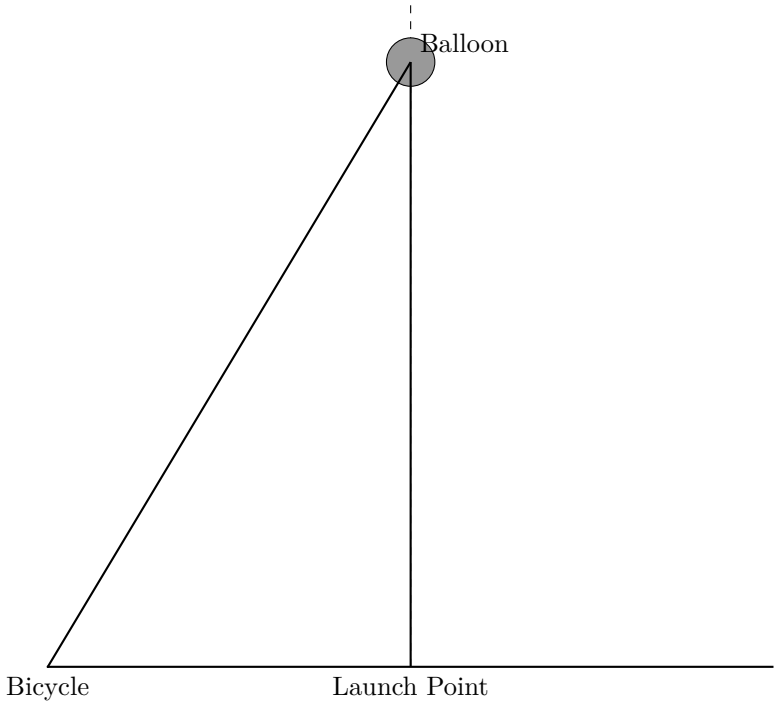
$$\boxed{\frac{dz}{dt} = 32.65\,\text{ft/s}}$$

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Question 10

Problem:

A hot air balloon is descending vertically at a rate of 2 ft/s, while a bicyclist passes directly beneath it, traveling in a straight line at a speed of 15 ft/s. How fast is the distance between the bicyclist and the balloon changing when the bicyclist is 60 feet from the point directly below the balloon and the balloon is 100 feet above the ground?



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Question 10 - Solution

Let:

- $x(t)$: horizontal distance of bicycle from point below the balloon — $\frac{dx}{dt} = 15$
- $y(t)$: vertical distance of balloon from ground — $\frac{dy}{dt} = -2$
- $z(t)$: distance between balloon and bicycle

Using the Pythagorean Theorem:

$$z^2 = x^2 + y^2$$

Differentiate both sides with respect to time t:

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

Plug in known values:

$$x = 60, \quad y = 100, \quad z = \sqrt{60^2 + 100^2} = \sqrt{3600 + 10000} = \sqrt{13600} = 116.6$$

$$z\frac{dz}{dt} = 60(15) + 100(-2) = 900 - 200 = 700 \Rightarrow \frac{dz}{dt} = \frac{700}{116.6} \approx 6.01\,\text{ft/s}$$

Answer:

$$\boxed{\frac{dz}{dt} \approx 6.01\,\text{ft/s}}$$

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Question 1

Let

$$f(x) = x^3 - 6x^2 + 9x + 1.$$

(a) Find all critical points of $f(x)$.

(b) Classify each critical point as a local minimum, local maximum, or neither using the First Derivative Test.

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Question 1 - Solution

We are given

$$f(x) = x^3 - 6x^2 + 9x + 1.$$

(a) Find $f'(x)$:

$$f'(x) = 3x^2 - 12x + 9.$$

Set $f'(x) = 0$ to find critical points:

$$3x^2 - 12x + 9 = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0.$$

Critical points: $x = 1, 3$

(b) First Derivative Test:

- Choose test values around $x = 1$ and $x = 3$.
- Use $f'(x) = 3x^2 - 12x + 9$.

Evaluate the sign of $f'(x)$:

- On $(-\infty, 1)$, pick $x = 0$:
$$f'(0) = 9 > 0.$$
- On $(1, 3)$, pick $x = 2$:
$$f'(2) = 3(4) - 24 + 9 = -3 < 0.$$
- On $(3, \infty)$, pick $x = 4$:
$$f'(4) = 48 - 48 + 9 = 9 > 0.$$

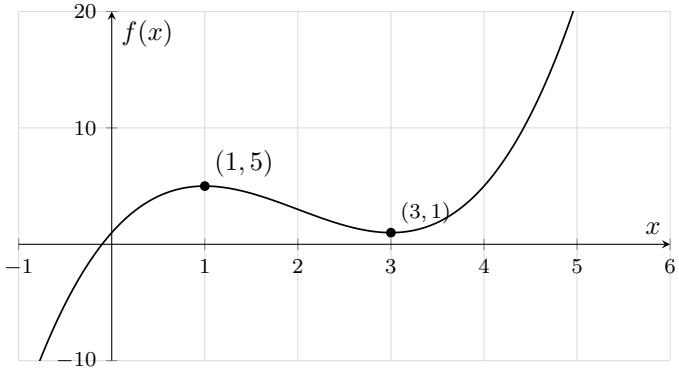
Conclusion:

- At $x = 1$, $f'(x)$ changes from positive to negative, so f has a local maximum.
- At $x = 3$, $f'(x)$ changes from negative to positive, so f has a local minimum.

Answer:

- Local maximum at $\boxed{x = 1}$ with $f(1) = 5$
- Local minimum at $\boxed{x = 3}$ with $f(3) = 1$

Graph of $f(x)$ with critical points marked



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Question 2

A company models its profit $P(x)$, in thousands of dollars, as a function of the number of units x (in hundreds) it produces and sells:

$$P(x) = -2x^3 + 15x^2 + 36x.$$

(a) Find the critical points of the function.
(b) Use the First Derivative Test to classify each critical point as a local maximum, minimum, or neither.
(c) How many units should be produced to maximize profit?

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Question 2 - Solution

We are given

$$P(x) = -2x^3 + 15x^2 + 36x.$$

(a) Find $P'(x)$:

$$P'(x) = -6x^2 + 30x + 36.$$

Set $P'(x) = 0$ to find critical points:

$$-6x^2 + 30x + 36 = 0 \Rightarrow x^2 - 5x - 6 = 0 \Rightarrow (x-6)(x+1) = 0.$$

Critical points: $x = -1,\ 6$

(b) First Derivative Test:
Evaluate the sign of $P'(x)$ on intervals determined by the critical points.

- On $(-\infty, -1)$, pick $x = -2$:

$$P'(-2) = -6(4) + 30(-2) + 36 = -24 - 60 + 36 = -48 < 0.$$

- On $(-1, 6)$, pick $x = 0$:

$$P'(0) = 36 > 0.$$

- On $(6, \infty)$, pick $x = 7$:

$$P'(7) = -6(49) + 30(7) + 36 = -294 + 210 + 36 = -48 < 0.$$

Conclusion:

- At $x = -1$, $P'(x)$ changes from negative to positive, so P has a local minimum.
- At $x = 6$, $P'(x)$ changes from positive to negative, so P has a local maximum.

(c) Maximizing profit:
Since $x = 6$ is a local maximum, the profit is maximized at $x = 6$ (hundreds of units), which is

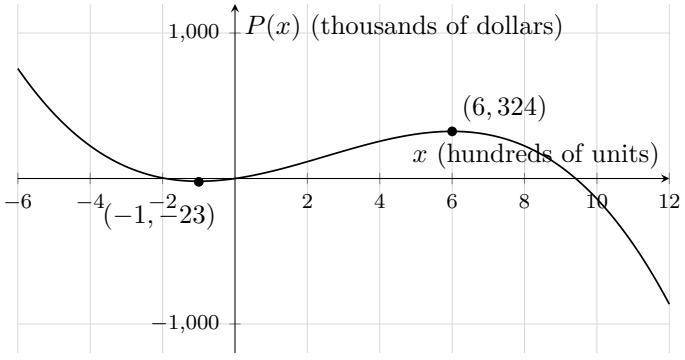
$$6 \times 100 = \boxed{600 \text{ units}}.$$

Compute the maximum profit:

$$P(6) = -2(6)^3 + 15(6)^2 + 36(6) = -432 + 540 + 216 = \boxed{324}.$$

This represents $\boxed{\$324{,}000}$ since P is in thousands of dollars.

Graph of $P(x)$ with critical points marked



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Question 3

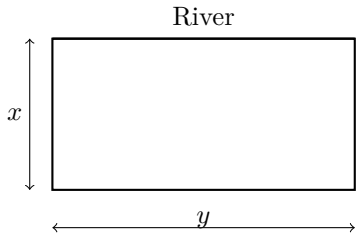
Problem:

A rectangular garden is to be built along a straight river. No fence is needed along the river, but fencing is required on the other three sides. If 120 meters of fencing is available, find the dimensions of the garden that maximize the enclosed area.

(a) Define the objective function and constraint.

(b) Use calculus to find the critical point(s).

(c) Determine the dimensions that maximize the area.



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Question 3 - Solution

Let x be the length perpendicular to the river, and y the length parallel to the river.

(a) Constraint:

Only three sides need fencing (two x's and one y):

$$2x + y = 120 \quad \Rightarrow \quad y = 120 - 2x$$

Objective: Maximize the area:

$$A(x) = x \cdot y = x(120 - 2x) = 120x - 2x^2$$

(b) Find critical points:

Take the derivative:

$$A'(x) = 120 - 4x$$

Set $A'(x) = 0$:

$$120 - 4x = 0 \Rightarrow x = 30$$

(c) Check dimensions:

$$y = 120 - 2(30) = 60$$

Answer: The garden has dimensions $\boxed{30 \text{ m} \times 60 \text{ m}}$, giving a maximum area.

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Question 4

Problem

A company models its revenue $R(x)$, in thousands of dollars, as a function of the number of items x (in hundreds) it produces and sells:

$$R(x) = -x^3 + 12x^2 + 9x.$$

(a) Find all critical points of $R(x)$.

(b) Use the First Derivative Test to classify each critical point as a local maximum, local minimum, or neither.

(c) How many items should be produced to maximize revenue?

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Question 4 - Solution

We are given

$$R(x) = -x^3 + 12x^2 + 9x.$$

(a) Find the derivative:

$$R'(x) = -3x^2 + 24x + 9.$$

Set $R'(x) = 0$:

$$-3x^2 + 24x + 9 = 0 \Rightarrow x^2 - 8x - 3 = 0.$$

Using the quadratic formula:

$$x = \frac{8 \pm \sqrt{64 + 12}}{2} = \frac{8 \pm \sqrt{76}}{2} = 4 \pm \sqrt{19}.$$

Critical points: $x = 4 - \sqrt{19},\ 4 + \sqrt{19}$

(b) First Derivative Test
Since $R'(x)$ is a downward-opening parabola, it is negative outside its roots and positive between them.

- At $x = 4 - \sqrt{19}$, $R'(x)$ changes from negative to positive, so $R(x)$ has a local minimum.
- At $x = 4 + \sqrt{19}$, $R'(x)$ changes from positive to negative, so $R(x)$ has a local maximum.

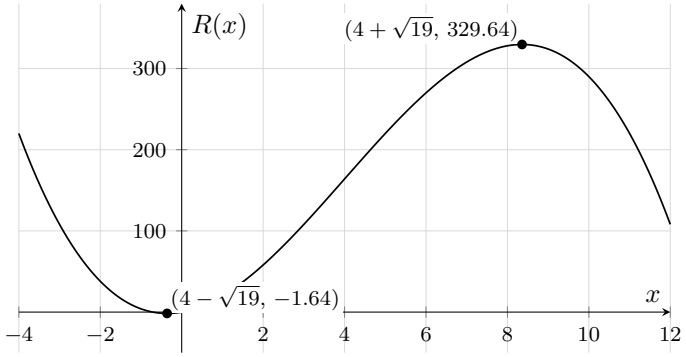
(c) Maximizing revenue
Revenue is maximized at

$$x = 4 + \sqrt{19} \approx 8.36.$$

Since x is measured in hundreds of items, this corresponds to

$$\boxed{836 \text{ items}}.$$

Graph of $R(x)$ with critical points marked



x: hundreds of items; $R(x)$: thousands of dollars.
Point heights are rounded to two decimal places.

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Question 5

Problem

Let

$$f(x) = x^4 - 4x^3 + 6x^2.$$

(a) Find the critical points of $f(x)$.

(b) Classify each critical point as a local maximum, local minimum, or neither.

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Question 5 - Solution

We are given

$$f(x) = x^4 - 4x^3 + 6x^2.$$

(a) Find the critical points.
Differentiate:

$$f'(x) = 4x^3 - 12x^2 + 12x.$$

Factor:

$$f'(x) = 4x(x^2 - 3x + 3).$$

Set $f'(x) = 0$:

$$4x(x^2 - 3x + 3) = 0.$$

This gives $x = 0$, or $x^2 - 3x + 3 = 0$. The quadratic has discriminant

$$(-3)^2 - 4(1)(3) = 9 - 12 = -3 < 0,$$

so it has no real solutions. Therefore,

$$\text{Only critical point: } x = 0.$$

(b) Classify the critical point.
Compute the second derivative:

$$f''(x) = 12x^2 - 24x + 12.$$

Evaluate at $x = 0$:

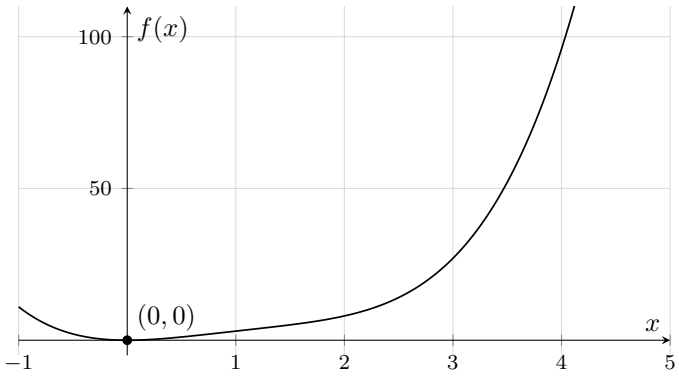
$$f''(0) = 12 > 0,$$

so $f(x)$ has a local minimum at $x = 0$. Also $f(0) = 0$, so the minimum point is $(0, 0)$.

Conclusion

$$\boxed{\text{Local minimum at } x = 0 \text{ (point } (0, 0))}$$

Graph of $f(x)$ with the critical point marked



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Question 6

Problem:

Let $f(x) = \frac{x^3}{3} - 2x^2 + x + 5$.

(a) Find all critical points of $f(x)$.

(b) Use the First Derivative Test to classify each critical point as a local maximum, local minimum, or neither.

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Question 6 - Solution

We are given:

$$f(x) = \frac{x^3}{3} - 2x^2 + x + 5$$

(a) Find the critical points:
Take the first derivative:

$$f'(x) = x^2 - 4x + 1$$

Set $f'(x) = 0$:

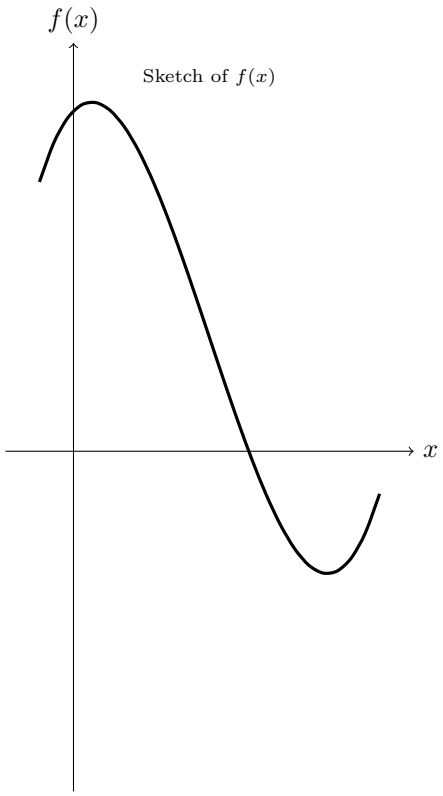
$$x^2 - 4x + 1 = 0 \Rightarrow x = 2 \pm \sqrt{3}$$

Critical points: $x = 2 + \sqrt{3}$, $x = 2 - \sqrt{3}$

(b) First Derivative Test:
Evaluate $f'(x)$ on intervals around the critical points:
Interval $(-\infty, 2 - \sqrt{3})$: pick $x = 0$, $f'(0) = 1 > 0$
Interval $(2 - \sqrt{3}, 2 + \sqrt{3})$: pick $x = 2$, $f'(2) = -3 < 0$
Interval $(2 + \sqrt{3}, \infty)$: pick $x = 4$, $f'(4) = 1 > 0$

Conclusion:
$f'(x)$ changes from positive to negative at $x = 2 - \sqrt{3} \rightarrow$ local maximum $f'(x)$ changes from negative to positive at $x = 2 + \sqrt{3} \rightarrow$ local minimum

$$\boxed{\begin{array}{l}\text{Local max at } x = 2 - \sqrt{3} \\ \text{Local min at } x = 2 + \sqrt{3}\end{array}}$$



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Question 7

Problem:

A company's profit function (in thousands of dollars) from selling x hundred units of a product is given by:

$$P(x) = -x^3 + 6x^2 + 60$$

(a) Find all critical points of $P(x)$.

(b) Determine whether each critical point is a local maximum or minimum using the First Derivative Test.

(c) Interpret your result in the context of the company's production.

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Question 7 - Solution

(a) Find critical points:

Take the derivative:

$$P'(x) = -3x^2 + 12x$$

Set $P'(x) = 0$:

$$-3x^2 + 12x = 0 \Rightarrow x(-3x + 12) = 0 \Rightarrow x = 0 \text{ or } x = 4$$

Critical points: $\boxed{x = 0}$, $\boxed{x = 4}$

(b) First Derivative Test:

Check intervals: For $x < 0$: not relevant (negative production)
Between $0 < x < 4$: pick $x = 2$, $P'(2) = -3(4) + 12(2) = -12 + 24 = 12 > 0$
For $x > 4$: pick $x = 5$, $P'(5) = -3(25) + 12(5) = -75 + 60 = -15 < 0$

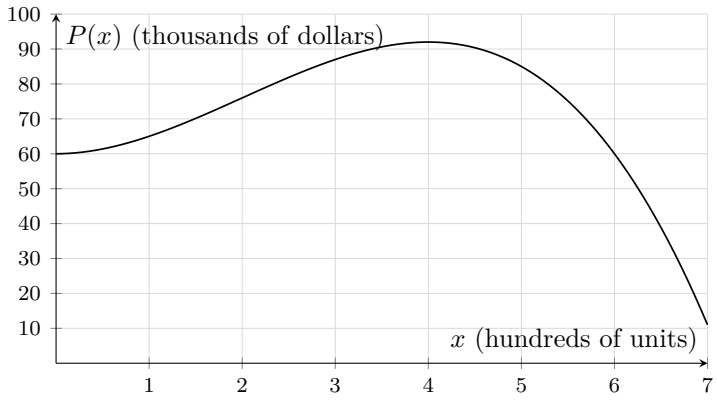
Conclusion: $P(x)$ increases before $x = 4$, decreases after $\rightarrow$ local max at $\boxed{x = 4}$
$x = 0$ is a stationary point, but no increase before it (since domain is $x \geq 0$)

(c) Interpretation:

Producing 400 units ($x = 4$) yields the maximum profit. Profit decreases beyond 400 units, and no benefit is gained producing less than 400.

$$\boxed{\text{Maximum profit occurs when the company produces 400 units.}}$$

Profit Function Graph:



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Question 8

Problem:

A ball is launched vertically upward, and its height in meters at time t seconds is modeled by:

$$h(t) = -5t^2 + 20t + 1$$

(a) Find all critical points of the function $h(t)$.

(b) Use the First Derivative Test to classify each critical point.

(c) What is the maximum height reached by the ball and at what time?

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Question 8 - Solution

(a) Find critical points:

Take the derivative:

$$h'(t) = -10t + 20$$

Set $h'(t) = 0$:

$$-10t + 20 = 0 \Rightarrow t = 2$$

Critical point: $\boxed{t = 2}$

(b) First Derivative Test:

For $t < 2$: pick $t = 1$, $h'(1) = -10(1) + 20 = 10 > 0$

For $t > 2$: pick $t = 3$, $h'(3) = -10(3) + 20 = -10 < 0$

Since $h'(t)$ changes from positive to negative at $t = 2$, this is a local maximum.

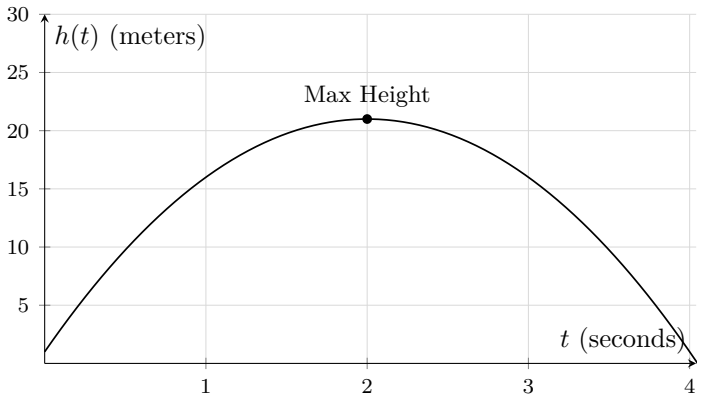
(c) Maximum Height:

Substitute $t = 2$ into $h(t)$:

$$h(2) = -5(4) + 20(2) + 1 = -20 + 40 + 1 = 21$$

Answer: The maximum height is $\boxed{21 \text{ meters}}$, reached at $\boxed{t = 2 \text{ seconds}}$.

Graph of the Trajectory:



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Question 9

Problem:

Let the function be defined as:

$$f(x) = x^3 - 3x^2 - 9x + 5$$

(a) Find all critical points of $f(x)$.

(b) Use the First Derivative Test to classify each critical point as a local maximum or local minimum.

(c) Determine the function value at each critical point.

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Question 9 - Solution

(a) Find critical points:

Differentiate:

$$f'(x) = 3x^2 - 6x - 9$$

Set $f'(x) = 0$:

$$3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1) = 0 \Rightarrow x = -1,\ 3$$

Critical points: $\boxed{x = -1,\ x = 3}$

(b) First Derivative Test:

Interval $(-\infty, -1)$: Choose $x = -2$, $f'(-2) = 3(4) + 12 - 9 = 15 > 0$

Interval $(-1, 3)$: Choose $x = 0$, $f'(0) = -9 < 0$

Interval $(3, \infty)$: Choose $x = 4$, $f'(4) = 3(16) - 24 - 9 = 15 > 0$

So: At $x = -1$, f' changes from positive to negative $\rightarrow$ local maximum At $x = 3$, f' changes from negative to positive $\rightarrow$ local minimum

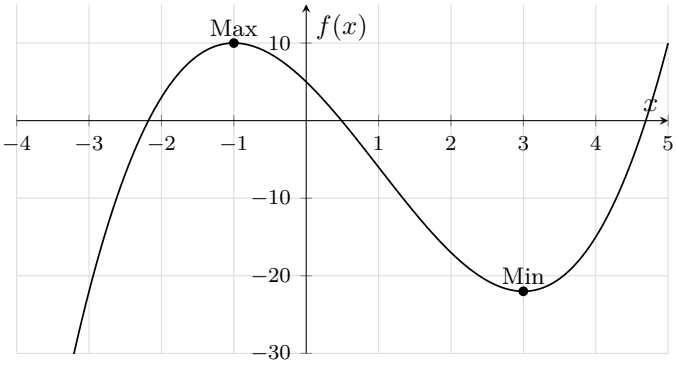
(c) Function values:

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = -1 - 3 + 9 + 5 = 10$$

$$f(3) = 27 - 27 - 27 + 5 = -22$$

Local maximum: $\boxed{f(-1) = 10}$ **Local minimum:** $\boxed{f(3) = -22}$

Graph of the function:



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Question 10

Problem:

Consider the function:

$$f(x) = x^4 - 4x^3 + 6x^2 + 1$$

(a) Find all critical points of $f(x)$.

(b) Classify each critical point using the First Derivative Test.

(c) Determine the function value at each critical point.

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Question 10 - Solution

(a) Find critical points:

First, compute the derivative:

$$f'(x) = 4x^3 - 12x^2 + 12x$$

Factor:

$$f'(x) = 4x(x^2 - 3x + 3)$$

Set derivative to zero:

$$4x(x^2 - 3x + 3) = 0 \Rightarrow x = 0 \quad \text{(real root)}$$

Solve $x^2 - 3x + 3 = 0$:

$$\text{Discriminant: } D = (-3)^2 - 4(1)(3) = 9 - 12 = -3 \quad \Rightarrow \text{no real roots}$$

Critical point: $\boxed{x = 0}$

(b) First Derivative Test: Test sign of $f'(x)$ around $x = 0$:

For $x < 0$: try $x = -1$

$$f'(-1) = 4(-1)^3 - 12(-1)^2 + 12(-1) = -4 - 12 - 12 = -28 < 0$$

For $x > 0$: try $x = 1$

$$f'(1) = 4(1)^3 - 12(1)^2 + 12(1) = 4 - 12 + 12 = 4 > 0$$

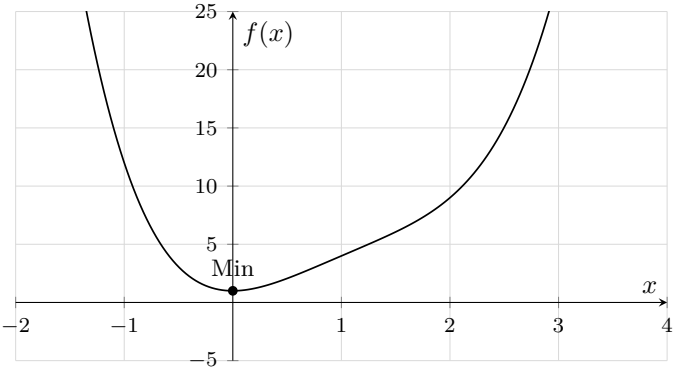
Since $f'(x)$ changes from negative to positive at $x = 0$, there is a local minimum.

(c) Function value at critical point:

$$f(0) = 0^4 - 4(0)^3 + 6(0)^2 + 1 = 1$$

Local minimum at: $\boxed{(0, 1)}$

Graph of the function:



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Question 1

Let:

$$f(x) = 2x^3 - 9x^2 + 12x + 1$$

(a) Find the critical points of $f(x)$.

(b) Use the second derivative test to determine whether each critical point is a local minimum, maximum, or neither.

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Question 1 - Solution

We are given:

$$f(x) = 2x^3 - 9x^2 + 12x + 1$$

(a) First derivative:

$$f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$$

Critical points: $x = 1, 2$

(b) Use the second derivative:

$$f''(x) = \frac{d}{dx}(f'(x)) = 12x - 18$$

Evaluate at each critical point:

$$f''(1) = 12(1) - 18 = -6 \;\Rightarrow\; \text{local maximum at } x = 1$$

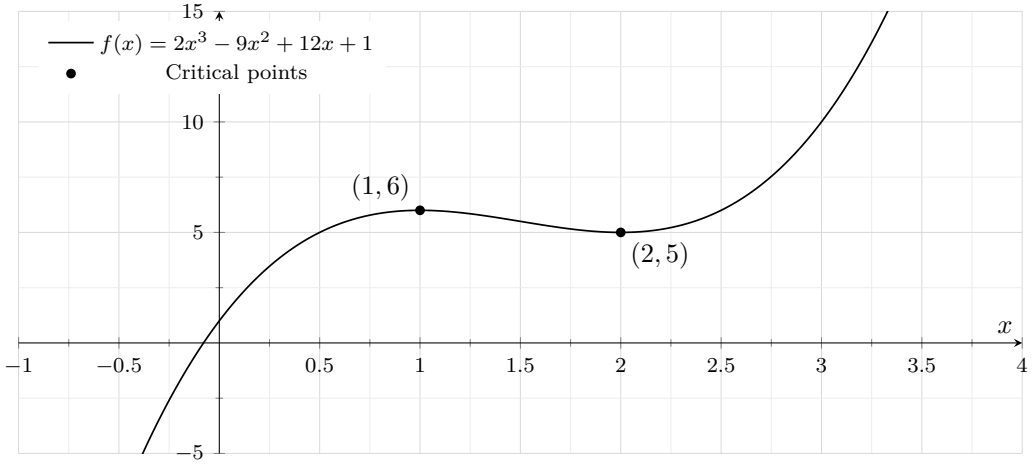
$$f''(2) = 12(2) - 18 = 6 \;\Rightarrow\; \text{local minimum at } x = 2$$

Answer:

Local maximum at $\boxed{x = 1}$

Local minimum at $\boxed{x = 2}$

Graph of $f(x)$



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Question 2

Problem:

Let $f(x) = x^3 - 6x^2 + 9x + 2$ on the closed interval $[0, 4]$.

(a) Find the critical points of $f(x)$ in the interval.

(b) Determine the absolute maximum and minimum values of $f(x)$ on the interval $[0, 4]$.

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Question 2 - Solution

(a) Find critical points:

First, compute the derivative:

$$f'(x) = 3x^2 - 12x + 9$$

Set the derivative to zero:

$$3x^2 - 12x + 9 = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1,\ x = 3$$

Both critical points are in the interval $[0, 4]$.

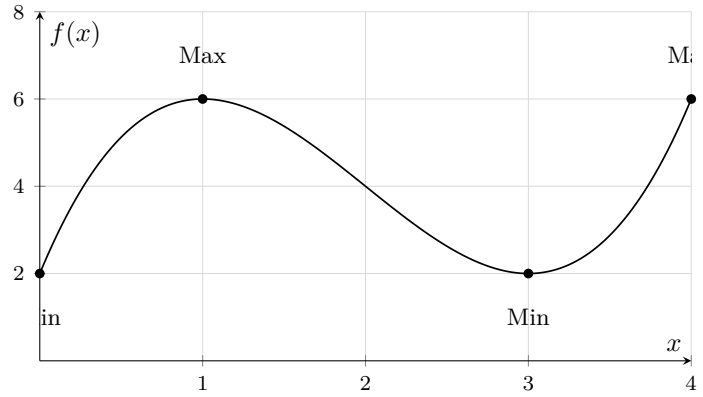
(b) Evaluate function at critical points and endpoints:

$$f(0) = 0 - 0 + 0 + 2 = 2$$
$$f(1) = 1 - 6 + 9 + 2 = 6$$
$$f(3) = 27 - 54 + 27 + 2 = 2$$
$$f(4) = 64 - 96 + 36 + 2 = 6$$

Conclusion:

- Maximum value: $\boxed{6}$ at $x = 1$ and $x = 4$ - Minimum value: $\boxed{2}$ at $x = 0$ and $x = 3$

Graph of the function on $[0, 4]$:



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Question 3

Problem:

Consider the function $f(x) = x^4 - 4x^3 + 10$ on the interval $[0, 4]$.

(a) Find the critical points of $f(x)$ on the interval.

(b) Determine the absolute maximum and minimum values of $f(x)$ on the interval $[0, 4]$.

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Question 3 - Solution

(a) Find critical points:

First, compute the derivative:

$$f'(x) = 4x^3 - 12x^2$$

Factor:

$$f'(x) = 4x^2(x-3) \Rightarrow f'(x) = 0 \text{ when } x = 0 \text{ or } x = 3$$

Both are within the interval $[0, 4]$, so check both.

(b) Evaluate function at critical points and endpoints:

$$f(0) = 0 - 0 + 10 = 10$$

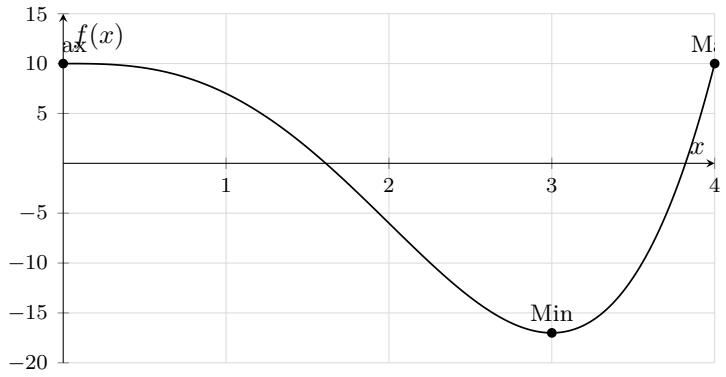
$$f(3) = 81 - 108 + 10 = -17$$

$$f(4) = 256 - 256 + 10 = 10$$

Conclusion:

- Minimum value: $\boxed{-17}$ at $x = 3$ - Maximum value: $\boxed{10}$ at $x = 0$ and $x = 4$

Graph of the function on $[0, 4]$:



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Question 4

Problem:

Let $f(x) = \frac{x}{x^2+1}$, and consider the interval $[-2, 2]$.

(a) Find the critical points of $f(x)$ on the interval.

(b) Determine the absolute maximum and minimum values of $f(x)$ on $[-2, 2]$.

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Question 4 - Solution

(a) Find the critical points:

We find $f'(x)$ using the quotient rule:

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

Set the numerator equal to zero:

$$1 - x^2 = 0 \Rightarrow x = \pm 1$$

Both $x = -1$ and $x = 1$ are in the interval $[-2, 2]$.

(b) Evaluate at critical points and endpoints:

$$f(-2) = \frac{-2}{4+1} = \frac{-2}{5}$$

$$f(-1) = \frac{-1}{1+1} = \frac{-1}{2}$$

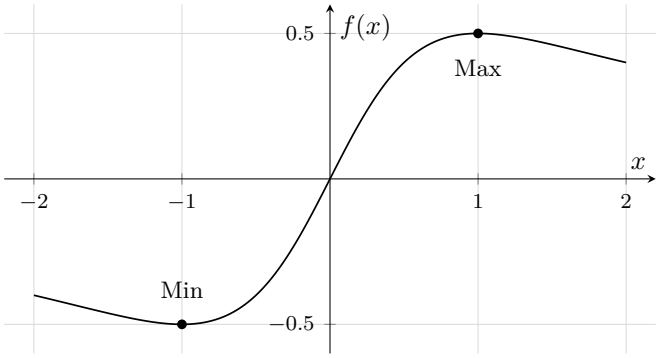
$$f(1) = \frac{1}{1+1} = \frac{1}{2}$$

$$f(2) = \frac{2}{4+1} = \frac{2}{5}$$

Conclusion:

- Maximum value: $\boxed{\frac{1}{2}}$ at $x = 1$ - Minimum value: $\boxed{-\frac{1}{2}}$ at $x = -1$

Graph of $f(x) = \frac{x}{x^2+1}$ on $[-2, 2]$:



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Question 5

Problem:

Let $f(x) = x^3 - 3x + 1$ on the interval $[-3, 3]$.

(a) Find all critical points of $f(x)$ on the interval.

(b) Determine the absolute maximum and minimum values of $f(x)$ on $[-3, 3]$.

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Question 5 - Solution

(a) Find critical points:

Take the derivative:

$$f'(x) = 3x^2 - 3$$

Set derivative equal to 0:

$$3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

Critical points: $x = -1, x = 1$

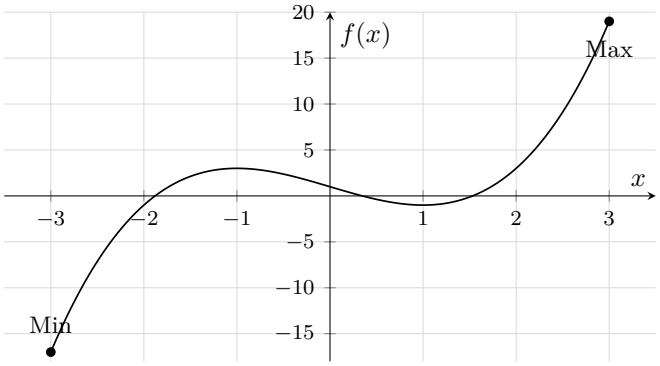
(b) Evaluate at critical points and endpoints:

$$f(-3) = (-3)^3 - 3(-3) + 1 = -27 + 9 + 1 = -17$$
$$f(-1) = (-1)^3 - 3(-1) + 1 = -1 + 3 + 1 = 3$$
$$f(1) = 1^3 - 3(1) + 1 = 1 - 3 + 1 = -1$$
$$f(3) = 27 - 9 + 1 = 19$$

Conclusion:

- Absolute maximum: $\boxed{19}$ at $x = 3$ - Absolute minimum: $\boxed{-17}$ at $x = -3$

Graph of $f(x) = x^3 - 3x + 1$ on $[-3, 3]$:



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Question 6

Problem:

Consider the function $f(x) = x^4 - 4x^2$ on the interval $[-3, 3]$.

(a) Find all critical points of $f(x)$ in the interval.

(b) Determine the absolute maximum and minimum values of $f(x)$ on $[-3, 3]$.

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Question 6 - Solution

(a) Find critical points:

Differentiate:

$$f'(x) = 4x^3 - 8x$$

Set derivative to zero:

$$4x^3 - 8x = 0 \Rightarrow 4x(x^2 - 2) = 0 \Rightarrow x = 0, \pm\sqrt{2}$$

Critical points in $[-3, 3]$: $x = -\sqrt{2}, 0, \sqrt{2}$

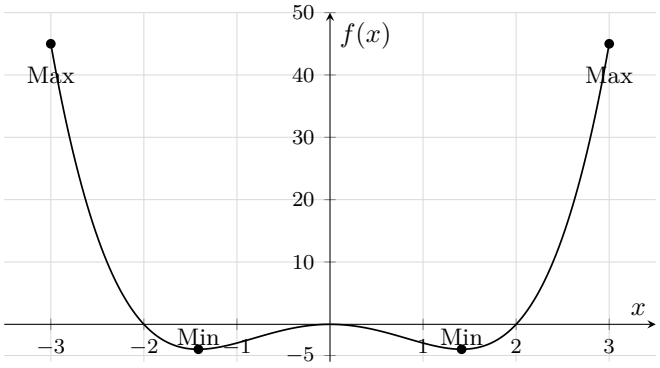
(b) Evaluate $f(x)$ at critical points and endpoints:

$$f(-3) = (-3)^4 - 4(-3)^2 = 81 - 36 = 45$$
$$f(-\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2 = 4 - 8 = -4$$
$$f(0) = 0^4 - 4(0)^2 = 0$$
$$f(\sqrt{2}) = 4 - 8 = -4$$
$$f(3) = 81 - 36 = 45$$

Conclusion:

- Absolute maximum: $\boxed{45}$ at $x = -3$ and $x = 3$ - Absolute minimum: $\boxed{-4}$ at $x = \pm\sqrt{2}$

Graph of $f(x) = x^4 - 4x^2$ on $[-3, 3]$:



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Question 7

Problem:

Let $f(x) = x^3 - 3x^2 + 1$ on the interval $[0, 4]$.

(a) Find all critical points of $f(x)$ in the interval.

(b) Determine the absolute maximum and minimum values of $f(x)$ on the given interval.

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Question 7 - Solution

(a) Find critical points:

Differentiate:

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

Set $f'(x) = 0$ to find critical points:

$$3x(x-2) = 0 \Rightarrow x = 0,\ x = 2$$

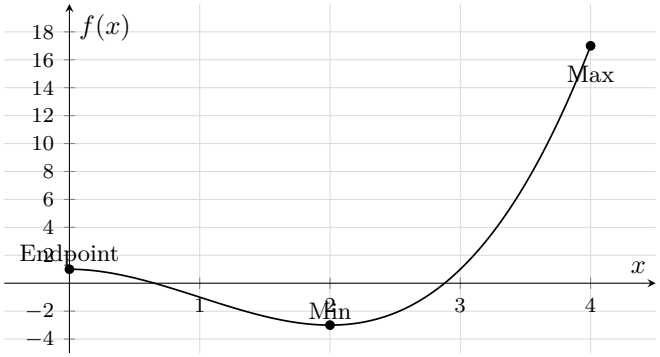
(b) Evaluate function at critical points and endpoints:

$$f(0) = 0^3 - 3(0)^2 + 1 = 1$$
$$f(2) = 8 - 12 + 1 = -3$$
$$f(4) = 64 - 48 + 1 = 17$$

Conclusion:

- Absolute maximum: $\boxed{17}$ at $x = 4$ - Absolute minimum: $\boxed{-3}$ at $x = 2$

Graph of $f(x) = x^3 - 3x^2 + 1$ on $[0, 4]$:



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Question 8

Problem:

Let $f(x) = \frac{x}{x^2+1}$ on the interval $[-3, 3]$.

(a) Find all critical points of $f(x)$ in the interval.

(b) Determine the absolute maximum and minimum values of $f(x)$ on the given interval.

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Question 8 - Solution

(a) Find critical points:

Differentiate using the quotient rule:

$$f(x) = \frac{x}{x^2+1}, \quad f'(x) = \frac{(1)(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

Set $f'(x) = 0$:

$$1 - x^2 = 0 \Rightarrow x = \pm 1$$

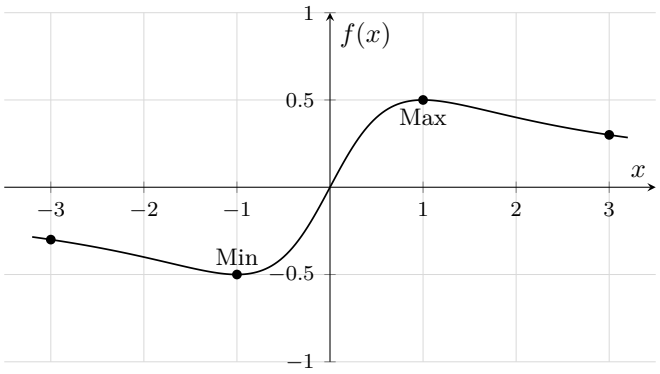
(b) Evaluate function at endpoints and critical points:

$$f(-3) = \frac{-3}{9+1} = -\frac{3}{10} \quad f(-1) = \frac{-1}{2} = -\frac{1}{2} \quad f(1) = \frac{1}{2} \quad f(3) = \frac{3}{10}$$

Conclusion:

- Absolute maximum: $\boxed{\frac{1}{2}}$ at $x = 1$ - Absolute minimum: $\boxed{-\frac{1}{2}}$ at $x = -1$

Graph of $f(x) = \frac{x}{x^2+1}$ on $[-3, 3]$:



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Question 9

Problem:

A farmer has 240 meters of fencing to enclose a rectangular field. One side of the rectangle lies along a river and does not require fencing.

(a) Express the area of the field as a function of one variable.

(b) Find the dimensions of the field that maximize the area.

(c) What is the maximum area?

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Question 9 - Solution

(a) Express the area as a function:

Let x be the length of the side perpendicular to the river. Since only three sides are fenced (two x's and one along the length), the total fencing is:

$$2x + y = 240 \Rightarrow y = 240 - 2x$$

Area:

$$A(x) = x \cdot y = x(240 - 2x) = 240x - 2x^2$$

(b) Maximize the area:

Differentiate:

$$A'(x) = 240 - 4x$$

Set derivative to zero:

$$240 - 4x = 0 \Rightarrow x = 60$$

Check that this gives a maximum using second derivative:

$$A''(x) = -4 < 0 \quad \Rightarrow \text{Maximum}$$

Find y:

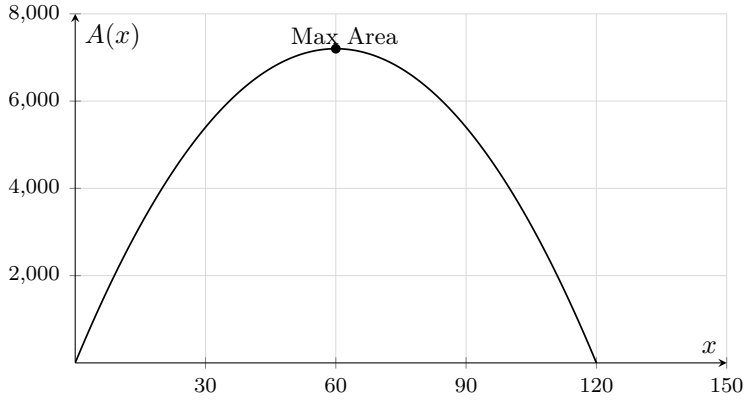
$$y = 240 - 2(60) = 120$$

Dimensions: $\boxed{60\,\text{m} \times 120\,\text{m}}$

(c) Maximum area:

$$A = 60 \cdot 120 = \boxed{7200\,\text{m}^2}$$

Graph of $A(x) = 240x - 2x^2$:



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Question 10

Problem:

A box with a square base and open top must have a volume of $500\,\text{cm}^3$. Find the dimensions of the box that minimize the surface area.

(a) Express the surface area as a function of one variable.

(b) Find the dimensions that minimize the surface area.

(c) What is the minimum surface area?

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Question 10 - Solution

Let x be the side length of the square base, and h be the height of the box.

(a) Surface area as a function of one variable:

Volume constraint:

$$x^2 h = 500 \Rightarrow h = \frac{500}{x^2}$$

Surface area (no top):

$$A(x) = x^2 + 4xh = x^2 + 4x \cdot \frac{500}{x^2} = x^2 + \frac{2000}{x}$$

(b) Minimize the surface area:

Differentiate:

$$A'(x) = 2x - \frac{2000}{x^2}$$

Set derivative to 0:

$$2x - \frac{2000}{x^2} = 0 \Rightarrow 2x^3 = 2000 \Rightarrow x^3 = 1000 \Rightarrow x = 10$$

Find h:

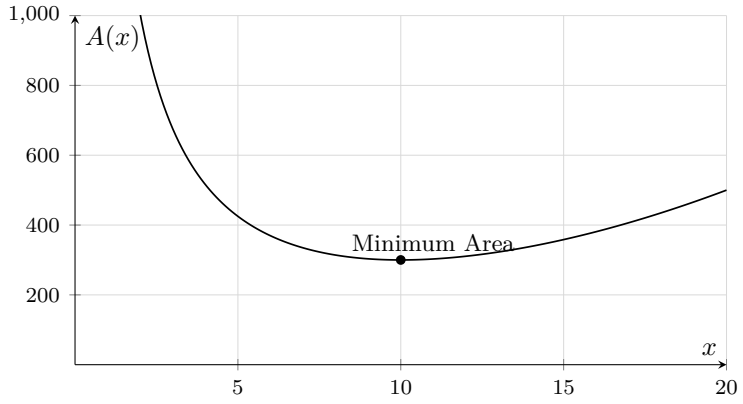
$$h = \frac{500}{10^2} = 5$$

Dimensions: $\boxed{x = 10\,\text{cm},\ h = 5\,\text{cm}}$

(c) Minimum surface area:

$$A = 10^2 + \frac{2000}{10} = 100 + 200 = \boxed{300\,\text{cm}^2}$$

Graph of $A(x) = x^2 + \frac{2000}{x}$:



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Question 1

Let:

$$f(x) = x^3 - 6x^2 + 9x + 2$$

Find the absolute maximum and minimum values of $f(x)$ on the closed interval $[0, 4]$.

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Question 1 - Solution

We are given:

$$f(x) = x^3 - 6x^2 + 9x + 2$$

Step 1: Find critical points

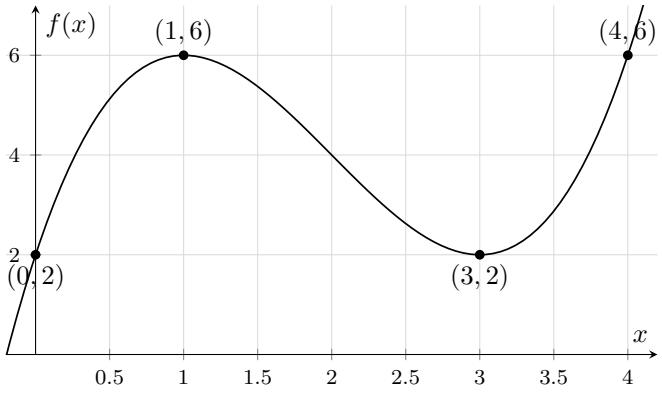
$$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0 \Rightarrow x = 1,\ 3$$

Step 2: Evaluate at endpoints and critical points in $[0, 4]$

$$f(0) = 2$$
$$f(1) = 6$$
$$f(3) = 2$$
$$f(4) = 6$$

Absolute Maximum: $\boxed{6}$ at $x = 1$ and $x = 4$
Absolute Minimum: $\boxed{2}$ at $x = 0$ and $x = 3$

Graph of the function on $[0, 4]$



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Question 2

Problem:

Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1$$

on the interval $[0, 4]$.

(a) Identify all critical points of $f(x)$ in the interval.

(b) Evaluate $f(x)$ at the endpoints and critical points.

(c) Determine the absolute maximum and minimum values.

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Question 2 - Solution

We are given:

$$f(x) = x^3 - 3x^2 + 1$$

(a) Find critical points:
Take the derivative:

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

Set $f'(x) = 0$:

$$3x(x-2) = 0 \Rightarrow x = 0,\ x = 2$$

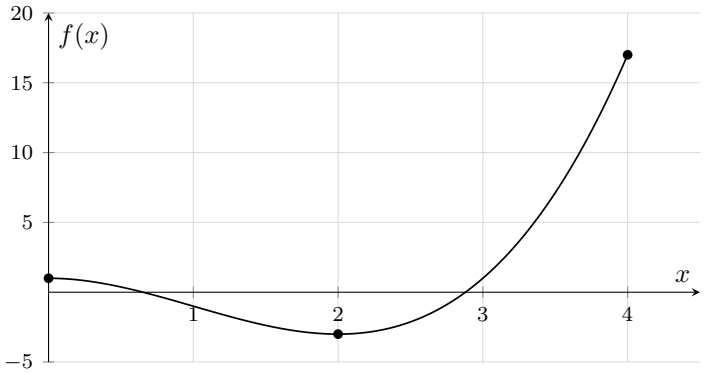
Both are in the interval $[0, 4]$.
(b) Evaluate function at critical points and endpoints:

$$f(0) = 0^3 - 3(0)^2 + 1 = 1$$
$$f(2) = 8 - 12 + 1 = -3$$
$$f(4) = 64 - 48 + 1 = 17$$

(c) Absolute extrema:

- Absolute minimum: $\boxed{f(2) = -3}$
- Absolute maximum: $\boxed{f(4) = 17}$

Graph of $f(x) = x^3 - 3x^2 + 1$ on $[0, 4]$:



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Question 3

Problem:

Find the absolute maximum and minimum values of the function

$$f(x) = \frac{x}{x^2 + 1}$$

on the interval $[-2, 2]$.

(a) Find the critical points of $f(x)$ in the interval.

(b) Evaluate $f(x)$ at the endpoints and at critical points.

(c) Determine the absolute maximum and minimum values.

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Question 3 - Solution

We are given:

$$f(x) = \frac{x}{x^2+1}$$

(a) Find critical points:
Use quotient rule:

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

Set $f'(x) = 0$:

$$\frac{1-x^2}{(x^2+1)^2} = 0 \Rightarrow 1-x^2 = 0 \Rightarrow x = \pm 1$$

So, critical points: $x = -1$, $x = 1$

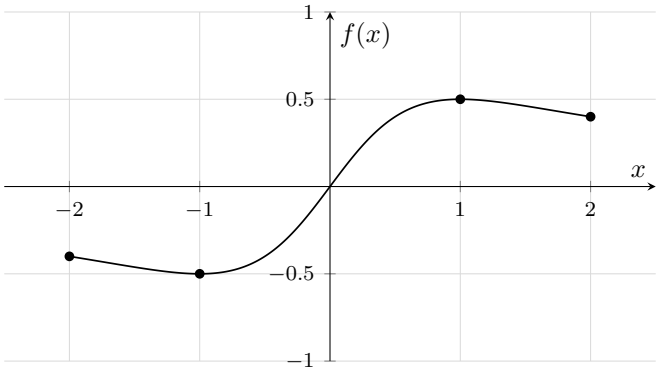
(b) Evaluate at endpoints and critical points:

$$f(-2) = \frac{-2}{4+1} = -\frac{2}{5}, \quad f(-1) = \frac{-1}{1+1} = -\frac{1}{2}$$
$$f(1) = \frac{1}{2}, \quad f(2) = \frac{2}{5}$$

(c) Conclusion:

- Absolute minimum: $\boxed{f(-1) = -\frac{1}{2}}$
- Absolute maximum: $\boxed{f(1) = \frac{1}{2}}$

Graph of $f(x) = \frac{x}{x^2+1}$ on $[-2, 2]$:



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Question 4

Problem:

Find the absolute maximum and minimum values of the function

$$f(x) = x^2 \ln(x)$$

on the interval $[1, 4]$.

(a) Find the critical points of $f(x)$ in the interval.

(b) Evaluate $f(x)$ at the endpoints and critical points.

(c) Determine the absolute maximum and minimum values.

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Question 4 - Solution

We are given:

$$f(x) = x^2 \ln(x)$$

(a) Find critical points:
Use product rule:

$$f'(x) = \frac{d}{dx}[x^2] \cdot \ln(x) + x^2 \cdot \frac{d}{dx}[\ln(x)] = 2x \ln(x) + x$$

Set derivative to zero:

$$f'(x) = 2x \ln(x) + x = x(2 \ln(x) + 1) = 0 \Rightarrow 2 \ln(x) + 1 = 0 \Rightarrow \ln(x) = -\frac{1}{2} \Rightarrow x = e^{-1/2}$$

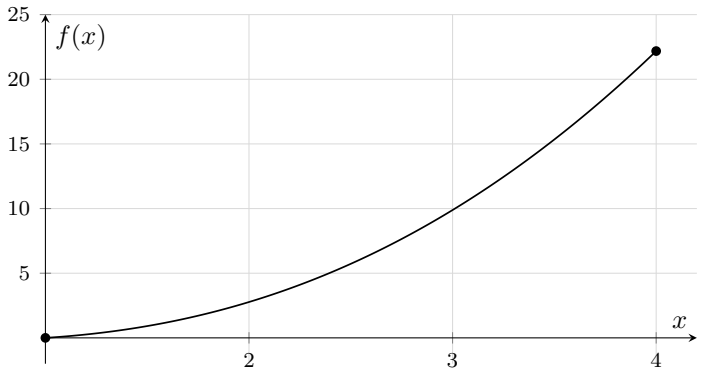
Since $e^{-1/2} \approx 0.606 \notin [1, 4]$, there are no critical points in the interval.
(b) Evaluate at endpoints:

$$f(1) = 1^2 \ln(1) = 0, \quad f(4) = 16 \ln(4) = 16 \cdot \ln(2^2) = 16 \cdot 2 \ln(2) \approx 16 \cdot 1.386 = 22.18$$

(c) Conclusion:

- Absolute minimum: $\boxed{f(1) = 0}$
- Absolute maximum: $\boxed{f(4) \approx 22.18}$

Graph of $f(x) = x^2 \ln(x)$ on $[1, 4]$:



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Question 5

Problem:

Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x + 1$$

on the interval $[-2, 2]$.

(a) Find the critical points of $f(x)$ in the interval.

(b) Evaluate $f(x)$ at the endpoints and critical points.

(c) Identify the absolute maximum and minimum values.

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Question 5 - Solution

We are given:

$$f(x) = x^3 - 3x + 1$$

(a) Find critical points:

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x - 1)(x + 1)$$

Set derivative to zero:

$$f'(x) = 0 \Rightarrow x = \pm 1$$

Critical points: $x = -1$, $x = 1$

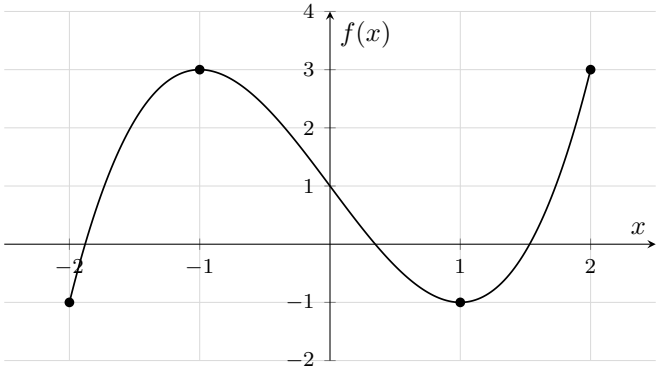
(b) Evaluate function:

$$f(-2) = (-2)^3 - 3(-2) + 1 = -8 + 6 + 1 = -1$$
$$f(-1) = (-1)^3 - 3(-1) + 1 = -1 + 3 + 1 = 3$$
$$f(1) = 1 - 3 + 1 = -1$$
$$f(2) = 8 - 6 + 1 = 3$$

(c) Conclusion:

- Absolute maximum: $\boxed{f(-1) = f(2) = 3}$
- Absolute minimum: $\boxed{f(-2) = f(1) = -1}$

Graph of $f(x) = x^3 - 3x + 1$ on $[-2, 2]$:



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Question 6

Problem:

Let $f(x) = x^4 - 4x^2$.

(a) Find the critical points of $f(x)$ in the interval $[-3, 3]$.

(b) Evaluate the function at the critical points and endpoints.

(c) Identify the absolute maximum and minimum values of $f(x)$ on the given interval.

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Question 6 - Solution

We are given:

$$f(x) = x^4 - 4x^2$$

(a) Find critical points:

$$f'(x) = 4x^3 - 8x = 4x(x^2 - 2)$$

Set derivative to zero:

$$f'(x) = 0 \Rightarrow x = 0, \pm\sqrt{2}$$

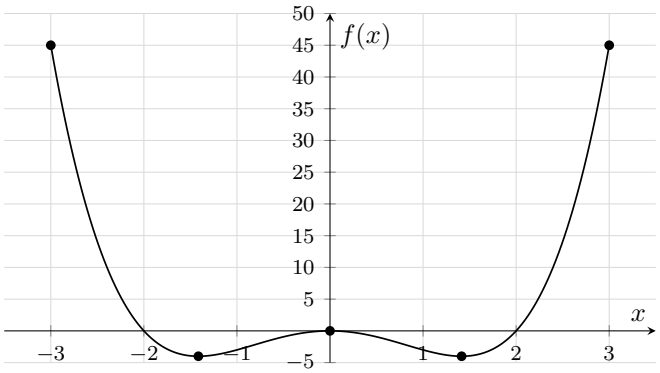
(b) Evaluate at critical points and endpoints:

$$f(-3) = (-3)^4 - 4(-3)^2 = 81 - 36 = 45$$
$$f(-\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2 = 4 - 8 = -4$$
$$f(0) = 0^4 - 4(0)^2 = 0$$
$$f(\sqrt{2}) = \text{same as } f(-\sqrt{2}) = -4$$
$$f(3) = 81 - 36 = 45$$

(c) Conclusion:

- Absolute maximum: $\boxed{f(-3) = f(3) = 45}$
- Absolute minimum: $\boxed{f(\pm\sqrt{2}) = -4}$

Graph of $f(x) = x^4 - 4x^2$ on $[-3, 3]$:



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Question 7

Problem:

A company models its daily profit (in hundreds of dollars) as a function of the number of units produced, using the function:

$$P(x) = -x^3 + 6x^2 + 15$$

where x is the number of units (in hundreds) produced per day, and $x \in [0, 5]$.

(a) Find all critical points of $P(x)$ on the interval.

(b) Determine the absolute maximum and minimum profit on the interval $[0, 5]$.

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Question 7 - Solution

We are given:

$$P(x) = -x^3 + 6x^2 + 15$$

(a) Find critical points:
Differentiate:

$$P'(x) = -3x^2 + 12x = -3x(x-4)$$

Set derivative equal to zero:

$$-3x(x-4) = 0 \Rightarrow x = 0,\ x = 4$$

These are critical points in the interval $[0, 5]$.

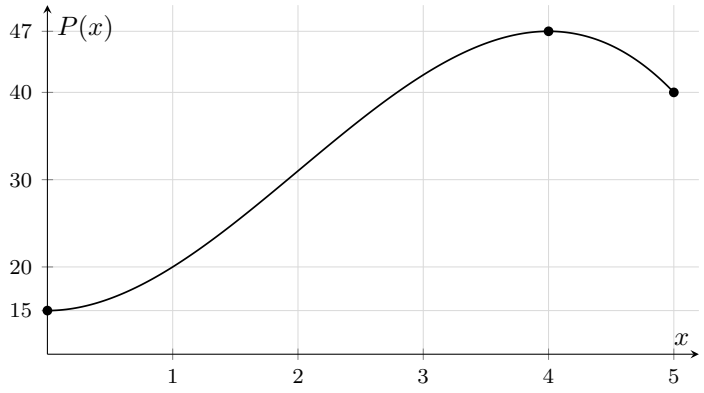
(b) Evaluate endpoints and critical points:

$$P(0) = -0 + 0 + 15 = 15$$
$$P(4) = -64 + 96 + 15 = 47$$
$$P(5) = -125 + 150 + 15 = 40$$

Conclusion:

- Absolute Maximum: $\boxed{P(4) = 47}$
- Absolute Minimum: $\boxed{P(0) = 15}$

Graph of $P(x) = -x^3 + 6x^2 + 15$ on $[0, 5]$:



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Question 8

Problem:

Let $f(x) = \frac{x}{x^2+1}$.

(a) Find all critical points of $f(x)$ on the interval $[-3, 3]$.

(b) Evaluate $f(x)$ at the critical points and endpoints.

(c) Determine the absolute maximum and minimum values of $f(x)$ on the given interval.

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Question 8 - Solution

We are given:

$$f(x) = \frac{x}{x^2+1}$$

(a) Find critical points:
Use the quotient rule:

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

Set numerator to 0:

$$-x^2+1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

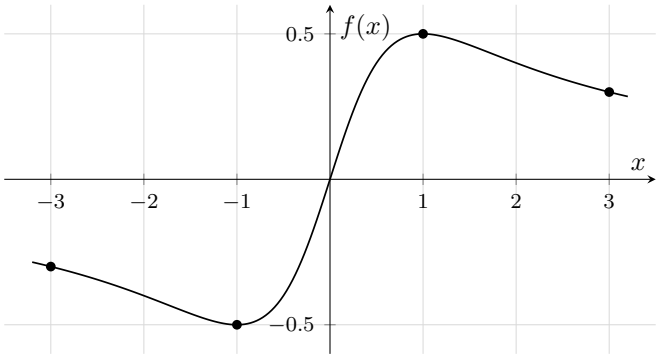
(b) Evaluate at endpoints and critical points:

$$f(-3) = \frac{-3}{9+1} = -\frac{3}{10}$$
$$f(-1) = \frac{-1}{1+1} = -\frac{1}{2}$$
$$f(1) = \frac{1}{1+1} = \frac{1}{2}$$
$$f(3) = \frac{3}{9+1} = \frac{3}{10}$$

(c) Conclusion:

- Absolute maximum: $\boxed{f(1) = \frac{1}{2}}$
- Absolute minimum: $\boxed{f(-1) = -\frac{1}{2}}$

Graph of $f(x) = \frac{x}{x^2+1}$ on $[-3, 3]$:



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Question 9

Problem:

Let $f(x) = x^3 - 3x^2 + 1$.

(a) Find the critical points of $f(x)$ on the interval $[0, 3]$.

(b) Evaluate the function at the critical points and endpoints.

(c) Determine the absolute maximum and minimum values of $f(x)$ on the interval.

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Question 9 - Solution

We are given:

$$f(x) = x^3 - 3x^2 + 1$$

(a) Find critical points:

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

Set derivative to zero:

$$3x(x-2) = 0 \Rightarrow x = 0,\ x = 2$$

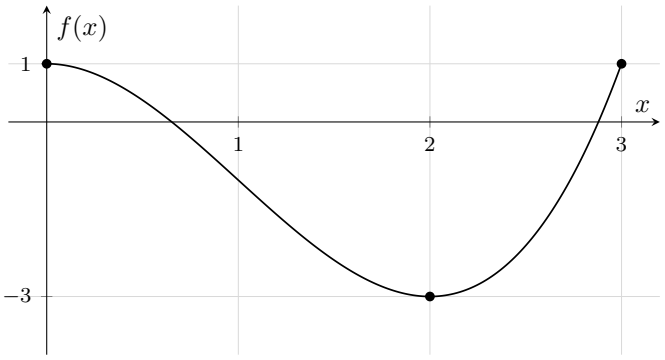
(b) Evaluate $f(x)$ at critical points and endpoints:

$$f(0) = 0 - 0 + 1 = 1$$
$$f(2) = 8 - 12 + 1 = -3$$
$$f(3) = 27 - 27 + 1 = 1$$

(c) Absolute extrema:

- Absolute minimum: $\boxed{f(2) = -3}$
- Absolute maximum: $\boxed{f(0) = f(3) = 1}$

Graph of $f(x) = x^3 - 3x^2 + 1$ on $[0, 3]$:



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Question 10

Problem:

Let $f(x) = \frac{x}{x^2+1}$.

(a) Find the critical points of $f(x)$ on the interval $[-2, 2]$.

(b) Evaluate the function at the critical points and endpoints.

(c) Determine the absolute maximum and minimum values of $f(x)$ on the interval.

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Question 10 - Solution

We are given:

$$f(x) = \frac{x}{x^2+1}$$

(a) Find critical points:
Use quotient rule:

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

Set $f'(x) = 0$:

$$1 - x^2 = 0 \Rightarrow x = \pm 1$$

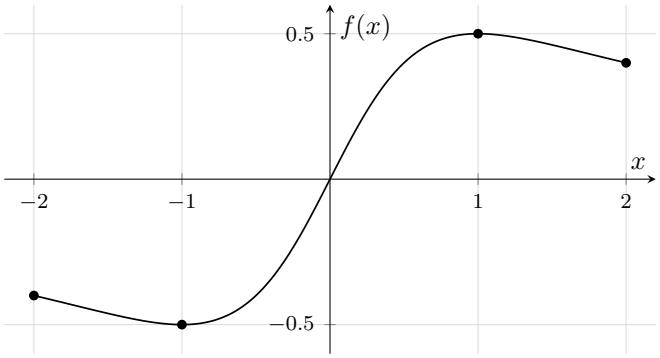
(b) Evaluate $f(x)$:

$$f(-2) = \frac{-2}{4+1} = -\frac{2}{5}, \quad f(-1) = \frac{-1}{1+1} = -\frac{1}{2}, \quad f(1) = \frac{1}{1+1} = \frac{1}{2}, \quad f(2) = \frac{2}{4+1} = \frac{2}{5}$$

(c) Absolute extrema:

- Absolute maximum: $\boxed{f(1) = \frac{1}{2}}$
- Absolute minimum: $\boxed{f(-1) = -\frac{1}{2}}$

Graph of $f(x) = \frac{x}{x^2+1}$ on $[-2, 2]$:



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Question 1

Let:

$$f(x) = x^4 - 4x^3 + 6x^2$$

(a) Find the intervals where $f(x)$ is concave up or concave down.
(b) Determine the inflection points of $f(x)$, if any.

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Question 1 - Solution

We are given:

$$f(x) = x^4 - 4x^3 + 6x^2$$

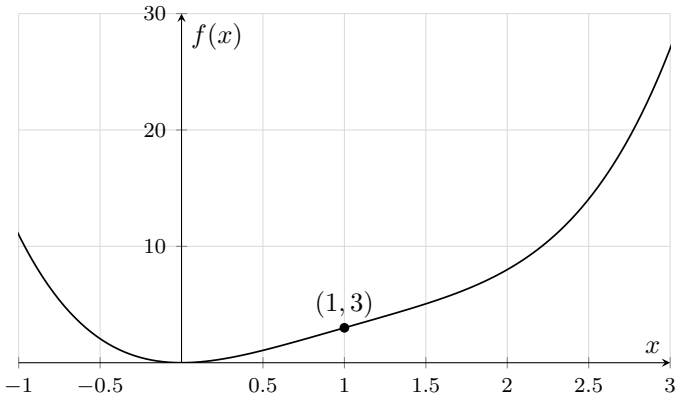
Step 1: Find the second derivative

$$f'(x) = 4x^3 - 12x^2 + 12x$$
$$f''(x) = 12x^2 - 24x + 12 = 12(x-1)^2$$

(a) Since $f''(x) = 12(x-1)^2 \geq 0$, the second derivative is never negative.
For all $x \neq 1$, $f''(x) > 0$, so the function is concave up.
At $x = 1$, the second derivative is zero, but the sign of $f''(x)$ does not change.
Answer
Concave up on $\boxed{(-\infty, \infty)}$

Concave down: $\boxed{\text{None}}$
(b) Because the concavity does not change at $x = 1$, there is no inflection point.
Answer
$\boxed{\text{No inflection points}}$

Graph of the function



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Question 2

Problem

Given the function

$$f(x) = x^3 - 3x^2 - 9x + 5$$

(a) Find the intervals where the function is increasing and decreasing.
(b) Identify and classify all local extrema.

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Question 2 - Solution

We are given

$$f(x) = x^3 - 3x^2 - 9x + 5$$

(a) First derivative

$$f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1)$$

The critical points are

$$x = -1 \quad \text{and} \quad x = 3$$

We test the sign of $f'(x)$.
For $x < -1$, $f'(x) > 0$. For $-1 < x < 3$, $f'(x) < 0$. For $x > 3$, $f'(x) > 0$.
Therefore, the function is increasing on $\boxed{(-\infty, -1) \cup (3, \infty)}$
and decreasing on $\boxed{(-1, 3)}$.

(b) Local extrema
Since the function changes from increasing to decreasing at $x = -1$, there is a local maximum:

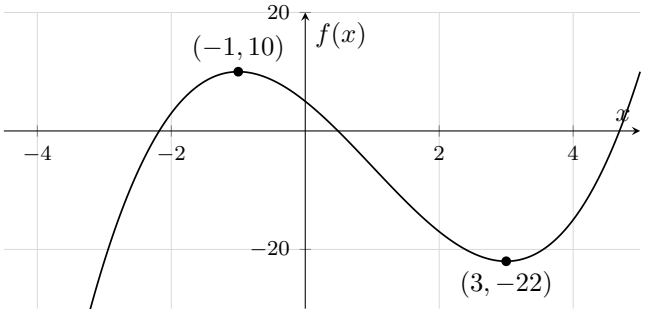
$$f(-1) = 10$$

Since the function changes from decreasing to increasing at $x = 3$, there is a local minimum:

$$f(3) = -22$$

Answer
Local maximum at $\boxed{(-1, 10)}$
Local minimum at $\boxed{(3, -22)}$

Graph of $f(x)$



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Question 3

Problem

Consider the function

$$f(x) = \frac{x^2 - 4x}{x + 1}$$

(a) Find the intervals where $f(x)$ is increasing or decreasing.
(b) Identify any local maximum or minimum points.

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Question 3 - Solution

We are given

$$f(x) = \frac{x^2 - 4x}{x+1} = \frac{x(x-4)}{x+1}, \qquad x \neq -1$$

(a) First derivative
Using the quotient rule,

$$f'(x) = \frac{(x+1)(2x-4) - (x^2-4x)}{(x+1)^2} = \frac{x^2+2x-4}{(x+1)^2}.$$

Setting the numerator equal to zero gives

$$x^2 + 2x - 4 = 0 \Rightarrow x = -1 \pm \sqrt{5}.$$

Since $(x+1)^2 > 0$ for $x \neq -1$, the sign of $f'(x)$ is determined by the numerator.
Answer
The function is increasing on

$$\boxed{(-\infty, -1-\sqrt{5}) \cup (-1+\sqrt{5}, \infty)}.$$

The function is decreasing on

$$\boxed{(-1-\sqrt{5}, -1) \cup (-1, -1+\sqrt{5})}.$$

(b) Local extrema
At $x = -1-\sqrt{5}$, the derivative changes from positive to negative, so there is a local maximum.
At $x = -1+\sqrt{5}$, the derivative changes from negative to positive, so there is a local minimum.

$$f(-1-\sqrt{5}) = -6-2\sqrt{5}, \qquad f(-1+\sqrt{5}) = -6+2\sqrt{5}.$$

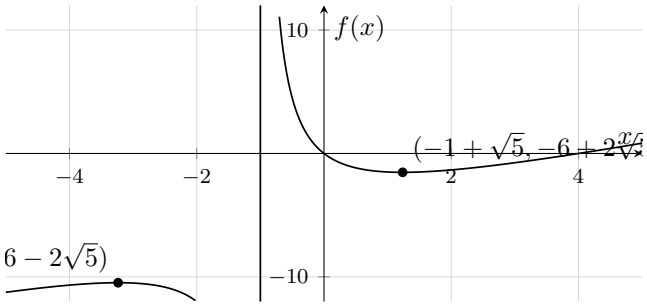
Answer
Local maximum at

$$\boxed{\left(-1-\sqrt{5}, -6-2\sqrt{5}\right)}$$

Local minimum at

$$\boxed{\left(-1+\sqrt{5}, -6+2\sqrt{5}\right)}$$

Graph of $f(x)$



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Question 4

Problem:

Let

$$f(x) = x^3 - 3x^2 - 9x + 5$$

(a) Find the intervals where $f(x)$ is increasing or decreasing.

(b) Identify any local maximum and minimum values.

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Question 4 - Solution

We are given:

$$f(x) = x^3 - 3x^2 - 9x + 5$$

(a) Find the first derivative:

$$f'(x) = 3x^2 - 6x - 9$$

Set $f'(x) = 0$ to find critical points:

$$3x^2 - 6x - 9 = 0 \quad \Rightarrow \quad x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0 \quad \Rightarrow \quad x = -1,\ 3$$

Test intervals for sign of $f'(x)$:
For $x < -1$: pick $x = -2$: $f'(-2) = 3(-2)^2 - 6(-2) - 9 = 12 + 12 - 9 = 15 > 0$
For $-1 < x < 3$: pick $x = 0$: $f'(0) = -9 < 0$
For $x > 3$: pick $x = 4$: $f'(4) = 3(16) - 6(4) - 9 = 48 - 24 - 9 = 15 > 0$
Conclusion:

$f(x)$ is increasing on $(-\infty, -1) \cup (3, \infty)$
$f(x)$ is decreasing on $(-1, 3)$

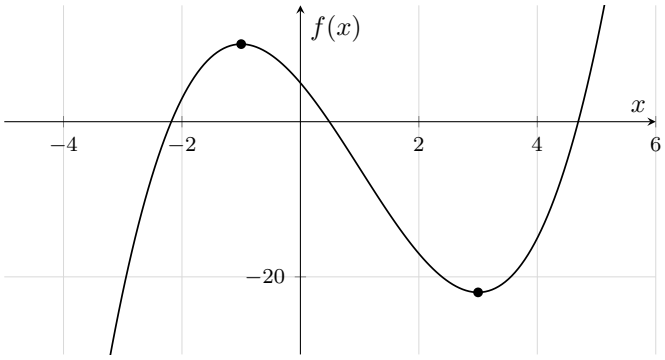
(b) Local Extrema:
At $x = -1$: changes from increasing to decreasing $\rightarrow$ local maximum At $x = 3$: changes from decreasing to increasing $\rightarrow$ local minimum
Evaluate $f(x)$ at critical points:

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = -1 - 3 + 9 + 5 = 10$$

$$f(3) = 27 - 27 - 27 + 5 = -22$$

Local Maximum: $\boxed{(-1,\ 10)}$
Local Minimum: $\boxed{(3,\ -22)}$

Graph of $f(x) = x^3 - 3x^2 - 9x + 5$:



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Question 5

Problem:

Let

$$f(x) = x^4 - 4x^3 + 6$$

(a) Find the intervals where $f(x)$ is increasing or decreasing.

(b) Identify any local maximum and minimum values.

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Question 5 - Solution

We are given:

$$f(x) = x^4 - 4x^3 + 6$$

(a) First Derivative:

$$f'(x) = 4x^3 - 12x^2$$

$$f'(x) = 4x^2(x-3)$$

Set $f'(x) = 0$:

$$4x^2(x-3) = 0 \quad \Rightarrow \quad x = 0,\ 3$$

Test intervals:
For $x < 0$: pick $x = -1$: $f'(-1) = 4(-1)^2(-1-3) = 4(1)(-4) = -16 < 0$
For $0 < x < 3$: pick $x = 1$: $f'(1) = 4(1)^2(1-3) = 4(-2) = -8 < 0$
For $x > 3$: pick $x = 4$: $f'(4) = 4(16)(1) = 64 > 0$
Conclusion:

$f(x)$ is decreasing on $(-\infty, 3)$

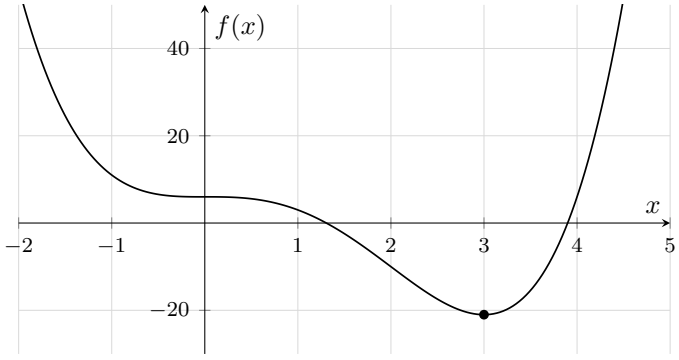
$f(x)$ is increasing on $(3, \infty)$

(b) Local Extrema:
$f'(x)$ does not change sign around $x = 0$ (negative on both sides) $\rightarrow$ no extremum at $x = 0$ At $x = 3$: from decreasing to increasing $\rightarrow$ local minimum
Evaluate:

$$f(3) = 3^4 - 4(3)^3 + 6 = 81 - 108 + 6 = -21$$

Local Minimum: $\boxed{(3,\ -21)}$

Graph of $f(x) = x^4 - 4x^3 + 6$:



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Question 6

Problem:

Let

$$f(x) = \frac{x^2}{x+1}$$

(a) Find the critical points of $f(x)$ and determine where $f(x)$ is increasing or decreasing.

(b) Identify any local maximum or minimum values.

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Question 6 - Solution

We are given:

$$f(x) = \frac{x^2}{x+1}$$

(a) First Derivative:
Use the quotient rule:

$$f'(x) = \frac{(2x)(x+1) - x^2(1)}{(x+1)^2} = \frac{2x(x+1) - x^2}{(x+1)^2} = \frac{2x^2 + 2x - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}$$

Set $f'(x) = 0$:

$$\frac{x(x+2)}{(x+1)^2} = 0 \Rightarrow x = 0,\ -2$$

Also, note $f(x)$ is undefined at $x = -1$

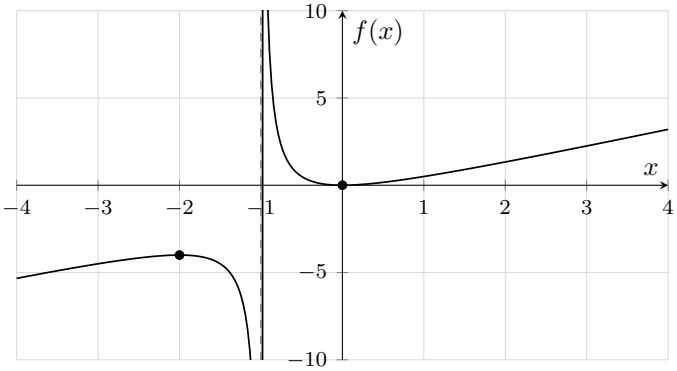
Test intervals:
$(-\infty, -2)$: pick $x = -3$: numerator is positive, denominator is positive $\rightarrow f' > 0$
$(-2, -1)$: pick $x = -1.5$: numerator is negative, denominator is positive $\rightarrow f' < 0$
$(-1, 0)$: pick $x = -0.5$: numerator is negative, denominator is positive $\rightarrow f' < 0$
$(0, \infty)$: pick $x = 1$: numerator is positive, denominator is positive $\rightarrow f' > 0$
Conclusion:

$$f(x) \text{ increases on } (-\infty, -2) \cup (0, \infty)$$
$$f(x) \text{ decreases on } (-2, -1) \cup (-1, 0)$$

(b) Local Extrema:
At $x = -2$: from increasing to decreasing $\rightarrow$ local max $f(-2) = \frac{(-2)^2}{-2+1} = \frac{4}{-1} = -4$
At $x = 0$: from decreasing to increasing $\rightarrow$ local min $f(0) = 0$
Answer:

$$\text{Local maximum at } (-2, -4), \quad \text{Local minimum at } (0, 0)$$

Graph of $f(x) = \frac{x^2}{x+1}$:



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Question 7

Problem:

Let

$$f(x) = x^3 - 3x^2 - 9x + 5$$

(a) Find the critical points of $f(x)$ and determine the intervals where $f(x)$ is increasing or decreasing.

(b) Identify all local maximum and minimum values.

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Question 7 - Solution

We are given:

$$f(x) = x^3 - 3x^2 - 9x + 5$$

(a) First Derivative:

$$f'(x) = 3x^2 - 6x - 9$$

Set $f'(x) = 0$ to find critical points:

$$3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1) = 0 \Rightarrow x = -1,\ 3$$

Test Intervals:
$(-\infty, -1)$: pick $x = -2$: $f'(-2) = 3(-2)^2 - 6(-2) - 9 = 12 + 12 - 9 = 15 > 0$
$(-1, 3)$: pick $x = 0$: $f'(0) = -9 < 0$
$(3, \infty)$: pick $x = 4$: $f'(4) = 3(16) - 6(4) - 9 = 48 - 24 - 9 = 15 > 0$

Conclusion:

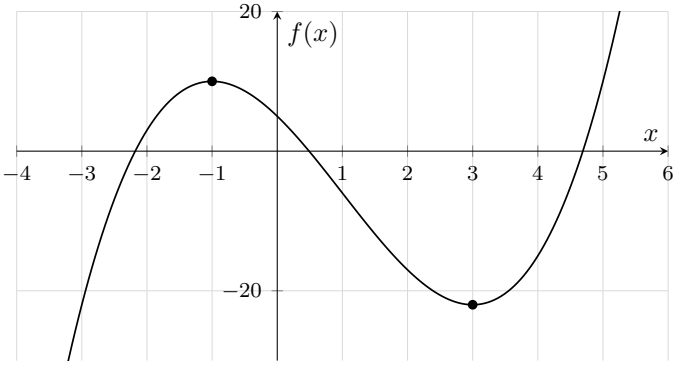
$f(x)$ is increasing on $(-\infty, -1) \cup (3, \infty)$
$f(x)$ is decreasing on $(-1, 3)$

(b) Local Extrema:
At $x = -1$: changes from increasing to decreasing $\rightarrow$ local maximum
$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = -1 - 3 + 9 + 5 = 10$
At $x = 3$: changes from decreasing to increasing $\rightarrow$ local minimum
$f(3) = 27 - 27 - 27 + 5 = -22$

Answer:

Local maximum at $(-1, 10)$, Local minimum at $(3, -22)$

Graph of $f(x) = x^3 - 3x^2 - 9x + 5$:



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Question 8

Problem:

Let

$$f(x) = \frac{x^2}{x+2}$$

(a) Find the domain of $f(x)$ and determine all critical points.

(b) Identify intervals where $f(x)$ is increasing or decreasing.

(c) Classify each critical point as a local maximum, local minimum, or neither.

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Question 8 - Solution

We are given:

$$f(x) = \frac{x^2}{x+2}$$

(a) Domain and Critical Points:
The function is undefined at $x = -2$, so the domain is:

$$\boxed{(-\infty, -2) \cup (-2, \infty)}$$

Find the derivative using the quotient rule:

$$f'(x) = \frac{(2x)(x+2) - x^2(1)}{(x+2)^2} = \frac{2x(x+2) - x^2}{(x+2)^2} = \frac{2x^2 + 4x - x^2}{(x+2)^2} = \frac{x^2 + 4x}{(x+2)^2}$$

Set the derivative to zero:

$$f'(x) = 0 \Rightarrow x^2 + 4x = 0 \Rightarrow x(x+4) = 0 \Rightarrow x = 0,\ x = -4$$

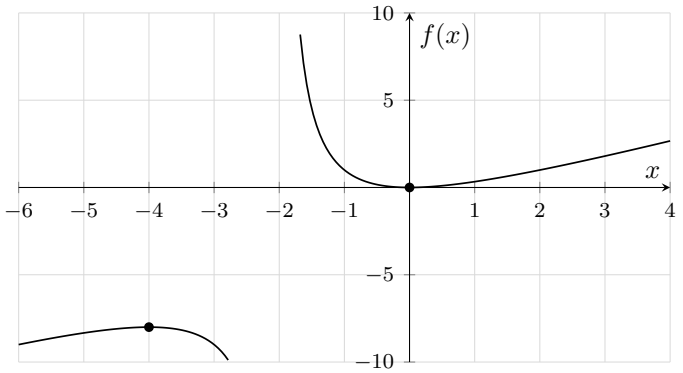
Critical Points: $x = -4$ and $x = 0$

(b) Test Intervals for Increasing/Decreasing:
Test sign of $f'(x) = \frac{x(x+4)}{(x+2)^2}$
Interval $(-\infty, -4)$: pick $x = -5$: $f'(-5) = \frac{-5(-1)}{(-3)^2} = \frac{5}{9} > 0$
Interval $(-4, -2)$: pick $x = -3$: $f'(-3) = \frac{-3(1)}{(1)^2} = -3 < 0$
Interval $(-2, 0)$: pick $x = -1$: $f'(-1) = \frac{-1(3)}{(1)^2} = -3 < 0$
Interval $(0, \infty)$: pick $x = 1$: $f'(1) = \frac{1(5)}{(3)^2} = \frac{5}{9} > 0$
Increasing: $(-\infty, -4) \cup (0, \infty)$ **Decreasing:** $(-4, -2) \cup (-2, 0)$

(c) Classify Critical Points:
At $x = -4$: changes from increasing to decreasing $\rightarrow$ local max $f(-4) = \frac{16}{-2} = -8$
At $x = 0$: changes from decreasing to increasing $\rightarrow$ local min $f(0) = 0$
Conclusion:

$$\text{Local maximum at } (-4, -8), \quad \text{Local minimum at } (0, 0)$$

Graph of $f(x) = \frac{x^2}{x+2}$:



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Question 9

Problem:

Let

$$f(x) = x^3 - 6x^2 + 9x + 1$$

(a) Find all critical points of $f(x)$.

(b) Determine the intervals where $f(x)$ is increasing or decreasing.

(c) Identify and classify all local extrema.

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Question 9 - Solution

We are given:

$$f(x) = x^3 - 6x^2 + 9x + 1$$

(a) Critical Points:
Compute the first derivative:

$$f'(x) = 3x^2 - 12x + 9$$

Set derivative equal to 0:

$$3x^2 - 12x + 9 = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow x = 1,\ x = 3$$

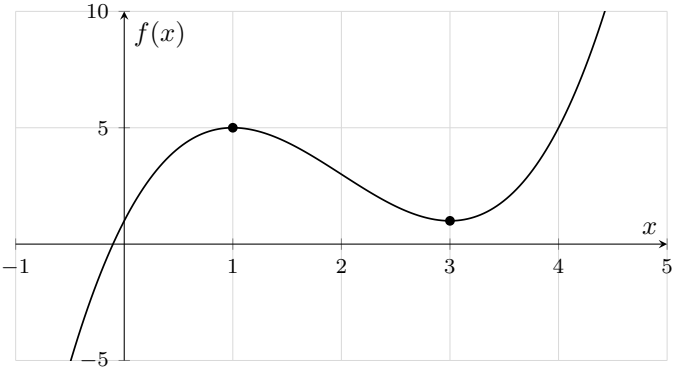
Critical points: $x = 1,\ x = 3$

(b) Increasing/Decreasing Intervals:
Test sign of $f'(x)$ in each interval:
$(-\infty, 1)$: pick $x = 0$, $f'(0) = 9 > 0$
$(1, 3)$: pick $x = 2$, $f'(2) = 3(4) - 12(2) + 9 = 12 - 24 + 9 = -3 < 0$
$(3, \infty)$: pick $x = 4$, $f'(4) = 3(16) - 48 + 9 = 48 - 48 + 9 = 9 > 0$
Increasing: $(-\infty, 1) \cup (3, \infty)$ **Decreasing:** $(1, 3)$

(c) Local Extrema:
At $x = 1$: increasing to decreasing $\rightarrow$ local max $f(1) = 1 - 6 + 9 + 1 = 5$
At $x = 3$: decreasing to increasing $\rightarrow$ local min $f(3) = 27 - 54 + 27 + 1 = 1$
Conclusion:

$$\text{Local maximum at } (1, 5), \quad \text{Local minimum at } (3, 1)$$

Graph of $f(x) = x^3 - 6x^2 + 9x + 1$:



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Question 10

Problem:

Let

$$f(x) = \frac{x}{x^2 + 1}$$

(a) Find the critical points of $f(x)$.

(b) Determine where the function is increasing and decreasing.

(c) Identify and classify any local extrema.

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Question 10 - Solution

We are given:

$$f(x) = \frac{x}{x^2+1}$$

(a) Critical Points:
Use the quotient rule:

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

Set numerator equal to zero:

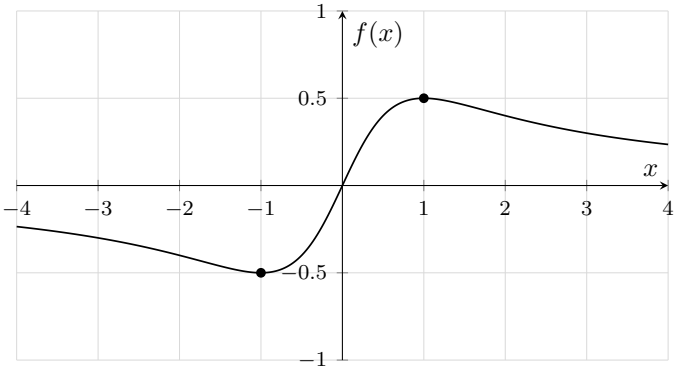
$$-x^2+1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

Critical points: $x = -1$, $x = 1$

(b) Increasing/Decreasing Intervals:
Analyze the sign of $f'(x) = \frac{1-x^2}{(x^2+1)^2}$
Denominator is always positive. So sign depends on numerator $1 - x^2$:
$x < -1$: $f'(x) < 0$ $-1 < x < 1$: $f'(x) > 0$ $x > 1$: $f'(x) < 0$
Increasing: $(-1, 1)$ **Decreasing:** $(-\infty, -1) \cup (1, \infty)$

(c) Local Extrema:
$f'(x)$ changes from negative to positive at $x = -1 \rightarrow$ local min $f(-1) = \frac{-1}{1+1} = -\frac{1}{2}$
$f'(x)$ changes from positive to negative at $x = 1 \rightarrow$ local max $f(1) = \frac{1}{1+1} = \frac{1}{2}$
Conclusion: Local minimum at $(-1, -\frac{1}{2})$, Local maximum at $(1, \frac{1}{2})$

Graph of $f(x) = \frac{x}{x^2+1}$:



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Question 1

Let

$$f(x) = x^3 - 3x^2 - 9x + 5$$

(a) Find the intervals where $f(x)$ is increasing or decreasing.
(b) Identify all local maxima and minima.
(c) Determine intervals of concavity and any inflection points.

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Question 1 - Solution

We are given

$$f(x) = x^3 - 3x^2 - 9x + 5$$

(a) First derivative

$$f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1)$$

The critical points are $x = -1$ and $x = 3$.

For $x < -1$, the derivative is positive. For $-1 < x < 3$, the derivative is negative. For $x > 3$, the derivative is positive.

Therefore, the function is increasing on

$$\boxed{(-\infty, -1) \cup (3, \infty)}$$

and decreasing on

$$\boxed{(-1, 3)}.$$

(b) Local extrema

At $x = -1$, the function changes from increasing to decreasing, so there is a local maximum.

$$f(-1) = 10$$

At $x = 3$, the function changes from decreasing to increasing, so there is a local minimum.

$$f(3) = -22$$

Local maximum at $\boxed{(-1, 10)}$

Local minimum at $\boxed{(3, -22)}$

(c) Second derivative

$$f''(x) = 6x - 6$$

Setting $f''(x) = 0$ gives $x = 1$.

For $x < 1$, the second derivative is negative, so the graph is concave down. For $x > 1$, the second derivative is positive, so the graph is concave up.

The inflection point occurs at

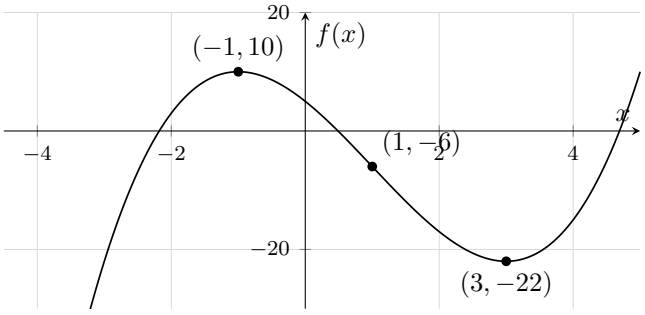
$$f(1) = -6$$

Concave down on $\boxed{(-\infty, 1)}$

Concave up on $\boxed{(1, \infty)}$

Inflection point at $\boxed{(1, -6)}$

Graph of $f(x)$



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Question 2

Let

$$f(x) = x^4 - 4x^3$$

(a) Find the intervals where the graph of f is concave up and concave down.
(b) Identify any inflection points.

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Question 2 - Solution

We are given

$$f(x) = x^4 - 4x^3$$

First derivative

$$f'(x) = 4x^3 - 12x^2$$

Second derivative

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

(a) Concavity
Setting $f''(x) = 0$ gives the critical values

$$x = 0 \quad \text{and} \quad x = 2.$$

For $x < 0$, the second derivative is positive, so the graph is concave up. For $0 < x < 2$, the second derivative is negative, so the graph is concave down. For $x > 2$, the second derivative is positive, so the graph is concave up.
Concave up on

$$\boxed{(-\infty, 0) \cup (2, \infty)}$$

Concave down on

$$\boxed{(0, 2)}$$

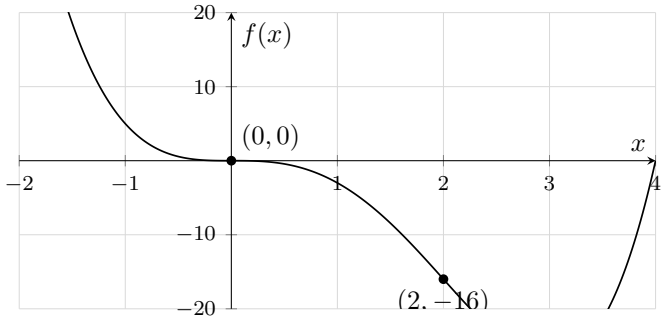
(b) Inflection points
Because the concavity changes at both $x = 0$ and $x = 2$, inflection points occur at these values.

$$f(0) = 0 \qquad f(2) = 16 - 32 = -16$$

Inflection points at

$$\boxed{(0, 0) \quad \text{and} \quad (2, -16)}$$

Graph of $f(x)$



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Question 3

Let

$$f(x) = \frac{x^2}{x^2 + 1}$$

(a) Determine the intervals where the graph of f is concave up and concave down.
(b) Identify all inflection points.

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Question 3 - Solution

We are given

$$f(x) = \frac{x^2}{x^2+1}$$

First derivative
Using the quotient rule,

$$f'(x) = \frac{(2x)(x^2+1) - x^2(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}.$$

Second derivative
Differentiating $f'(x)$ gives

$$f''(x) = \frac{2(1-3x^2)}{(x^2+1)^3}.$$

(a) Concavity
Setting $f''(x) = 0$ yields

$$1 - 3x^2 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}}.$$

For $|x| < \frac{1}{\sqrt{3}}$, the second derivative is positive, so the graph is concave up.
For $|x| > \frac{1}{\sqrt{3}}$, the second derivative is negative, so the graph is concave down.
Concave up on

$$\boxed{\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$$

Concave down on

$$\boxed{\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)}$$

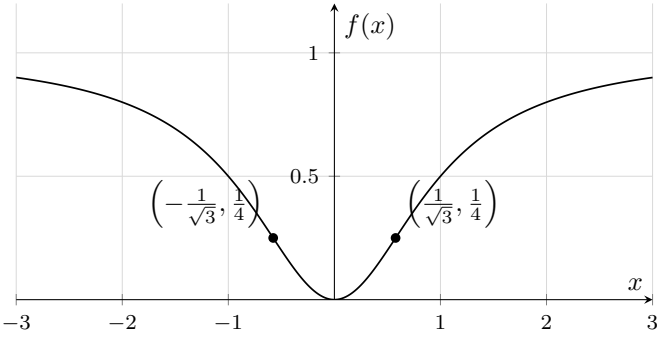
(b) Inflection points
Because the concavity changes at $x = \pm\frac{1}{\sqrt{3}}$, inflection points occur there.

$$f\left(\pm\frac{1}{\sqrt{3}}\right) = \frac{1}{4}$$

Inflection points at

$$\boxed{\left(-\frac{1}{\sqrt{3}}, \frac{1}{4}\right) \quad \text{and} \quad \left(\frac{1}{\sqrt{3}}, \frac{1}{4}\right)}$$

Graph of $f(x)$



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Question 4

Let

$$f(x) = x^3 - 3x$$

(a) Determine the intervals where the graph of f is concave up and concave down.
(b) Identify all inflection points.

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Question 4 - Solution

We are given

$$f(x) = x^3 - 3x$$

First derivative

$$f'(x) = 3x^2 - 3$$

Second derivative

$$f''(x) = 6x$$

(a) Concavity
Setting $f''(x) = 0$ gives

$$x = 0.$$

For $x < 0$, the second derivative is negative, so the graph is concave down.
For $x > 0$, the second derivative is positive, so the graph is concave up.
Concave down on

$$\boxed{(-\infty, 0)}$$

Concave up on

$$\boxed{(0, \infty)}$$

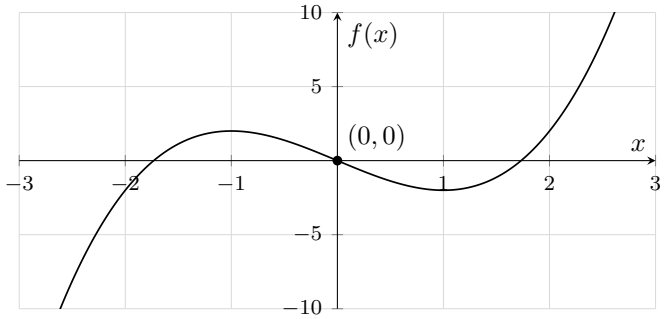
(b) Inflection point
Because the concavity changes at $x = 0$, an inflection point occurs there.

$$f(0) = 0$$

Inflection point at

$$\boxed{(0, 0)}$$

Graph of $f(x)$



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Question 5

Let

$$f(x) = \frac{x}{x^2 + 1}$$

(a) Determine the intervals where the graph of f is concave up and concave down.
(b) Identify all inflection points.

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Question 5 - Solution

We are given

$$f(x) = \frac{x}{x^2+1}$$

First derivative
Using the quotient rule,

$$f'(x) = \frac{(x^2+1) - 2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$$

Second derivative
Differentiating $f'(x)$,

$$f''(x) = \frac{-2x(3-x^2)}{(x^2+1)^3}.$$

(a) Concavity
Setting $f''(x) = 0$ gives

$$x = -\sqrt{3}, \quad x = 0, \quad x = \sqrt{3}.$$

For $x < -\sqrt{3}$, the second derivative is negative, so the graph is concave down.
For $-\sqrt{3} < x < 0$, the second derivative is positive, so the graph is concave up.
For $0 < x < \sqrt{3}$, the second derivative is negative, so the graph is concave down.
For $x > \sqrt{3}$, the second derivative is positive, so the graph is concave up.
Concave up on

$$\boxed{(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)}$$

Concave down on

$$\boxed{(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})}$$

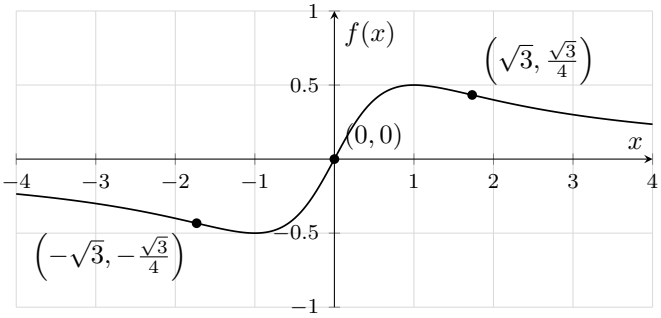
(b) Inflection points
Because the concavity changes at each critical value, inflection points occur at all three values.

$$f(0) = 0, \qquad f(\pm\sqrt{3}) = \pm\frac{\sqrt{3}}{4}$$

Inflection points at

$$\boxed{\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), \quad (0,0), \quad \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)}$$

Graph of $f(x)$



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Question 6

Let

$$f(x) = x^4 - 4x^3 + 10$$

(a) Determine the intervals where the graph of f is concave up and concave down.
(b) Identify all inflection points.

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Question 6 - Solution

We are given

$$f(x) = x^4 - 4x^3 + 10$$

First derivative

$$f'(x) = 4x^3 - 12x^2$$

Second derivative

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

(a) Concavity
Setting $f''(x) = 0$ gives

$$x = 0 \quad \text{and} \quad x = 2.$$

For $x < 0$, the second derivative is positive, so the graph is concave up.
For $0 < x < 2$, the second derivative is negative, so the graph is concave down.
For $x > 2$, the second derivative is positive, so the graph is concave up.
Concave up on

$$\boxed{(-\infty, 0) \cup (2, \infty)}$$

Concave down on

$$\boxed{(0, 2)}$$

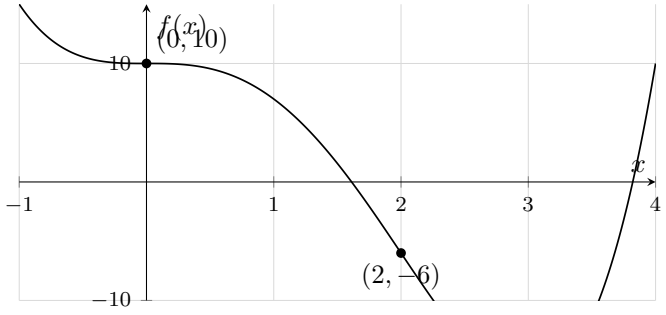
(b) Inflection points
Because the concavity changes at both values, inflection points occur at $x = 0$ and $x = 2$.

$$f(0) = 10, \qquad f(2) = -6$$

Inflection points at

$$\boxed{(0, 10) \quad \text{and} \quad (2, -6)}$$

Graph of $f(x)$



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Question 7

Let

$$f(x) = x^3 - 6x^2 + 12x - 5$$

(a) Determine the intervals where the graph of f is concave up and concave down.
(b) Find all inflection points.

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Question 7 - Solution

We are given

$$f(x) = x^3 - 6x^2 + 12x - 5$$

First derivative

$$f'(x) = 3x^2 - 12x + 12$$

Second derivative

$$f''(x) = 6x - 12$$

(a) Concavity
Setting $f''(x) = 0$ gives

$$x = 2.$$

For $x < 2$, the second derivative is negative, so the graph is concave down.
For $x > 2$, the second derivative is positive, so the graph is concave up.
Concave down on

$$\boxed{(-\infty, 2)}$$

Concave up on

$$\boxed{(2, \infty)}$$

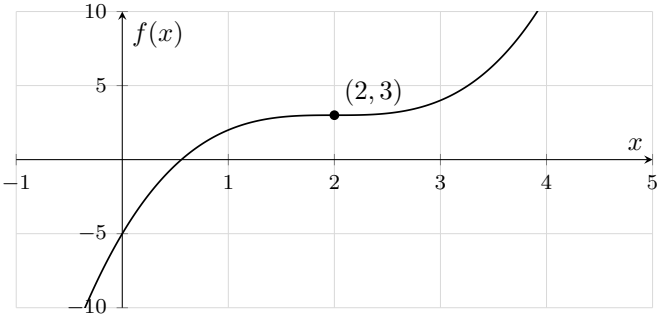
(b) Inflection point
Because the concavity changes at $x = 2$, an inflection point occurs there.

$$f(2) = 3$$

Inflection point at

$$\boxed{(2, 3)}$$

Graph of $f(x)$



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Question 8

Let

$$f(x) = \frac{x^2}{x^2 + 1}$$

(a) Determine the intervals where the graph of f is concave up and concave down.
(b) Find all inflection points, if any.

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Question 8 - Solution

We are given

$$f(x) = \frac{x^2}{x^2+1}$$

First derivative
Using the quotient rule,

$$f'(x) = \frac{(2x)(x^2+1) - x^2(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}.$$

Second derivative
Differentiating $f'(x)$,

$$f''(x) = \frac{2(1-3x^2)}{(x^2+1)^3}.$$

(a) Concavity
Setting $f''(x) = 0$ gives

$$1 - 3x^2 = 0 \quad \Rightarrow \quad x = \pm\frac{1}{\sqrt{3}}.$$

For $|x| < \frac{1}{\sqrt{3}}$, the second derivative is positive, so the graph is concave up.
For $|x| > \frac{1}{\sqrt{3}}$, the second derivative is negative, so the graph is concave down.
Concave up on

$$\boxed{\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$$

Concave down on

$$\boxed{\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)}$$

(b) Inflection points
Because the concavity changes at both values, inflection points occur at

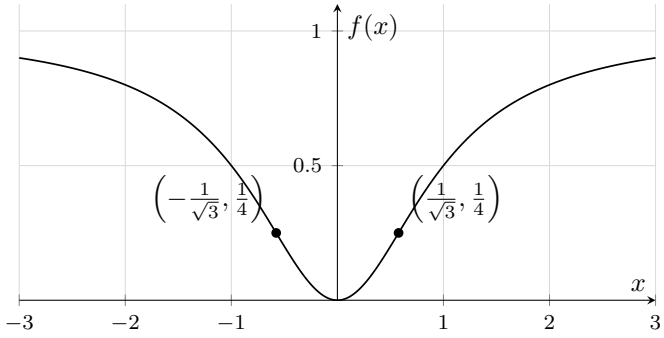
$$x = \pm\frac{1}{\sqrt{3}}.$$

$$f\left(\pm\frac{1}{\sqrt{3}}\right) = \frac{1}{4}$$

Inflection points at

$$\boxed{\left(-\frac{1}{\sqrt{3}}, \frac{1}{4}\right) \quad \text{and} \quad \left(\frac{1}{\sqrt{3}}, \frac{1}{4}\right)}$$

Graph of $f(x)$



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Question 9

Let

$$f(x) = x^3 - 3x$$

(a) Determine the intervals where the graph of f is concave up and concave down.
(b) Find all inflection points.

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Question 9 - Solution

We are given

$$f(x) = x^3 - 3x$$

First derivative

$$f'(x) = 3x^2 - 3$$

Second derivative

$$f''(x) = 6x$$

(a) Concavity
Setting $f''(x) = 0$ gives

$$x = 0.$$

For $x < 0$, the second derivative is negative, so the graph is concave down.
For $x > 0$, the second derivative is positive, so the graph is concave up.
Concave down on

$$\boxed{(-\infty, 0)}$$

Concave up on

$$\boxed{(0, \infty)}$$

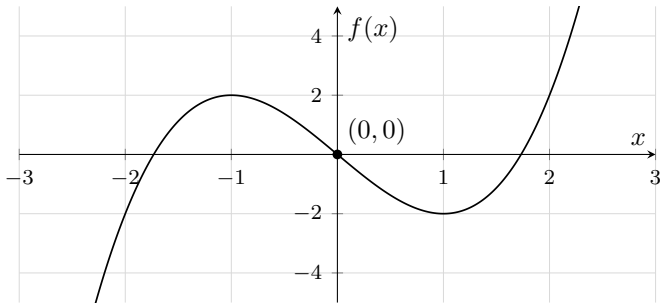
(b) Inflection point
Because the concavity changes at $x = 0$, an inflection point occurs there.

$$f(0) = 0$$

Inflection point at

$$\boxed{(0, 0)}$$

Graph of $f(x)$



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Question 10

Let

$$f(x) = \frac{x}{x^2 + 1}$$

(a) Determine the intervals where f is concave up and concave down.
(b) Find the inflection points of f, if any.

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Question 10 - Solution

We are given

$$f(x) = \frac{x}{x^2+1}$$

First derivative
Using the quotient rule,

$$f'(x) = \frac{(x^2+1) - 2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$$

Second derivative
Differentiating $f'(x)$,

$$f''(x) = \frac{-2x(3-x^2)}{(x^2+1)^3}.$$

(a) Concavity
Setting $f''(x) = 0$ gives

$$x = -\sqrt{3}, \quad x = 0, \quad x = \sqrt{3}.$$

For $x < -\sqrt{3}$, the second derivative is negative, so the graph is concave down.
For $-\sqrt{3} < x < 0$, the second derivative is positive, so the graph is concave up.
For $0 < x < \sqrt{3}$, the second derivative is negative, so the graph is concave down.
For $x > \sqrt{3}$, the second derivative is positive, so the graph is concave up.
Concave up on

$$\boxed{(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)}$$

Concave down on

$$\boxed{(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})}$$

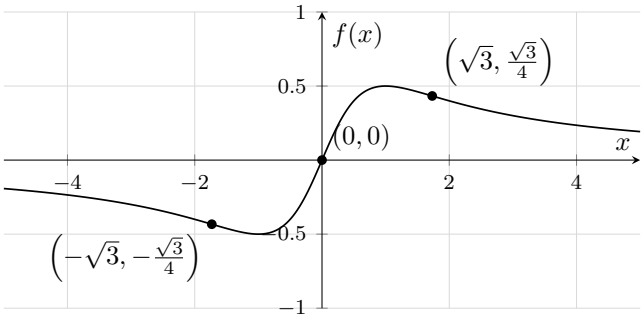
(b) Inflection points
Because the concavity changes at all three values, inflection points occur at each one.

$$f(0) = 0, \qquad f(\pm\sqrt{3}) = \pm\frac{\sqrt{3}}{4}$$

Inflection points at

$$\boxed{\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), \quad (0,0), \quad \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)}$$

Graph of $f(x)$



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Question 1

Let

$$f(x) = x^2 + 2x$$

(a) Verify that $f(x)$ satisfies the hypotheses of the Mean Value Theorem on the interval $[1, 4]$.
(b) Find all numbers $c \in (1, 4)$ that satisfy the conclusion of the Mean Value Theorem.

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Question 1 - Solution

We are given

$$f(x) = x^2 + 2x$$

(a) Verifying the hypotheses

The function $f(x)$ is a polynomial, so it is continuous on the closed interval $[1, 4]$ and differentiable on the open interval $(1, 4)$.

Therefore, the hypotheses of the Mean Value Theorem are satisfied.

(b) Applying the Mean Value Theorem

First compute the derivative:

$$f'(x) = 2x + 2$$

Next compute the average rate of change of f on $[1, 4]$:

$$\frac{f(4) - f(1)}{4 - 1} = \frac{(16 + 8) - (1 + 2)}{3} = \frac{21}{3} = 7$$

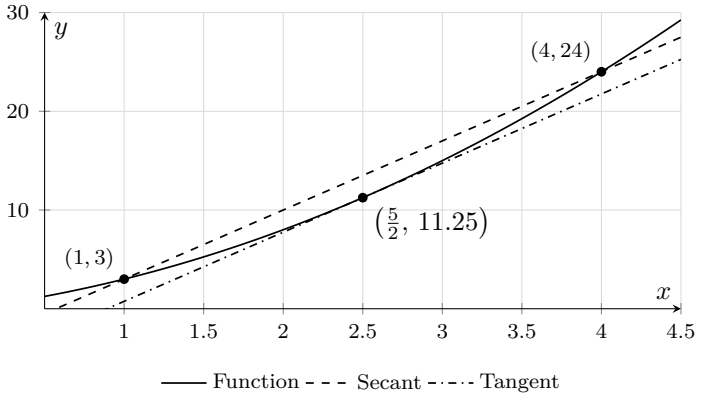
Set the derivative equal to this value:

$$2c + 2 = 7 \Rightarrow 2c = 5 \Rightarrow c = \frac{5}{2}$$

Thus, the number that satisfies the conclusion of the Mean Value Theorem is

$$\boxed{c = \frac{5}{2}}$$

Graph of $f(x)$, the secant line, and the tangent line at c



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Question 2

Problem:

Let

$$f(x) = \ln(x), \quad \text{for } x \in [1, 4]$$

(a) Verify that f satisfies the hypotheses of the Mean Value Theorem on the interval.

(b) Find all values of $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}$$

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Question 2 - Solution

We are given:

$$f(x) = \ln(x)$$

(a) Check MVT hypotheses:
1. $f(x)$ is continuous on $[1, 4]$: Yes, $\ln(x)$ is continuous for $x > 0$, so continuous on $[1, 4]$
2. $f(x)$ is differentiable on $(1, 4)$: Yes, $\ln(x)$ is differentiable for $x > 0$, so differentiable on $(1, 4)$

The Mean Value Theorem applies.

(b) Find the value(s) of c:
Compute average rate of change:

$$\frac{f(4) - f(1)}{4 - 1} = \frac{\ln(4) - \ln(1)}{3} = \frac{\ln(4)}{3} = \frac{2\ln(2)}{3}$$

Now compute $f'(x)$:

$$f'(x) = \frac{1}{x}$$

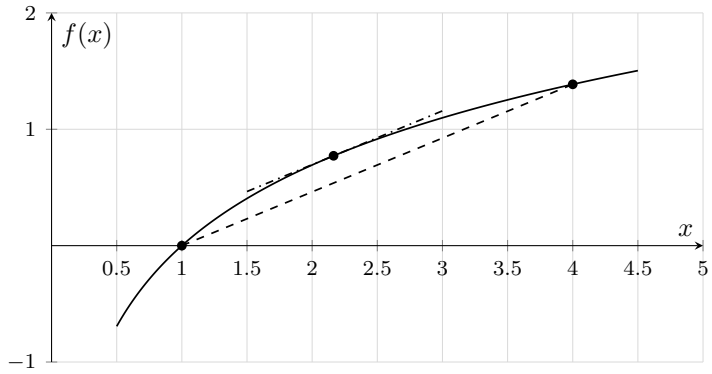
Set:

$$f'(c) = \frac{1}{c} = \frac{2\ln(2)}{3} \quad \Rightarrow \quad c = \frac{3}{2\ln(2)}$$

Answer:

$$\boxed{c = \frac{3}{2\ln(2)} \approx 2.164}$$

Graph of $f(x) = \ln(x)$ on $[1, 4]$:



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Question 2

Problem:

Let

$$f(x) = x^3 - 3x^2 + 2x, \quad \text{on the interval } [1, 3]$$

(a) Verify that f satisfies the hypotheses of the Mean Value Theorem on $[1, 3]$.

(b) Find all values of $c \in (1, 3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

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Question 2 - Solution

We are given:

$$f(x) = x^3 - 3x^2 + 2x$$

(a) Hypotheses of the MVT:
- $f(x)$ is a polynomial continuous on $[1,3]$ - $f(x)$ is differentiable on $(1,3)$
Therefore, MVT applies.

(b) Apply the MVT:
Compute:

$$f(3) = 27 - 27 + 6 = 6, \quad f(1) = 1 - 3 + 2 = 0$$

$$\text{Average rate of change} = \frac{6-0}{3-1} = \frac{6}{2} = 3$$

Compute derivative:

$$f'(x) = 3x^2 - 6x + 2$$

Set $f'(c) = 3$:

$$3c^2 - 6c + 2 = 3 \quad \Rightarrow \quad 3c^2 - 6c - 1 = 0$$

Solve:

$$c = \frac{6 \pm \sqrt{(-6)^2 - 4(3)(-1)}}{2(3)} = \frac{6 \pm \sqrt{36+12}}{6} = \frac{6 \pm \sqrt{48}}{6} = \frac{6 \pm 4\sqrt{3}}{6} = \frac{3 \pm 2\sqrt{3}}{3}$$

Answer:

$$\boxed{c = \frac{3 \pm 2\sqrt{3}}{3}}$$ Only the value(s) in $(1,3)$ are valid.

Since:

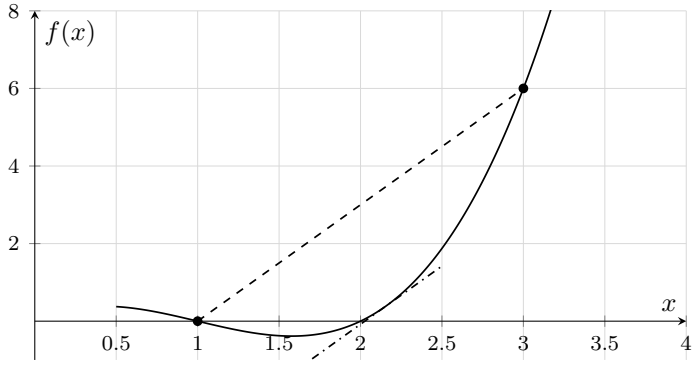
$$\frac{3 - 2\sqrt{3}}{3} \approx 0.845 < 1 \quad \text{(not valid)}$$

$$\frac{3 + 2\sqrt{3}}{3} \approx 2.155 \in (1,3)$$

Final answer:

$$\boxed{c = \frac{3 + 2\sqrt{3}}{3}}$$

Graph of $f(x) = x^3 - 3x^2 + 2x$ on $[1,3]$:



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Question 4

Problem:

Let

$$f(x) = \ln(x^2 + 1) \quad \text{on the interval } [-1, 2].$$

(a) Verify that f satisfies the hypotheses of the Mean Value Theorem on $[-1, 2]$.

(b) Find all values of $c \in (-1, 2)$ such that

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}.$$

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Question 4 - Solution

The function $f(x) = \ln(1 + x^2)$ is continuous on $[-1, 2]$ and differentiable on $(-1, 2)$, so the Mean Value Theorem applies.

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{\ln(5/2)}{3}, \qquad f'(c) = \frac{2c}{1 + c^2}.$$

Let $L = \ln(5/2)$. Then $Lc^2 - 6c + L = 0$, so

$$c = \frac{3 \pm \sqrt{9 - L^2}}{L}.$$

The plus root exceeds 2. The only admissible value is

$$\boxed{c = \frac{3 - \sqrt{9 - [\ln(5/2)]^2}}{\ln(5/2)} \approx 0.156453.}$$

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Question 5

Problem:

Let

$$f(x) = \frac{x}{x+2} \quad \text{on the interval } [1, 4].$$

(a) Verify that $f(x)$ satisfies the hypotheses of the Mean Value Theorem on $[1, 4]$.

(b) Find all values $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

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Question 5 - Solution

We are given:

$$f(x) = \frac{x}{x+2}$$

(a) Check MVT conditions:

- $f(x)$ is a rational function with denominator $x+2 \neq 0$ on $[1,4]$ continuous on $[1,4]$. - It is differentiable on $(1,4)$ as it's smooth where defined.

MVT applies.

(b) Compute average rate of change:

$$f(1) = \frac{1}{3}, \quad f(4) = \frac{4}{6} = \frac{2}{3}$$

$$\frac{f(4) - f(1)}{4-1} = \frac{\frac{2}{3} - \frac{1}{3}}{3} = \frac{1}{9}$$

Find derivative:

Use quotient rule:

$$f'(x) = \frac{(x+2)(1) - x(1)}{(x+2)^2} = \frac{x+2-x}{(x+2)^2} = \frac{2}{(x+2)^2}$$

Set:

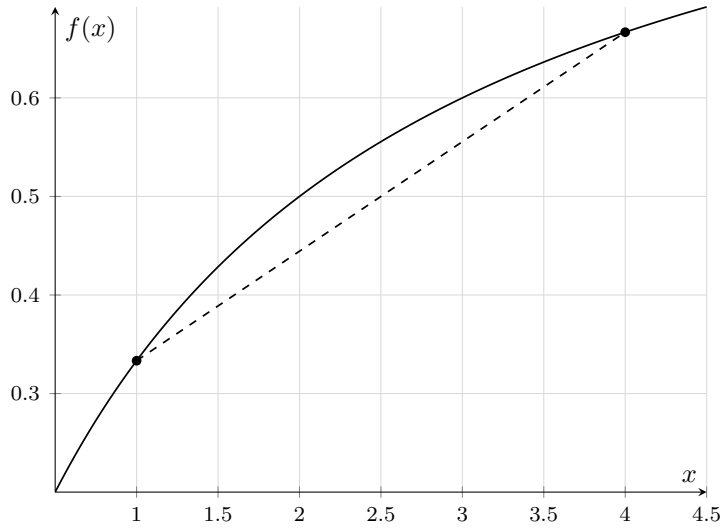
$$\frac{2}{(c+2)^2} = \frac{1}{9} \Rightarrow (c+2)^2 = 18 \Rightarrow c+2 = \pm\sqrt{18} = \pm 3\sqrt{2} \Rightarrow c = -2 \pm 3\sqrt{2}$$

Now, only one of these is in the interval $(1,4)$:

$$c = -2 + 3\sqrt{2} \approx -2 + 4.24 = 2.24$$

$$\boxed{c = -2 + 3\sqrt{2}} \text{ is in } (1,4)$$

Graph of $f(x)$ and secant line:



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Question 6

Problem:

Let

$$f(x) = \ln(x^2 + 1)$$

on the interval $[-1, 2]$.

(a) Verify that $f(x)$ satisfies the hypotheses of the Mean Value Theorem on the interval.

(b) Find all values of $c \in (-1, 2)$ such that

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}.$$

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Question 6 - Solution

The function $f(x) = \ln(1 + x^2)$ is continuous on $[-1, 2]$ and differentiable on $(-1, 2)$, so the Mean Value Theorem applies.

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{\ln(5/2)}{3}, \qquad f'(c) = \frac{2c}{1 + c^2}.$$

Let $L = \ln(5/2)$. Then $Lc^2 - 6c + L = 0$, so

$$c = \frac{3 \pm \sqrt{9 - L^2}}{L}.$$

The plus root exceeds 2. The only admissible value is

$$\boxed{c = \frac{3 - \sqrt{9 - [\ln(5/2)]^2}}{\ln(5/2)} \approx 0.156453.}$$

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Question 7

Problem:

Let

$$f(x) = \frac{x}{x+1}$$

on the interval $[1, 4]$.

(a) Show that the hypotheses of the Mean Value Theorem are satisfied.

(b) Find all values of $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

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Question 7 - Solution

We are given:

$$f(x) = \frac{x}{x+1}$$

(a) MVT Conditions:

- $f(x)$ is continuous on $[1, 4]$: The function is a rational function with no discontinuities on this interval.
- $f(x)$ is differentiable on $(1, 4)$: Differentiable as a quotient of differentiable functions, and denominator never zero.

MVT applies.

(b) Compute the average rate of change:

$$f(1) = \frac{1}{2}, \quad f(4) = \frac{4}{5}$$

$$\frac{f(4) - f(1)}{4 - 1} = \frac{\frac{4}{5} - \frac{1}{2}}{3} = \frac{\frac{8-5}{10}}{3} = \frac{3}{10} \cdot \frac{1}{3} = \frac{1}{10}$$

Find derivative:

$$f'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$$

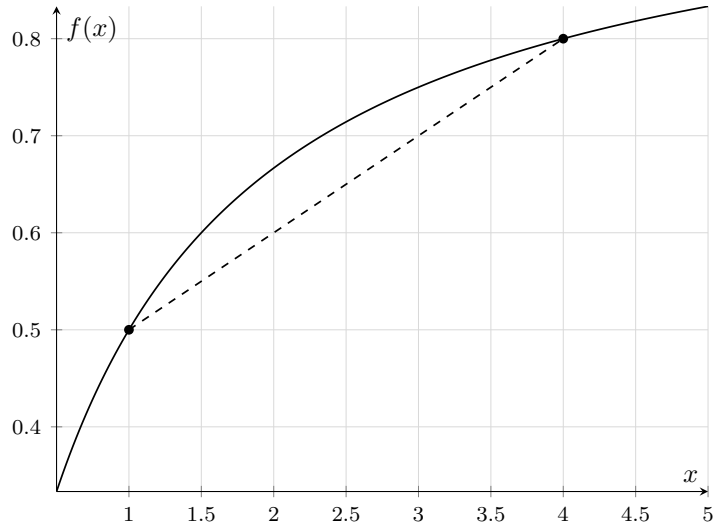
Set:

$$\frac{1}{(c+1)^2} = \frac{1}{10} \Rightarrow (c+1)^2 = 10 \Rightarrow c + 1 = \sqrt{10} \Rightarrow c = \sqrt{10} - 1$$

Since $\sqrt{10} \approx 3.16$, we find $c \approx 2.16 \in (1, 4)$

$$\boxed{c = \sqrt{10} - 1}$$

Graph of $f(x)$ and Secant Line:



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Question 8

Problem:

Let

$$f(x) = \sin(x)$$

on the interval $[0, \pi]$.

(a) Verify that $f(x)$ satisfies the hypotheses of the Mean Value Theorem.

(b) Find all numbers $c \in (0, \pi)$ such that

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}.$$

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Question 8 - Solution

We are given:

$$f(x) = \sin(x)$$

(a) MVT Conditions:
- $f(x)$ is continuous on $[0, \pi]$: Sine is continuous everywhere.
- $f(x)$ is differentiable on $(0, \pi)$: Sine is differentiable everywhere.
  MVT applies.

(b) Apply MVT:
Average rate of change:

$$f(\pi) = \sin(\pi) = 0, \quad f(0) = \sin(0) = 0$$

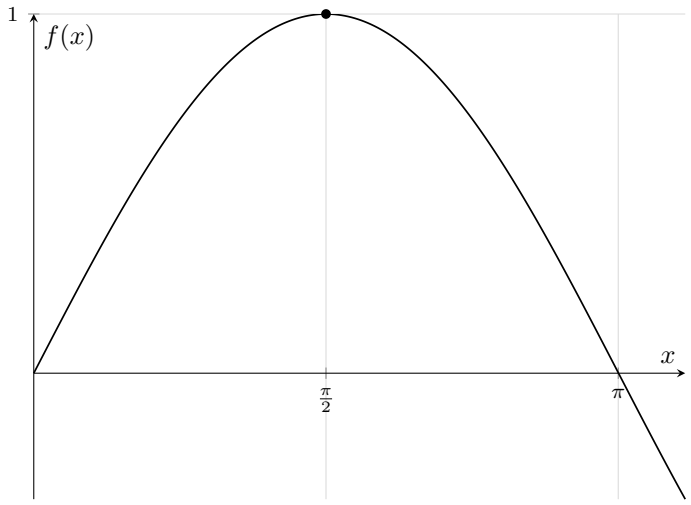
$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

We want:

$$f'(c) = \cos(c) = 0 \Rightarrow c = \frac{\pi}{2}$$

$$\boxed{c = \frac{\pi}{2}}$$

Graph of $f(x) = \sin(x)$ on $[0, \pi]$:



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Question 9

Problem:

Let

$$f(x) = \frac{1}{x}$$

on the interval $[1, 3]$.

(a) Verify that $f(x)$ satisfies the hypotheses of the Mean Value Theorem.

(b) Find all numbers $c \in (1, 3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}.$$

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Question 9 - Solution

We are given:

$$f(x) = \frac{1}{x}$$

(a) Conditions for MVT:
- $f(x)$ is continuous on $[1, 3]$ - $f(x)$ is differentiable on $(1, 3)$
Therefore, MVT applies.

(b) Apply MVT:

$$f(3) = \frac{1}{3}, \quad f(1) = 1$$

$$\frac{f(3) - f(1)}{3 - 1} = \frac{\frac{1}{3} - 1}{2} = \frac{-\frac{2}{3}}{2} = -\frac{1}{3}$$

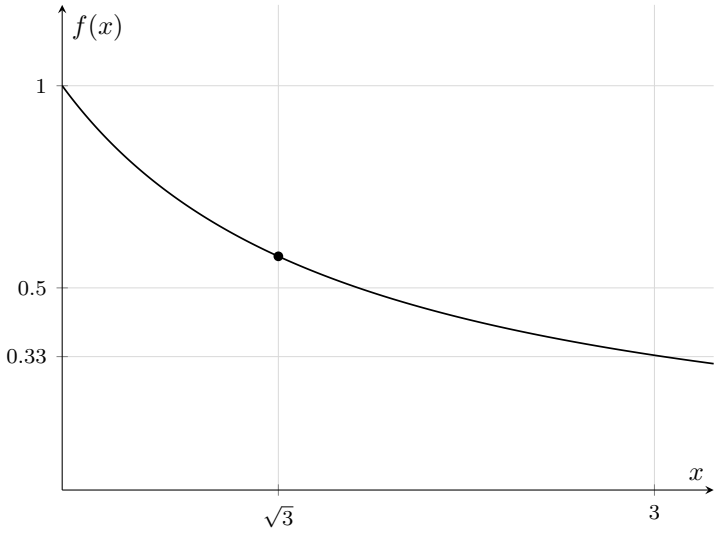
Now compute derivative:

$$f'(x) = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$f'(c) = -\frac{1}{c^2} = -\frac{1}{3} \Rightarrow c^2 = 3 \Rightarrow c = \sqrt{3}$$

$$\boxed{c = \sqrt{3} \in (1, 3)}$$

Graph of $f(x) = \frac{1}{x}$ on $[1, 3]$:



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Question 10

Problem:

Let

$$f(x) = \sin(x)$$

on the interval $[0, \pi]$.

(a) Verify that $f(x)$ satisfies the conditions of the Mean Value Theorem on $[0, \pi]$.

(b) Find all values of $c \in (0, \pi)$ such that

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}.$$

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Question 10 - Solution

We are given:

$$f(x) = \sin(x)$$

(a) Conditions for MVT:
- $f(x)$ is continuous on $[0, \pi]$ - $f(x)$ is differentiable on $(0, \pi)$
MVT applies.

(b) Apply MVT:

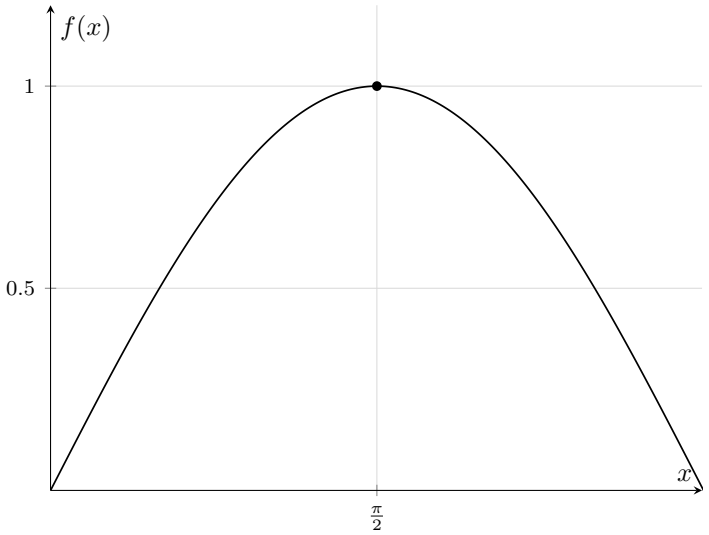
$$f(\pi) = 0, \quad f(0) = 0$$
$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

We need:

$$f'(c) = \cos(c) = 0 \Rightarrow c = \frac{\pi}{2}$$

$$\boxed{c = \frac{\pi}{2}}$$

Graph of $f(x) = \sin(x)$ on $[0, \pi]$:



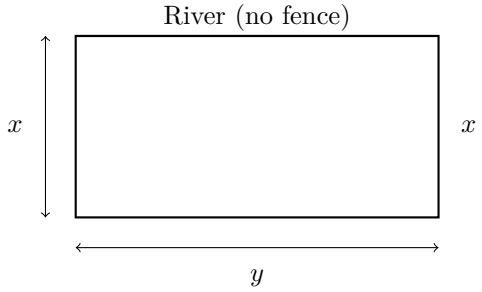
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Question 1

A rectangular garden is to be fenced on three sides (two widths and one length), with the fourth side along a straight river needing no fence. The area of the garden must be 120 m^2. What dimensions will minimize the amount of fencing used?

Clearly define variables, write the function to be minimized, and find the minimum using calculus.

Diagram:



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Question 1 - Solution

Let:

- x = width (perpendicular to the river) - y = length (along the river)

Given: - Area constraint: $A = xy = 120 \Rightarrow y = \frac{120}{x}$ - Fence needed: two widths and one length $\Rightarrow F = 2x + y$

Substitute into fencing function:

$$F(x) = 2x + \frac{120}{x}$$

Minimize $F(x)$:
Take derivative:

$$F'(x) = 2 - \frac{120}{x^2}$$

Set $F'(x) = 0$:

$$2 = \frac{120}{x^2} \Rightarrow x^2 = 60 \Rightarrow x = \sqrt{60} = 2\sqrt{15}$$

Then:

$$y = \frac{120}{x} = \frac{120}{2\sqrt{15}} = \frac{60}{\sqrt{15}} = \frac{60\sqrt{15}}{15} = 4\sqrt{15}$$

Answer: Width = $\boxed{2\sqrt{15}\text{ m}}$, Length = $\boxed{4\sqrt{15}\text{ m}}$

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Question 2

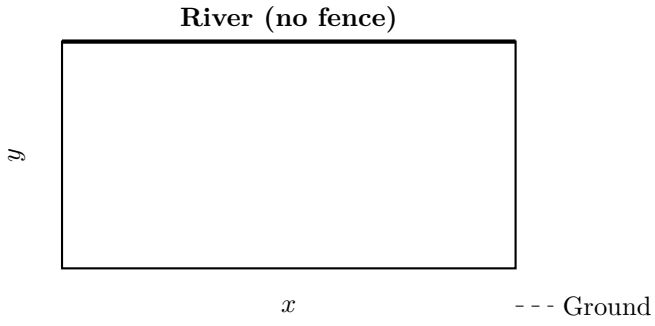
Problem:

A rectangular field is to be enclosed on three sides with fencing (the fourth side lies along a river and requires no fence). The total length of fencing available is 240 meters.

(a) What dimensions will maximize the area of the field?

(b) What is the maximum area?

Diagram:



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Question 2 - Solution

Let: - x = length parallel to the river (no fence needed) - y = width (two sides to be fenced)

From the constraint:

$$x + 2y = 240 \quad \Rightarrow \quad x = 240 - 2y$$

The area is:

$$A = x \cdot y = (240 - 2y)y = 240y - 2y^2$$

Differentiate:

$$\frac{dA}{dy} = 240 - 4y$$

Set derivative to zero:

$$240 - 4y = 0 \quad \Rightarrow \quad y = 60$$

Back-substitute:

$$x = 240 - 2(60) = 120$$

(a) Optimal Dimensions:

$$\boxed{x = 120 \text{ m}, \quad y = 60 \text{ m}}$$

(b) Maximum Area:

$$A = 120 \times 60 = \boxed{7200 \text{ m}^2}$$

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Question 3

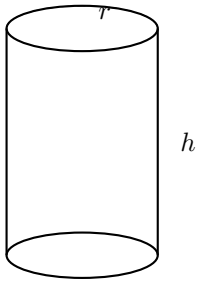
Problem:

A cylindrical can is to be made to hold a volume of $1000\,\text{cm}^3$. The material for the top and bottom costs twice as much per square centimeter as the material for the side.

(a) Find the dimensions (radius and height) of the can that minimize the cost of the material.

(b) What is the minimum total cost in terms of material area?

Diagram:



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Question 3 - Solution

Let: - r: radius of the cylinder - h: height of the cylinder

Volume constraint:

$$V = \pi r^2 h = 1000 \quad \Rightarrow \quad h = \frac{1000}{\pi r^2}$$

Cost: - Side area $= 2\pi rh$, unit cost $= 1$ - Top and bottom area $= 2\pi r^2$, unit cost $= 2$
Total cost C (proportional to material area):

$$C = 2\pi rh + 2 \cdot 2\pi r^2 = 2\pi rh + 4\pi r^2$$

Substitute h:

$$C(r) = 2\pi r \cdot \frac{1000}{\pi r^2} + 4\pi r^2 = \frac{2000}{r} + 4\pi r^2$$

Differentiate:

$$C'(r) = -\frac{2000}{r^2} + 8\pi r$$

Set derivative to 0:

$$-\frac{2000}{r^2} + 8\pi r = 0 \Rightarrow 8\pi r^3 = 2000 \Rightarrow r^3 = \frac{250}{\pi}$$

$$r = \sqrt[3]{\frac{250}{\pi}}, \quad h = \frac{1000}{\pi r^2}$$

(a) Optimal Dimensions:

$$\boxed{r = \sqrt[3]{\frac{250}{\pi}}, \quad h = \frac{1000}{\pi r^2}}$$

(b) Minimum Cost (in material area units):

$$\boxed{C = \frac{2000}{r} + 4\pi r^2} \quad \text{where } r = \sqrt[3]{\frac{250}{\pi}}$$

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Question 4

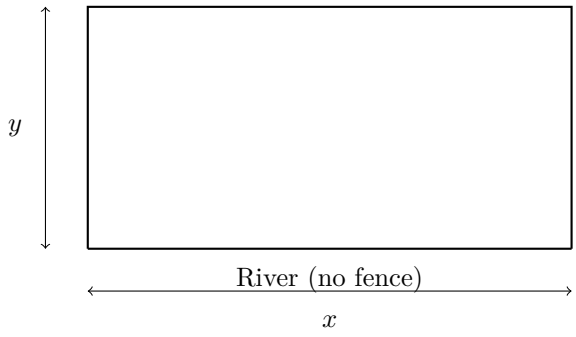
Problem:

A farmer wants to build a rectangular fence next to a straight river. No fencing is needed along the river. The farmer has 240 meters of fencing available.

(a) What dimensions will maximize the area of the enclosure?

(b) What is the maximum area that can be enclosed?

Diagram:



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Question 4 - Solution

Let: - x: length of the fence parallel to the river (only 1 needed) - y: width of the fence perpendicular to the river (2 needed)

Fencing constraint:

$$x + 2y = 240 \quad \Rightarrow \quad x = 240 - 2y$$

Area:

$$A = x \cdot y = (240 - 2y)y = 240y - 2y^2$$

Differentiate:

$$A'(y) = 240 - 4y$$

Set derivative to 0:

$$240 - 4y = 0 \quad \Rightarrow \quad y = 60$$

$$x = 240 - 2(60) = 120$$

(a) Optimal Dimensions:

$$\boxed{x = 120\,\text{m}, \quad y = 60\,\text{m}}$$

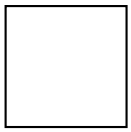
(b) Maximum Area:

$$A = x \cdot y = 120 \cdot 60 = \boxed{7200\,\text{m}^2}$$

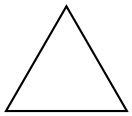
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Question 5

A square and an equilateral triangle are to be constructed from a single wire 100 cm long. The square and the triangle do not overlap. How much wire should be used for each shape in order to minimize the total enclosed area?



Square



Equilateral Triangle

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Question 5 - Solution

Let:

$$x = \text{length of wire used for the square}, \quad 0 < x < 100$$

Then:

$$100 - x = \text{length of wire used for the triangle}$$

Step 1: Expressions for Area

Square:

Let $s = \frac{x}{4}$ be the side of the square. Then area of the square is:

$$A_{\text{square}} = s^2 = \left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$$

Triangle:

Let each side of the triangle be $t = \frac{100-x}{3}$. Then area of equilateral triangle is:

$$A_{\text{triangle}} = \frac{\sqrt{3}}{4}t^2 = \frac{\sqrt{3}}{4}\left(\frac{100-x}{3}\right)^2 = \frac{\sqrt{3}}{36}(100-x)^2$$

Total Area:

$$A(x) = \frac{x^2}{16} + \frac{\sqrt{3}}{36}(100-x)^2$$

Step 2: Minimize Total Area

Differentiate:

$$A'(x) = \frac{1}{8}x - \frac{\sqrt{3}}{18}(100-x)$$

Set $A'(x) = 0$:

$$\frac{1}{8}x = \frac{\sqrt{3}}{18}(100-x) \Rightarrow 18x = 8\sqrt{3}(100-x) \Rightarrow 18x = 800\sqrt{3} - 8\sqrt{3}x \Rightarrow (18+8\sqrt{3})x = 800\sqrt{3} \Rightarrow x = \frac{800\sqrt{3}}{18+8\sqrt{3}}$$

Rationalizing denominator:

$$x = \frac{800\sqrt{3}(18-8\sqrt{3})}{(18+8\sqrt{3})(18-8\sqrt{3})} = \frac{800\sqrt{3}(18-8\sqrt{3})}{324-192} = \frac{800\sqrt{3}(18-8\sqrt{3})}{132}$$

Approximate:

$$x \approx 43.5 \Rightarrow \text{Use about } \boxed{43.5 \text{ cm for the square}}, \quad \boxed{56.5 \text{ cm for the triangle}}$$

Step 3: Justification for Minimum

We compute the second derivative:

$$A''(x) = \frac{1}{8} + \frac{\sqrt{3}}{18} > 0$$

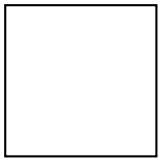
Since $A''(x) > 0$, this critical point gives a local minimum.

$$\boxed{\text{The wire distribution minimizes total area.}}$$

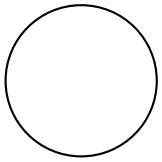
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Question 6

A 100 cm long wire is to be cut into two pieces. One piece is bent to form a square, and the other is bent to form a circle. How should the wire be cut to minimize the total enclosed area?



Square



Circle

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Question 6 - Solution

Let:

$$x = \text{length used for the square}, \quad 0 < x < 100 \Rightarrow 100 - x = \text{length used for the circle}$$

Step 1: Area Expressions

Square:

$$\text{Perimeter} = x \Rightarrow \text{Side} = \frac{x}{4}, \quad A_{\text{square}} = \left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$$

Circle:

$$\text{Circumference} = 100 - x = 2\pi r \Rightarrow r = \frac{100 - x}{2\pi} \Rightarrow A_{\text{circle}} = \pi r^2 = \pi \left(\frac{100 - x}{2\pi}\right)^2 = \frac{(100 - x)^2}{4\pi}$$

Total Area:

$$A(x) = \frac{x^2}{16} + \frac{(100 - x)^2}{4\pi}$$

Step 2: Minimize Area

Differentiate:

$$A'(x) = \frac{1}{8}x - \frac{1}{2\pi}(100 - x)$$

Set $A'(x) = 0$:

$$\frac{1}{8}x = \frac{1}{2\pi}(100 - x) \Rightarrow \pi x = 4(100 - x) \Rightarrow \pi x = 400 - 4x \Rightarrow x(\pi + 4) = 400 \Rightarrow x = \frac{400}{\pi + 4}$$

Step 3: Approximate and Justify Minimum

$$x \approx \frac{400}{3.1416 + 4} = \frac{400}{7.1416} \approx 56.0$$

$$\boxed{x \approx 56.0 \text{ cm for the square}}, \quad \boxed{44.0 \text{ cm for the circle}}$$

Second derivative:

$$A''(x) = \frac{1}{8} + \frac{1}{2\pi} > 0 \Rightarrow \text{This critical point is a minimum.}$$

$$\boxed{\text{This wire division minimizes the total area.}}$$

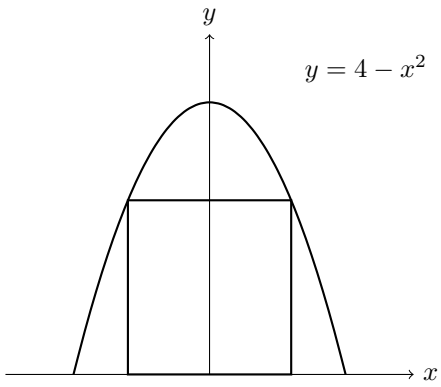
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Question 7

Find the dimensions of the rectangle of maximum area that can be inscribed under the parabola

$$y = 4 - x^2$$

with its base lying along the x-axis and the top two corners touching the curve.



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Question 7 - Solution

Let the rectangle extend from $-x$ to x, so its width is $2x$, and its height is the y-value of the curve:

$$\text{Height} = y = 4 - x^2$$

Step 1: Area Function

$$A(x) = \text{width} \times \text{height} = 2x(4 - x^2) = 8x - 2x^3$$

Step 2: Maximize the Area

Differentiate:

$$A'(x) = 8 - 6x^2$$

Set $A'(x) = 0$:

$$8 - 6x^2 = 0 \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \frac{2}{\sqrt{3}} \approx 1.1547$$

Step 3: Dimensions of the Rectangle

$$\text{Width} = 2x = \frac{4}{\sqrt{3}} \approx 2.309$$

$$\text{Height} = 4 - x^2 = 4 - \frac{4}{3} = \frac{8}{3} \approx 2.667$$

Maximum Area Rectangle Dimensions: Width ≈ 2.309, Height ≈ 2.667

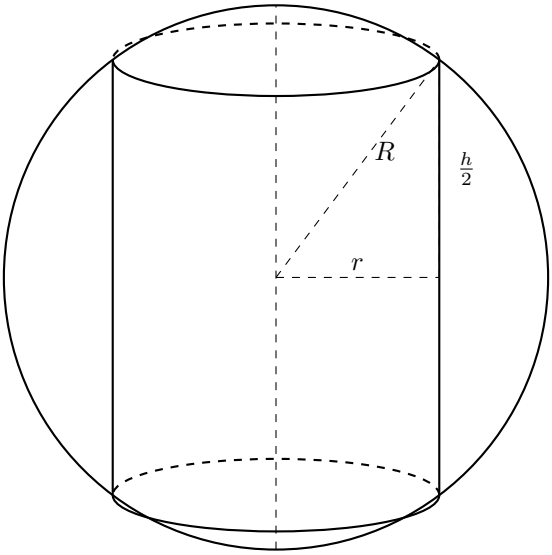
Step 4: Confirm Maximum

$$A''(x) = -12x < 0 \quad \text{at } x > 0 \Rightarrow \text{maximum confirmed.}$$

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Question 8

A cylinder is inscribed in a sphere of radius R. Find the maximum possible volume of such a cylinder and determine its dimensions.



Cylinder inscribed in a sphere of radius R

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Question 8 - Solution

Let the height of the cylinder be h and its radius be r. From the diagram, the diagonal of the right triangle gives the constraint:

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2 \Rightarrow h = 2\sqrt{R^2 - r^2}$$

Step 1: Volume Function

$$V = \pi r^2 h = \pi r^2 \cdot 2\sqrt{R^2 - r^2} = 2\pi r^2 \sqrt{R^2 - r^2}$$

Step 2: Maximize $V(r)$

Let:

$$V(r) = 2\pi r^2 (R^2 - r^2)^{1/2}$$

Differentiate using the product and chain rule:

$$V'(r) = 2\pi \left[2r(R^2 - r^2)^{1/2} - \frac{r^3}{(R^2 - r^2)^{1/2}}\right] = \frac{2\pi}{(R^2 - r^2)^{1/2}} \left(2r(R^2 - r^2) - r^3\right)$$

Set $V'(r) = 0$:

$$2r(R^2 - r^2) - r^3 = 0 \Rightarrow r(2R^2 - 3r^2) = 0 \Rightarrow r = 0 \text{ or } r^2 = \frac{2R^2}{3}$$

Step 3: Dimensions

$$r = \sqrt{\frac{2}{3}}R, \quad h = 2\sqrt{R^2 - r^2} = 2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}$$

Step 4: Max Volume

$$V = \pi r^2 h = \pi \cdot \left(\frac{2R^2}{3}\right) \cdot \frac{2R}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}$$

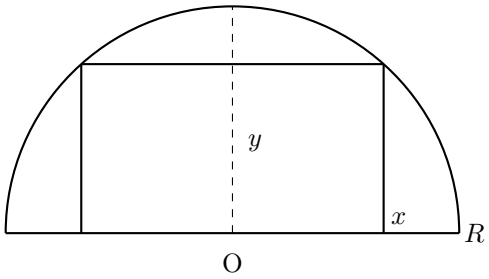
$$\boxed{\begin{array}{l} \text{Radius: } r = \sqrt{\frac{2}{3}}R \\ \text{Height: } h = \frac{2R}{\sqrt{3}} \\ \text{Max Volume: } V = \frac{4\pi R^3}{3\sqrt{3}} \end{array}}$$

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Question 1

A rectangle is inscribed in a semicircle of radius R, with the rectangle's base lying along the diameter of the semicircle.

Find the dimensions of the rectangle that has the maximum area.



Rectangle inscribed in a semicircle

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Question 1 - Solution

Let the coordinates of the top-right corner of the rectangle be (x, y). Since the rectangle is symmetric and lies in a semicircle of radius R centered at the origin:

$$x^2 + y^2 = R^2 \quad \text{(semicircle equation)} \Rightarrow y = \sqrt{R^2 - x^2}$$

Step 1: Area of Rectangle

Since the base is $2x$ and the height is y, area is:

$$A(x) = 2x \cdot y = 2x\sqrt{R^2 - x^2}$$

Step 2: Maximize Area

$$A(x) = 2x\sqrt{R^2 - x^2}$$

Differentiate using the product rule:

$$A'(x) = 2\sqrt{R^2 - x^2} + 2x \cdot \frac{-x}{\sqrt{R^2 - x^2}} = \frac{2(R^2 - x^2) - 2x^2}{\sqrt{R^2 - x^2}} = \frac{2R^2 - 4x^2}{\sqrt{R^2 - x^2}}$$

Set $A'(x) = 0$:

$$2R^2 - 4x^2 = 0 \Rightarrow x^2 = \frac{R^2}{2} \Rightarrow x = \frac{R}{\sqrt{2}}, \quad y = \sqrt{R^2 - x^2} = \frac{R}{\sqrt{2}}$$

Step 3: Max Area and Dimensions

$$\text{Width} = 2x = \sqrt{2}R, \quad \text{Height} = y = \frac{R}{\sqrt{2}}, \quad \text{Area} = 2x \cdot y = R^2$$

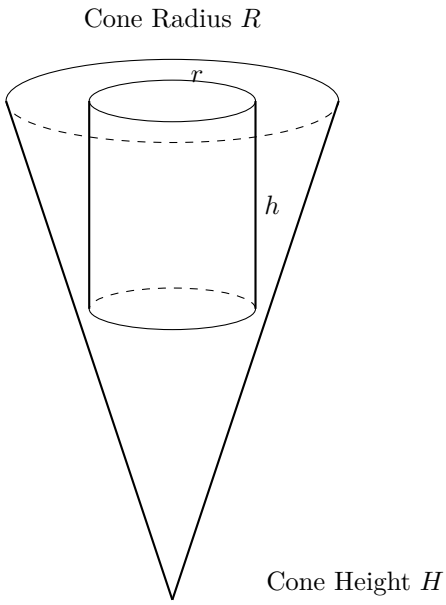
Maximum area: $A = R^2$
Width: $2x = \sqrt{2}R$
Height: $y = \dfrac{R}{\sqrt{2}}$

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Question 10

A right circular cylinder is inscribed in a right circular cone of height H and base radius R, such that the cylinder shares the same central axis as the cone.

Find the dimensions of the cylinder that maximize its volume.



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Question 10 - Solution

Let the height of the cylinder be h, and its base radius be r. We want to maximize:

$$V = \pi r^2 h$$

Step 1: Relationship between r and h

From similar triangles (cone vs cylinder section), we have:

$$\frac{r}{H-h} = \frac{R}{H} \Rightarrow r = \frac{R(H-h)}{H}$$

Substitute into volume:

$$V(h) = \pi \left(\frac{R(H-h)}{H}\right)^2 h = \pi \cdot \frac{R^2(H-h)^2 h}{H^2}$$

Step 2: Maximize $V(h)$

Let's simplify:

$$V(h) = \frac{\pi R^2}{H^2}(H-h)^2 h$$

Let $k = \frac{\pi R^2}{H^2}$, then:

$$V(h) = k(H-h)^2 h$$

Use the product rule:

$$V'(h) = k\left[2(H-h)(-1)\cdot h + (H-h)^2\right] = k\left[-2h(H-h) + (H-h)^2\right]$$

$$= k(H-h)[(H-h) - 2h] = k(H-h)(H-3h)$$

Set $V'(h) = 0$:

$$(H-h)(H-3h) = 0 \Rightarrow h = H \text{ or } h = \frac{H}{3}$$

But $h = H$ gives 0 volume. So maximum occurs at $\boxed{h = \frac{H}{3}}$

Substitute into r:

$$r = \frac{R(H - H/3)}{H} = \frac{2R}{3}$$

Step 3: Final Answer

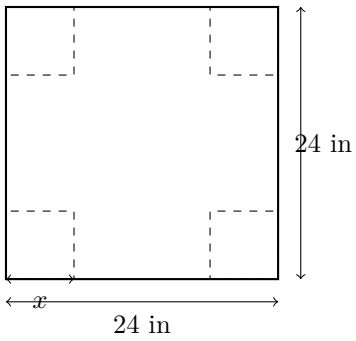
Height of cylinder: $h = \frac{H}{3}$
Radius of cylinder: $r = \frac{2R}{3}$
Maximum Volume: $V = \pi r^2 h = \pi \cdot \left(\frac{2R}{3}\right)^2 \cdot \frac{H}{3} = \frac{4\pi R^2 H}{27}$

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Question 1

A square piece of cardboard 24 inches on each side is to be made into an open-top box by cutting equal squares from each corner and folding up the sides. What size square should be cut from each corner to maximize the volume of the box?

Diagram:



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Question 1 - Solution

Let x be the length of the square cut from each corner.

After folding, the dimensions of the box will be:
Height $= x$
Length and width $= 24 - 2x$

Volume:

$$V(x) = x(24 - 2x)^2$$

Step 1: Expand volume function

$$V(x) = x(576 - 96x + 4x^2) = 576x - 96x^2 + 4x^3$$

Step 2: Find critical points

$$V'(x) = 576 - 192x + 12x^2 \Rightarrow 12x^2 - 192x + 576 = 0 \Rightarrow x^2 - 16x + 48 = 0 \Rightarrow x = 8 \pm \sqrt{16} \Rightarrow x = 8 \pm 4$$

$$x = 4 \quad \text{or} \quad x = 12$$

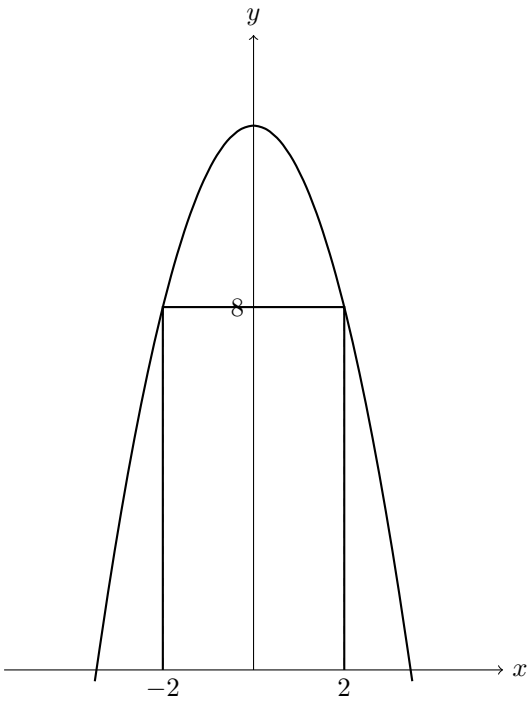
But $x = 12$ makes $24 - 2x = 0 \rightarrow$ not valid
Valid critical point: $x = 4$

Answer: Cut out squares of size $\boxed{4 \text{ in}}$ to maximize volume

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Question 2

A rectangle is inscribed under the parabola $y = 12 - x^2$ such that its base lies on the x-axis and its upper corners lie on the parabola.



Find the dimensions of the rectangle of maximum area.

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Question 2 - Solution

Let the top corners of the rectangle be (x, y) and $(-x, y)$, where the rectangle is symmetric about the y-axis.

From the parabola:

$$y = 12 - x^2$$

The width of the rectangle is $2x$, and the height is $y = 12 - x^2$

Area Function:

$$A(x) = \text{width} \times \text{height} = 2x(12 - x^2) = 24x - 2x^3$$

Differentiate:

$$A'(x) = 24 - 6x^2$$

Set derivative to zero:

$$24 - 6x^2 = 0 \Rightarrow x^2 = 4 \Rightarrow x = 2$$

Check Maximum:

Second derivative:

$$A''(x) = -12x \Rightarrow A''(2) = -24 < 0$$

So, $x = 2$ gives a maximum.

Final Dimensions:

$$x = 2 \Rightarrow \text{width} = 2x = 4$$

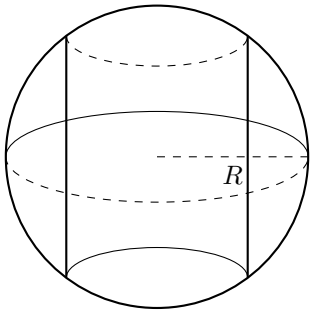
$$y = 12 - x^2 = 12 - 4 = 8$$

$$\boxed{\begin{aligned} \text{Width} &= 4 \\ \text{Height} &= 8 \\ \text{Maximum Area} &= 32 \end{aligned}}$$

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Question 3

A right circular cylinder is inscribed inside a sphere of radius R.



Find the dimensions (radius and height) of the cylinder that **maximize its volume**, and express the maximum volume in terms of R.

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Question 3 - Solution

Let the radius of the cylinder be r and height h. Since the cylinder fits in the sphere of radius R, we use the Pythagorean relation from the triangle formed by r, $h/2$, and R:

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2 \Rightarrow h = 2\sqrt{R^2 - r^2}$$

Volume of the Cylinder:

$$V = \pi r^2 h = \pi r^2 \cdot 2\sqrt{R^2 - r^2} = 2\pi r^2 \sqrt{R^2 - r^2}$$

Maximize Volume:

Let:

$$V(r) = 2\pi r^2 \sqrt{R^2 - r^2}$$

Take the derivative (using product and chain rule):

$$V'(r) = 2\pi \left[2r\sqrt{R^2 - r^2} + r^2 \cdot \frac{1}{2}(R^2 - r^2)^{-1/2}(-2r)\right]$$

$$= 2\pi \left[2r\sqrt{R^2 - r^2} - \frac{r^3}{\sqrt{R^2 - r^2}}\right]$$

Set $V'(r) = 0$:

$$2r\sqrt{R^2 - r^2} = \frac{r^3}{\sqrt{R^2 - r^2}} \Rightarrow 2(R^2 - r^2) = r^2 \Rightarrow 2R^2 - 2r^2 = r^2 \Rightarrow 3r^2 = 2R^2 \Rightarrow r^2 = \frac{2}{3}R^2$$

$$r = R\sqrt{\frac{2}{3}}, \quad h = 2\sqrt{R^2 - r^2} = 2\sqrt{R^2 - \frac{2}{3}R^2} = 2\sqrt{\frac{1}{3}R^2} = 2R\sqrt{\frac{1}{3}}$$

Maximum Volume:

$$V = \pi r^2 h = \pi \cdot \frac{2}{3}R^2 \cdot 2R\sqrt{\frac{1}{3}} = \frac{4\pi R^3}{3} \cdot \frac{1}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}$$

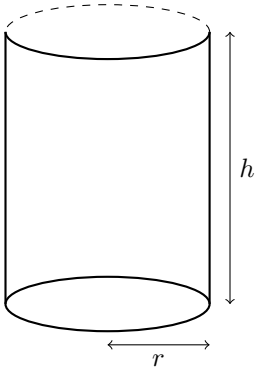
$$\boxed{\begin{aligned} r &= R\sqrt{\frac{2}{3}} \\ h &= 2R\sqrt{\frac{1}{3}} \\ V_{\max} &= \frac{4\pi R^3}{3\sqrt{3}} \end{aligned}}$$

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Question 4

A closed right circular cylinder has a fixed volume of $V = 1000$ cm^3.

Find the radius and height of the cylinder that minimizes its surface area. Also determine the minimum surface area.



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Question 4 - Solution

Let the radius be r, and the height be h. We are given:

$$V = \pi r^2 h = 1000 \Rightarrow h = \frac{1000}{\pi r^2}$$

Surface Area of the Cylinder:

$$A = 2\pi r^2 + 2\pi r h \Rightarrow A(r) = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}$$

Minimize $A(r)$:

Take derivative:

$$A'(r) = 4\pi r - \frac{2000}{r^2}$$

Set $A'(r) = 0$:

$$4\pi r = \frac{2000}{r^2} \Rightarrow 4\pi r^3 = 2000 \Rightarrow r^3 = \frac{500}{\pi} \Rightarrow r = \sqrt[3]{\frac{500}{\pi}}$$

Now compute h:

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi \left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = \frac{1000}{\pi} \cdot \left(\frac{\pi}{500}\right)^{2/3} = 1000 \cdot \pi^{-1/3} \cdot 500^{-2/3}$$

Minimum Surface Area:

$$A = 2\pi r^2 + \frac{2000}{r} = 2\pi \left(\sqrt[3]{\frac{500}{\pi}}\right)^2 + \frac{2000}{\sqrt[3]{\frac{500}{\pi}}}$$

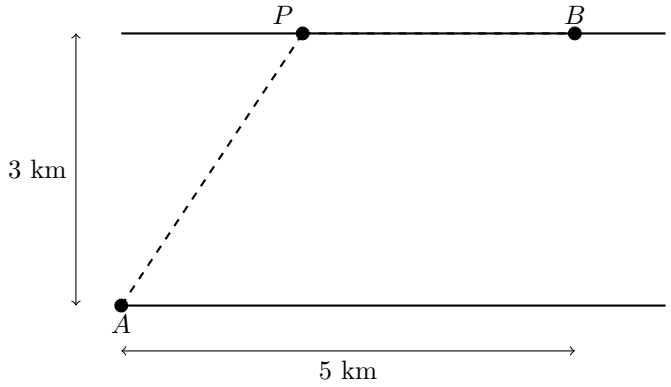
$$\boxed{\begin{aligned} r &= \sqrt[3]{\frac{500}{\pi}} \\ h &= \frac{1000}{\pi r^2} \\ A_{\min} &= 2\pi r^2 + \frac{2000}{r} \end{aligned}}$$

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Question 5

A person is standing at point A on the shore of a straight river that is 3 km wide. Point B lies on the opposite shore, 5 km downstream from the point directly opposite A. The person can run at 10 km/h along the shore and swim at 5 km/h directly across the river or at an angle.

The person first swims from A to P, then runs along the opposite shore to B. At what point P between the point directly opposite A and B should the person land to minimize total time to reach B?



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Question 5 - Solution

Let the swimmer land at point P, which is x km from the point directly across from A.

Step 1: Write distances:
Swim distance: $\sqrt{x^2+3^2} = \sqrt{x^2+9}$, Run distance: $5-x$

Step 2: Write time expression:
Swimming speed: 5 km/h , Running speed: 10 km/h

$$T(x) = \frac{\sqrt{x^2+9}}{5} + \frac{5-x}{10}$$

Step 3: Minimize time — take derivative:

$$T'(x) = \frac{x}{5\sqrt{x^2+9}} - \frac{1}{10}$$

Set $T'(x) = 0$:

$$\frac{x}{5\sqrt{x^2+9}} = \frac{1}{10} \Rightarrow \frac{2x}{\sqrt{x^2+9}} = 1 \Rightarrow 4x^2 = x^2+9 \Rightarrow 3x^2 = 9 \Rightarrow x^2 = 3 \Rightarrow x = \sqrt{3}$$

Step 4: Answer

$$x = \sqrt{3} \approx 1.732 \text{ km}$$

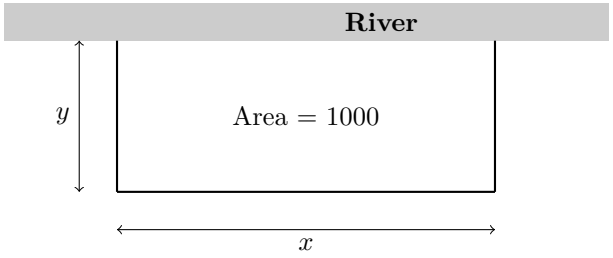
$T_{\min}$ = when the person lands $\sqrt{3}$ km from the point directly across from A.

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Question 6

A rectangular enclosure is to be built adjacent to a river (no fence is needed along the river side). The other three sides are to be fenced. The area of the enclosure must be 1000 square meters.

If the fencing along the two sides perpendicular to the river costs \$10 per meter and the fencing along the side parallel to the river costs \$5 per meter, what dimensions will minimize the total cost?



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Question 6 - Solution

Let: - x = length along the river (no fence needed) - y = depth away from river (fence on both sides)

Area constraint:

$$xy = 1000 \Rightarrow y = \frac{1000}{x}$$

Cost function: - Two sides of length y at \$10/m: cost $= 2(10)(y) = 20y$ - One side of length x at \$5/m: cost $= 5x$

$$C(x) = 5x + 20 \cdot \frac{1000}{x} = 5x + \frac{20000}{x}$$

Minimize cost — take derivative:

$$C'(x) = 5 - \frac{20000}{x^2}$$

Set derivative to zero:

$$5 = \frac{20000}{x^2} \Rightarrow x^2 = 4000 \Rightarrow x = \sqrt{4000} = 20\sqrt{10}$$

Find y:

$$y = \frac{1000}{x} = \frac{1000}{20\sqrt{10}} = \frac{50}{\sqrt{10}} = 5\sqrt{10}$$

Answer:

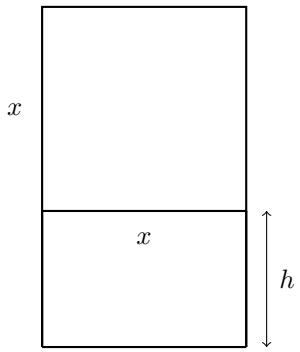
$$\boxed{\begin{aligned} x &= 20\sqrt{10} \approx 63.25 \text{ m} \\ y &= 5\sqrt{10} \approx 15.81 \text{ m} \end{aligned}}$$

These dimensions minimize the fencing cost.

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Question 7

A rectangular box with a square base and no top is to have a volume of 500 cm^3. What dimensions will minimize the surface area?



Open-top square box

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Question 7 - Solution

Let: - x = side length of the square base (in cm) - h = height of the box (in cm)

Volume constraint:

$$V = x^2 h = 500 \Rightarrow h = \frac{500}{x^2}$$

Surface area to minimize: - One bottom square: x^2 - Four sides: $4(xh)$

$$A(x) = x^2 + 4xh = x^2 + 4x \cdot \frac{500}{x^2} = x^2 + \frac{2000}{x}$$

Take derivative:

$$A'(x) = 2x - \frac{2000}{x^2}$$

Set derivative to zero:

$$2x = \frac{2000}{x^2} \Rightarrow 2x^3 = 2000 \Rightarrow x^3 = 1000 \Rightarrow x = 10$$

Find h:

$$h = \frac{500}{x^2} = \frac{500}{100} = 5$$

Answer:

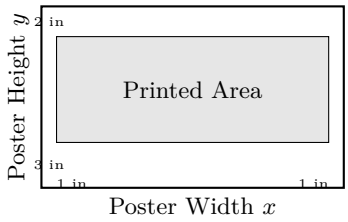
$$\boxed{x = 10 \text{ cm}, \quad h = 5 \text{ cm}}$$

These dimensions minimize the surface area.

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Question 8

A poster is to have an area of 192 in^2. It must have 1-inch margins on the sides, a 2-inch margin at the top, and a 3-inch margin at the bottom. What dimensions of the poster will allow the largest printed area?



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Question 8 - Solution

Let x be poster width and y its height. Then $xy = 192$, and positive printed dimensions require $2 < x < 192/5$.

$$A(x) = (x-2)(192/x - 5) = 202 - 5x - 384/x.$$

$$A'(x) = -5 + 384/x^2 = 0 \quad \Longrightarrow \quad x = \sqrt{384/5}.$$

Since $A''(x) = -768/x^3 < 0$, this is the unique maximum.

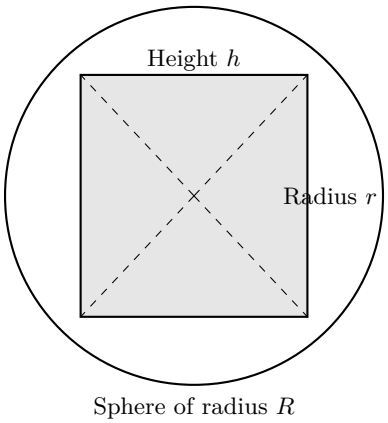
$$\boxed{x = \sqrt{384/5} \approx 8.76 \text{ in}, \qquad y = \sqrt{480} \approx 21.91 \text{ in}.}$$

The exact width, rather than its rounded value, is used to compute the height.

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Question 9

A cylinder is inscribed in a sphere of radius R. Find the dimensions (radius and height) of the cylinder that has the maximum volume.



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Question 9 - Solution

Let the cylinder have: - Radius: r - Height: h

It is inscribed in a sphere of radius R, so by the Pythagorean Theorem applied to the right triangle formed from the center of the sphere to the top edge of the cylinder:

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2$$

Volume of the cylinder:

$$V = \pi r^2 h$$

From the constraint:

$$r^2 = R^2 - \left(\frac{h}{2}\right)^2 = R^2 - \frac{h^2}{4}$$

Substitute into the volume:

$$V(h) = \pi\left(R^2 - \frac{h^2}{4}\right)h = \pi\left(R^2 h - \frac{h^3}{4}\right)$$

Differentiate and maximize:

$$V'(h) = \pi\left(R^2 - \frac{3h^2}{4}\right)$$

Set $V'(h) = 0$:

$$R^2 - \frac{3h^2}{4} = 0 \Rightarrow h^2 = \frac{4R^2}{3} \Rightarrow h = \frac{2R}{\sqrt{3}}$$

Then:

$$r^2 = R^2 - \left(\frac{h}{2}\right)^2 = R^2 - \frac{1}{4}\cdot\frac{4R^2}{3} = R^2 - \frac{R^2}{3} = \frac{2R^2}{3} \Rightarrow r = \sqrt{\frac{2}{3}}R$$

Answer:

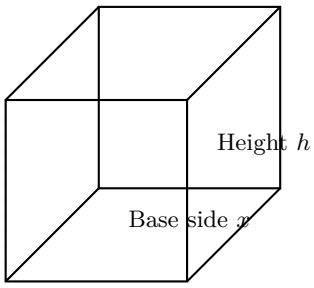
$$\boxed{h = \frac{2R}{\sqrt{3}}, \quad r = \sqrt{\frac{2}{3}}R}$$

These are the dimensions of the inscribed cylinder with maximum volume.

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Question 10

A box with a square base and an open top must have a volume of V cubic units. Find the dimensions of the box that minimize its surface area.



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Question 10 - Solution

Let: - x = length of the base side (square base) - h = height of the box - Volume: $V = x^2h \Rightarrow h = \frac{V}{x^2}$

Surface Area (SA): The box has no top, so:

$$A(x) = \text{Base} + 4 \text{ sides} = x^2 + 4xh$$

$$A(x) = x^2 + 4x\left(\frac{V}{x^2}\right) = x^2 + \frac{4V}{x}$$

Minimize $A(x)$:
Differentiate:

$$A'(x) = 2x - \frac{4V}{x^2}$$

Set $A'(x) = 0$:

$$2x - \frac{4V}{x^2} = 0 \Rightarrow 2x^3 = 4V \Rightarrow x^3 = 2V \Rightarrow x = \sqrt[3]{2V}$$

Then:

$$h = \frac{V}{x^2} = \frac{V}{\left(\sqrt[3]{2V}\right)^2} = \frac{V}{\sqrt[3]{4V^2}} = \sqrt[3]{\frac{V}{4}}$$

Answer:

$$\boxed{x = \sqrt[3]{2V}, \quad h = \sqrt[3]{\frac{V}{4}}}$$

These dimensions give the minimum surface area for the box.

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Question 1

Evaluate the limit:

$$\lim_{x \to 0} \frac{\sin(3x)}{x}$$

(a) Verify that the limit results in an indeterminate form.
(b) Use L'Hospital's Rule to evaluate the limit.

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Question 1 - Solution

(a) As $x \to 0$:
- $\sin(3x) \to 0$, and $x \to 0$ - So, the limit is of the indeterminate form $\frac{0}{0}$

(b) Apply L'Hospital's Rule:

$$\lim_{x\to 0} \frac{\sin(3x)}{x} = \lim_{x\to 0} \frac{d}{dx}[\sin(3x)] \Big/ \frac{d}{dx}[x]$$

$$= \lim_{x\to 0} \frac{3\cos(3x)}{1} = 3\cos(0) = 3(1) = \boxed{3}$$

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Question 2

Evaluate the limit:

$$\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$$

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Question 2 - Solution

We observe that both $\frac{1}{x} \to \infty$ and $\frac{1}{\sin x} \to \infty$ as $x \to 0^+$, so the expression is of the indeterminate form:

$$\infty - \infty$$

We first combine into a single fraction:

$$\frac{1}{x} - \frac{1}{\sin x} = \frac{\sin x - x}{x \sin x}$$

Now the limit becomes:

$$\lim_{x \to 0^+} \frac{\sin x - x}{x \sin x}$$

As $x \to 0^+$, both numerator and denominator $\to 0$. This is now a $\frac{0}{0}$ indeterminate form, so we apply L'Hospital's Rule:

$$\lim_{x \to 0^+} \frac{\sin x - x}{x \sin x} = \lim_{x \to 0^+} \frac{\cos x - 1}{\sin x + x \cos x}$$

Now evaluate the new limit:
- Numerator: $\cos x - 1 \to 0$ - Denominator: $\sin x + x \cos x \to 0$
Still indeterminate! Apply L'Hospital's Rule again:

$$= \lim_{x \to 0^+} \frac{-\sin x}{\cos x + \cos x - x \sin x} = \lim_{x \to 0^+} \frac{-\sin x}{2 \cos x - x \sin x}$$

Now evaluate: - Numerator: $-\sin x \to 0$ - Denominator: $2 \cos x - x \sin x \to 2$
So the final result is:

$$\boxed{0}$$

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Question 3

Evaluate the limit:

$$\lim_{x\to\infty}\left(x\ln\left(1+\frac{1}{x}\right)\right)$$

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Question 3 - Solution

We observe that as $x \to \infty$:
- $\ln\left(1+\frac{1}{x}\right) \to 0$ - $x \to \infty$
So the expression becomes:

$$\infty \cdot 0 \quad \text{(indeterminate form)}$$

We rewrite it as a quotient:

$$x \ln\left(1+\frac{1}{x}\right) = \frac{\ln\left(1+\frac{1}{x}\right)}{1/x}$$

Now as $x \to \infty$, we get a $\frac{0}{0}$ form, so we can apply L'Hospital's Rule.
Let's compute the limit using L'Hospital:

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x} = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln\left(1+\frac{1}{x}\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)}$$

Differentiate numerator and denominator:
- Numerator derivative:

$$\frac{d}{dx}\ln\left(1+\frac{1}{x}\right) = \frac{-1}{x^2(1+\frac{1}{x})} = \frac{-1}{x(x+1)}$$

- Denominator derivative:

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

Now plug in:

$$= \lim_{x\to\infty} \frac{-1/(x(x+1))}{-1/x^2} = \lim_{x\to\infty} \frac{x^2}{x(x+1)} = \lim_{x\to\infty} \frac{x}{x+1}$$

$$\lim_{x\to\infty} \frac{x}{x+1} = 1$$

Final Answer:

$$\boxed{1}$$

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Question 4

Evaluate the limit:

$$\lim_{x \to 0^+} (1 + 2x)^{\frac{1}{x}}$$

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Question 4 - Solution

We identify the limit as the indeterminate form:

$$\lim_{x\to 0^+} (1+2x)^{\frac{1}{x}} = 1^{\infty}$$

Let:

$$y = (1+2x)^{\frac{1}{x}} \Rightarrow \ln y = \frac{1}{x}\ln(1+2x)$$

Now compute the limit:

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1+2x)}{x}$$

This is a $\frac{0}{0}$ form. Apply L'Hospital's Rule:

$$= \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln(1+2x)}{\frac{d}{dx}x} = \lim_{x\to 0^+} \frac{\frac{2}{1+2x}}{1} = \frac{2}{1} = 2$$

So:

$$\ln y \to 2 \quad \Rightarrow \quad y \to e^2$$

Final Answer:

$$\boxed{e^2}$$

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Question 5

Evaluate the limit:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$$

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Question 5 - Solution

As $x \to \infty$, both numerator and denominator approach infinity:

$$\frac{\ln x}{\sqrt{x}} \to \frac{\infty}{\infty} \quad \text{(indeterminate form)}$$

Apply L'Hospital's Rule:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

Final Answer:

$$\boxed{0}$$

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Question 6

Evaluate the limit:

$$\lim_{x\to\infty} (x - \ln x)$$

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Question 6 - Solution

As $x \to \infty$, we observe:

$$x \to \infty, \quad \ln x \to \infty$$

So the limit is of the form:

$$\infty - \infty \quad \text{(indeterminate form)}$$

To apply L'Hospital's Rule, we rewrite the expression to form a quotient. Factor:

$$x - \ln x = x\left(1 - \frac{\ln x}{x}\right)$$

Now evaluate:

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0 \quad \text{(since exponential grows faster)}$$

So:

$$\lim_{x\to\infty} (x - \ln x) = \infty \cdot (1 - 0) = \infty$$

Final Answer: $\boxed{\infty}$

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Question 7

Evaluate the limit:

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}$$

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Question 7 - Solution

As $x \to 0$, we know:

$$\tan x \to 0, \quad x \to 0 \Rightarrow \tan x - x \to 0 \quad \text{and} \quad x^3 \to 0 \Rightarrow \text{Indeterminate form } \frac{0}{0}$$

Apply L'Hospital's Rule:

First derivative:

$$\lim_{x\to 0} \frac{\frac{d}{dx}[\tan x - x]}{\frac{d}{dx}[x^3]} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}$$

As $x \to 0$, $\sec^2 x \to 1$, so numerator $\to 0$, denominator $\to 0$, still $\frac{0}{0}$

Second derivative:

$$\lim_{x\to 0} \frac{\frac{d}{dx}[\sec^2 x - 1]}{\frac{d}{dx}[3x^2]} = \lim_{x\to 0} \frac{2\sec^2 x \cdot \sec x \tan x}{6x}$$

As $x \to 0$, $\tan x \to 0$, so numerator $\to 0$, denominator $\to 0$

Third derivative:

Differentiate numerator and denominator again. Let's simplify instead:

Use the series:

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \Rightarrow \tan x - x = \frac{x^3}{3} + \cdots$$

Then:

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\frac{x^3}{3} + \cdots}{x^3} = \frac{1}{3}$$

Final Answer: $\boxed{\frac{1}{3}}$

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Question 8

Evaluate the limit:

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$$

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Question 8 - Solution

As $x \to \infty$, we have:

$$\sqrt{x^2+x} \to \infty, \quad x \to \infty \Rightarrow \text{Indeterminate form: } \infty - \infty$$

To resolve this, multiply by the conjugate:

$$\lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right) = \lim_{x\to\infty} \frac{(\sqrt{x^2+x} - x)(\sqrt{x^2+x} + x)}{\sqrt{x^2+x} + x}$$

$$= \lim_{x\to\infty} \frac{(x^2+x) - x^2}{\sqrt{x^2+x} + x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x} + x}$$

Now simplify numerator and denominator:
Factor x from the square root:

$$= \lim_{x\to\infty} \frac{x}{x\left(\sqrt{1+\frac{1}{x}} + 1\right)} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}} + 1}$$

As $x \to \infty$, $\frac{1}{x} \to 0$, so:

$$\lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}} + 1} = \frac{1}{1+1} = \frac{1}{2}$$

Final Answer: $\boxed{\frac{1}{2}}$

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Question 9

Evaluate the limit:

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}$$

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Question 9 - Solution

As $x \to 0$, we have:

$$\tan x \to 0, \quad x \to 0 \quad \Rightarrow \quad \frac{\tan x - x}{x^3} \to \frac{0}{0} \quad \text{(indeterminate form)}$$

Apply L'Hospital's Rule:

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}$$

As $x \to 0$, $\sec^2 x \to 1$, so the numerator goes to 0, denominator to 0:

$$\Rightarrow \frac{0}{0} \quad \text{Apply L'Hospital's Rule again}$$

$$= \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} = \lim_{x\to 0} \frac{2 \cdot 1 \cdot 0}{6 \cdot 0} = \frac{0}{0} \quad \text{(Apply once more)}$$

Take third derivative:
Numerator derivative:

$$\frac{d}{dx}\left[2\sec^2 x \tan x\right] = 2\left(2\sec^2 x \tan^2 x + \sec^4 x\right)$$

Denominator derivative:

$$\frac{d}{dx}(6x) = 6$$

Now take the limit:

$$\lim_{x\to 0} \frac{2(2\sec^2 x \tan^2 x + \sec^4 x)}{6} = \frac{2(2 \cdot 1 \cdot 0 + 1)}{6} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

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Question 10

Evaluate the limit:

$$\lim_{x \to 0} \frac{\sin(5x)}{\tan(3x)}$$

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Question 10 - Solution

As $x \to 0$:

$$\sin(5x) \to 0, \quad \tan(3x) \to 0 \Rightarrow \text{Indeterminate form } \frac{0}{0}$$

Apply L'Hospital's Rule:

$$\lim_{x\to 0} \frac{\sin(5x)}{\tan(3x)} = \lim_{x\to 0} \frac{d}{dx}\left[\frac{\sin(5x)}{\tan(3x)}\right] = \lim_{x\to 0} \frac{5\cos(5x)}{3\sec^2(3x)}$$

$$= \lim_{x\to 0} \frac{5\cos(5x)}{3 \cdot \frac{1}{\cos^2(3x)}} = \lim_{x\to 0} \frac{5\cos(5x)\cos^2(3x)}{3}$$

As $x \to 0$:

$$\cos(5x) \to 1, \quad \cos(3x) \to 1 \Rightarrow \cos^2(3x) \to 1$$

So,

$$\frac{5 \cdot 1 \cdot 1}{3} = \frac{5}{3}$$

Final Answer: $\boxed{\frac{5}{3}}$

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Question 1

Use a linear approximation to estimate $\sqrt{4.1}$.

(a) Define the function and the point where you linearize.

(b) Find the linearization $L(x)$ at the chosen point.

(c) Use $L(x)$ to estimate $\sqrt{4.1}$.

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Question 1 - Solution

(a) Let $f(x) = \sqrt{x}$
Choose point $a = 4$, since $\sqrt{4} = 2$ is easy to compute.

(b) Linearization formula:

$$L(x) = f(a) + f'(a)(x - a)$$

Compute derivative:

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(4) = \frac{1}{2 \cdot 2} = \frac{1}{4}$$

$$L(x) = 2 + \frac{1}{4}(x - 4)$$

(c) Estimate $\sqrt{4.1} \approx L(4.1)$:

$$L(4.1) = 2 + \frac{1}{4}(0.1) = 2 + 0.025 = \boxed{2.025}$$

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Question 2

Use a linear approximation to estimate the value of $\sqrt[3]{8.1}$.

(a) Identify a function $f(x)$, a point a, and write the linear approximation $L(x)$.

(b) Use $L(x)$ to estimate $\sqrt[3]{8.1}$.

(c) Compare the estimate to the actual value using a calculator.

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Question 2 - Solution

Let $f(x) = \sqrt[3]{x} = x^{1/3}$. We choose $a = 8$, since $\sqrt[3]{8} = 2$ is easy to compute.

(a) Linear approximation formula:

$$L(x) = f(a) + f'(a)(x - a)$$

Compute:

$$f(x) = x^{1/3}, \quad f'(x) = \frac{1}{3}x^{-2/3} \Rightarrow f'(8) = \frac{1}{3(8)^{2/3}} = \frac{1}{3 \cdot 4} = \frac{1}{12}$$

So:

$$L(x) = f(8) + f'(8)(x - 8) = 2 + \frac{1}{12}(x - 8)$$

(b) Estimate:

$$L(8.1) = 2 + \frac{1}{12}(8.1 - 8) = 2 + \frac{1}{12}(0.1) = 2 + \frac{1}{120} \approx 2.0083$$

(c) Actual value:

$$\sqrt[3]{8.1} \approx 2.0083 \quad \text{(using calculator)}$$

Conclusion: The linear approximation gives a very close estimate:

$$\boxed{\sqrt[3]{8.1} \approx 2.0083}$$

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Question 3

Use a linear approximation to estimate the value of $\ln(1.05)$.

(a) Identify a function $f(x)$, choose a point a, and write the linear approximation $L(x)$.

(b) Use $L(x)$ to estimate $\ln(1.05)$.

(c) Compare the estimate to the actual value.

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Question 3 - Solution

Let $f(x) = \ln(x)$, and choose $a = 1$ because $\ln(1) = 0$ is easy to compute.

(a) Linear approximation formula:

$$L(x) = f(a) + f'(a)(x - a)$$

Compute:

$$f(x) = \ln(x), \quad f'(x) = \frac{1}{x}, \quad f'(1) = 1$$

So:

$$L(x) = \ln(1) + 1 \cdot (x - 1) = 0 + (x - 1) = x - 1$$

(b) Estimate:

$$L(1.05) = 1.05 - 1 = 0.05$$

(c) Actual value:

$$\ln(1.05) \approx 0.04879 \quad \text{(using calculator)}$$

Conclusion: The linear approximation gives:

$$\boxed{\ln(1.05) \approx 0.05}$$

which is a very good estimate.

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Question 4

Use a linear approximation to estimate $\sqrt[3]{8.1}$.

(a) Identify a function $f(x)$, choose a suitable point a, and write the linear approximation $L(x)$.

(b) Use $L(x)$ to estimate $\sqrt[3]{8.1}$.

(c) Compare the estimate to the actual value.

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Question 4 - Solution

Let $f(x) = \sqrt[3]{x} = x^{1/3}$, and choose $a = 8$, since $\sqrt[3]{8} = 2$.

(a) Linear approximation formula:

$$L(x) = f(a) + f'(a)(x - a)$$

Compute:

$$f'(x) = \frac{1}{3}x^{-2/3}, \quad f'(8) = \frac{1}{3} \cdot 8^{-2/3} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

So:

$$L(x) = 2 + \frac{1}{12}(x - 8)$$

(b) Estimate:

$$L(8.1) = 2 + \frac{1}{12}(0.1) = 2 + \frac{1}{120} = \frac{241}{120} \approx 2.0083$$

(c) Actual value:

$$\sqrt[3]{8.1} \approx 2.00828 \quad \text{(using calculator)}$$

Conclusion: The linear approximation gives:

$$\boxed{\sqrt[3]{8.1} \approx 2.0083}$$

which is very close to the actual value.

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Question 5

Use linear approximation to estimate $\ln(1.05)$.

(a) Define an appropriate function $f(x)$, and choose a convenient point a near 1.05.

(b) Find the linear approximation $L(x)$ of $f(x)$ at a.

(c) Use $L(x)$ to approximate $\ln(1.05)$, and compare it with the actual value.

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Question 5 - Solution

Let $f(x) = \ln(x)$. Choose $a = 1$, since $\ln(1) = 0$ is simple and near 1.05.

(a) Derivative of $f(x)$:

$$f'(x) = \frac{1}{x}, \quad f'(1) = 1$$

(b) Linear approximation:

$$L(x) = f(a) + f'(a)(x - a) = \ln(1) + 1(x - 1) = x - 1$$

(c) Estimate:

$$L(1.05) = 1.05 - 1 = 0.05$$

Actual value:

$$\ln(1.05) \approx 0.04879 \quad \text{(using calculator)}$$

Conclusion:

$$\boxed{\ln(1.05) \approx 0.05} \quad \text{(Linear Approximation)}$$

Actual: $\ln(1.05) \approx 0.04879 \Rightarrow$ Very close!

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Question 6

Use linear approximation to estimate $\sqrt[3]{8.2}$.

(a) Define an appropriate function $f(x)$, and choose a point a near 8.2.

(b) Find the linear approximation $L(x)$ of $f(x)$ at a.

(c) Use $L(x)$ to estimate $\sqrt[3]{8.2}$, and compare it to the actual value.

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Question 6 - Solution

Let $f(x) = \sqrt[3]{x} = x^{1/3}$, and choose $a = 8$ since $\sqrt[3]{8} = 2$ is exact and close to 8.2.

(a) Compute derivative:

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}, \quad \text{so} \quad f'(8) = \frac{1}{3\sqrt[3]{64}} = \frac{1}{3 \cdot 4} = \frac{1}{12}$$

(b) Linear approximation:

$$L(x) = f(a) + f'(a)(x - a) = 2 + \frac{1}{12}(x - 8)$$

(c) Estimate:

$$L(8.2) = 2 + \frac{1}{12}(0.2) = 2 + \frac{0.2}{12} = 2 + \frac{1}{60} \approx 2.0167$$

Actual value:

$$\sqrt[3]{8.2} \approx 2.0164 \quad \text{(from calculator)}$$

Conclusion:

$$\boxed{\sqrt[3]{8.2} \approx 2.0167} \quad \text{(Linear Approximation)}$$

Actual: $\sqrt[3]{8.2} \approx 2.0164 \Rightarrow$ Very close!

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Question 7

Use linear approximation to estimate $\ln(1.05)$.

(a) Define a function $f(x)$ and choose a suitable point a near 1.05.

(b) Find the linear approximation $L(x)$ of $f(x)$ at $x = a$.

(c) Use $L(x)$ to estimate $\ln(1.05)$ and compare to the actual value.

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Question 7 - Solution

Let $f(x) = \ln(x)$. We choose $a = 1$ since $\ln(1) = 0$ is easy to compute.

(a) Compute the derivative:

$$f'(x) = \frac{1}{x}, \quad \text{so } f'(1) = 1$$

(b) Linear approximation:

$$L(x) = f(a) + f'(a)(x - a) = 0 + 1(x - 1) = x - 1$$

(c) Estimate:

$$L(1.05) = 1.05 - 1 = 0.05$$

Compare to actual:

$$\ln(1.05) \approx 0.04879 \quad \text{(from calculator)}$$

Conclusion:

$$\boxed{\ln(1.05) \approx 0.05} \quad \text{(Linear Approximation)}$$

Actual: $\ln(1.05) \approx 0.04879 \Rightarrow$ Very close!

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Question 8

Use a linear approximation to estimate $\sqrt{10.2}$.

(a) Define an appropriate function $f(x)$ and choose a point a close to 10.2 where $f(a)$ is easy to compute.

(b) Find the linear approximation $L(x)$ to $f(x)$ at $x = a$.

(c) Use the approximation to estimate $\sqrt{10.2}$, and compare it with the actual value.

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Question 8 - Solution

For $f(x) = \sqrt{x}$, use the nearby perfect square $a = 9$.

$$f(9) = 3, \qquad f'(9) = \frac{1}{6}, \qquad L(x) = 3 + \frac{x-9}{6}.$$

Therefore

$$\boxed{\sqrt{10.2} \approx L(10.2) = 3.2.}$$

Since $f''(x) = -1/(4x^{3/2}) < 0$ for $x > 0$, the tangent line lies above the curve, so this is an overestimate. For comparison, $\sqrt{10.2} \approx 3.193743885$; the error is approximately 0.006256115.

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Question 9

Use a linear approximation to estimate $\ln(1.05)$.

(a) Define a function $f(x)$ such that $f(x) = \ln(x)$, and choose a value a near 1.05 where $f(a)$ is easy to compute.

(b) Find the linear approximation $L(x)$ at $x = a$.

(c) Use the approximation to estimate $\ln(1.05)$ and compare with the actual value.

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Question 9 - Solution

(a) Let $f(x) = \ln(x)$, and choose $a = 1$ since $\ln(1) = 0$ is easy to compute.

$$f(a) = \ln(1) = 0, \quad f'(x) = \frac{1}{x}, \quad f'(1) = 1$$

(b) Linear approximation:

$$L(x) = f(a) + f'(a)(x - a) = 0 + 1(x - 1) = x - 1$$

(c) Estimate $\ln(1.05)$ using:

$$L(1.05) = 1.05 - 1 = 0.05$$

Actual: $\ln(1.05) \approx 0.04879$

Conclusion:

$$\boxed{\ln(1.05) \approx 0.05} \quad \text{(Linear Approximation)}$$

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Question 10

Use a linear approximation to estimate $\sqrt{10.2}$.

(a) Let $f(x) = \sqrt{x}$. Choose a value a near 10.2 where the square root is easy to compute.

(b) Find the linear approximation $L(x)$ at $x = a$.

(c) Use the approximation to estimate $\sqrt{10.2}$ and compare with the actual value.

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Question 10 - Solution

For $f(x) = \sqrt{x}$, use the nearby perfect square $a = 9$.

$$f(9) = 3, \qquad f'(9) = \frac{1}{6}, \qquad L(x) = 3 + \frac{x-9}{6}.$$

Therefore

$$\boxed{\sqrt{10.2} \approx L(10.2) = 3.2.}$$

Since $f''(x) = -1/(4x^{3/2}) < 0$ for $x > 0$, the tangent line lies above the curve, so this is an overestimate. For comparison, $\sqrt{10.2} \approx 3.193743885$; the error is approximately 0.006256115.

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Question 1

Let $f(x) = x^3$.

(a) Use differentials to approximate the change in f as x changes from 2 to 2.05.

(b) Compare your result to the actual change $f(2.05) - f(2)$.

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Question 1 - Solution

We are given:

$$f(x) = x^3, \quad f'(x) = 3x^2$$

(a) Use:

$$df = f'(x)\,dx$$

Let $x = 2$, $dx = 0.05$:

$$f'(2) = 3(2)^2 = 12$$

$$df = 12 \cdot 0.05 = \boxed{0.6}$$

So, the approximate change in f is $\boxed{0.6}$

(b) Actual change:

$$f(2.05) = (2.05)^3 = 8.615125, \quad f(2) = 8$$

$$\text{Actual change} = 8.615125 - 8 = \boxed{0.615125}$$

Conclusion: The differential estimate 0.6 is close to the actual change 0.615125

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Question 2

Suppose the radius r of a sphere is measured to be 5 cm with a possible error of 0.1 cm.

(a) Use differentials to approximate the possible error in the calculated volume of the sphere.

(b) Estimate the relative error and percentage error in the volume.

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Question 2 - Solution

We use the volume formula of a sphere:

$$V = \frac{4}{3}\pi r^3$$

(a) Approximate error using differentials:

$$dV = \frac{dV}{dr}\,dr = 4\pi r^2\,dr$$

$$dV = 4\pi(5)^2(0.1) = 4\pi(25)(0.1) = 10\pi \approx 31.42 \text{ cm}^3$$

(b) Relative error:

$$\frac{dV}{V} = \frac{10\pi}{\frac{4}{3}\pi(5)^3} = \frac{10\pi}{\frac{500}{3}\pi} = \frac{10 \cdot 3}{500} = \frac{30}{500} = 0.06$$

Percentage error:

$$0.06 \times 100 = \boxed{6\%}$$

Final Answer:

- Approximate error in volume: $\boxed{31.42\,\text{cm}^3}$
- Relative error: $\boxed{0.06}$
- Percentage error: $\boxed{6\%}$

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Question 3

The length of the side of a cube is measured to be 10 cm with a possible error of 0.2 cm.

(a) Use differentials to approximate the maximum error in the calculated surface area of the cube.

(b) Estimate the relative error and the percentage error in the surface area.

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Question 3 - Solution

Let the surface area of the cube be:

$$S = 6x^2$$

where x is the length of a side.

(a) Approximate error using differentials:

$$dS = \frac{dS}{dx}\,dx = 12x\,dx$$

$$dS = 12(10)(0.2) = 24\,\text{cm}^2$$

(b) Relative error:

$$\frac{dS}{S} = \frac{24}{6(10)^2} = \frac{24}{600} = 0.04$$

Percentage error:

$$0.04 \times 100 = \boxed{4\%}$$

Final Answer:

- Maximum error in surface area: $\boxed{24\,\text{cm}^2}$
- Relative error: $\boxed{0.04}$
- Percentage error: $\boxed{4\%}$

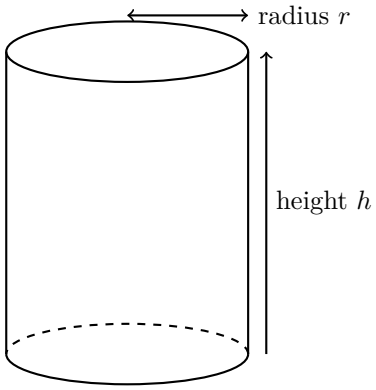
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Question 4

A right circular cylinder has a height of 12 cm, and its radius is measured to be 5 cm with a possible error of 0.1 cm.

(a) Use differentials to approximate the maximum error in the calculated volume of the cylinder.

(b) Estimate the relative error and percentage error in the volume.



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Question 4 - Solution

The volume of a right circular cylinder is given by:

$$V = \pi r^2 h$$

Given: $r = 5\,\text{cm}, \quad dr = 0.1\,\text{cm}, \quad h = 12\,\text{cm}$

(a) Use differentials to approximate the error:

$$dV = \frac{dV}{dr} \cdot dr = 2\pi r h \cdot dr$$

$$dV = 2\pi(5)(12)(0.1) = 12\pi\,\text{cm}^3 \approx \boxed{37.70\,\text{cm}^3}$$

(b) Estimate the relative error and percentage error:

$$V = \pi r^2 h = \pi(5)^2(12) = 300\pi \approx 942.48\,\text{cm}^3$$

$$\text{Relative error} = \frac{dV}{V} = \frac{12\pi}{300\pi} = \frac{12}{300} = 0.04$$

$$\text{Percentage error} = 0.04 \times 100 = \boxed{4\%}$$

Final Answer:

- Maximum error in volume: $\boxed{37.70\,\text{cm}^3}$
- Relative error: $\boxed{0.04}$
- Percentage error: $\boxed{4\%}$

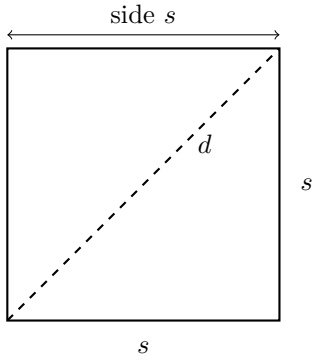
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Question 5

The length of the diagonal d of a square is measured to be 10 cm with a possible error of 0.2 cm. Use differentials to approximate:

(a) The maximum error in computing the area of the square.

(b) The relative and percentage errors.



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Question 5 - Solution

We are given that the diagonal of a square is $d = 10$ cm with an error of $dd = 0.2$ cm. The area of a square in terms of its diagonal is:

$$s = \frac{d}{\sqrt{2}}, \quad A = s^2 = \left(\frac{d}{\sqrt{2}}\right)^2 = \frac{d^2}{2}$$

Differentiate:

$$dA = \frac{d}{dx}\left(\frac{d^2}{2}\right) \cdot dd = d \cdot dd$$

(a) Maximum error in the area:

$$dA = d \cdot dd = 10 \cdot 0.2 = \boxed{2 \text{ cm}^2}$$

(b) Relative and percentage error:
Actual area:

$$A = \frac{10^2}{2} = 50 \text{ cm}^2$$

Relative error:

$$\frac{dA}{A} = \frac{2}{50} = 0.04$$

Percentage error:

$$0.04 \times 100 = \boxed{4\%}$$

Final Answer:

- Maximum error in area: $\boxed{2 \text{ cm}^2}$
- Relative error: $\boxed{0.04}$
- Percentage error: $\boxed{4\%}$

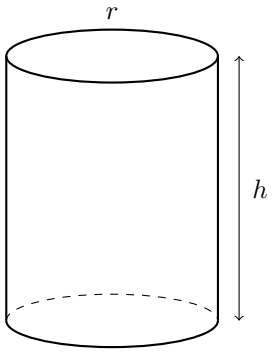
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Question 6

The radius r of a right circular cylinder is measured to be 5 cm with a possible error of 0.1 cm. The height h is 20 cm with a possible error of 0.2 cm.

(a) Use differentials to approximate the maximum error in calculating the volume of the cylinder.

(b) Find the relative and percentage errors in the volume.



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Question 6 - Solution

The volume of a cylinder is:

$$V = \pi r^2 h$$

We compute the differential dV:

$$dV = \frac{\partial V}{\partial r}\,dr + \frac{\partial V}{\partial h}\,dh = 2\pi r h\,dr + \pi r^2\,dh$$

Given:

$$r = 5 \text{ cm}, \quad dr = 0.1 \text{ cm}, \quad h = 20 \text{ cm}, \quad dh = 0.2 \text{ cm}$$

Substitute into the differential:

$$dV = 2\pi(5)(20)(0.1) + \pi(5)^2(0.2) = 20\pi + 5\pi = \boxed{25\pi \approx 78.54 \text{ cm}^3}$$

(a) Maximum error:

$$\boxed{dV \approx 78.54 \text{ cm}^3}$$

(b) Relative and percentage error:

Actual volume:

$$V = \pi(5)^2(20) = 500\pi \approx 1570.8 \text{ cm}^3$$

Relative error:

$$\frac{dV}{V} = \frac{25\pi}{500\pi} = \frac{1}{20} = 0.05$$

Percentage error:

$$0.05 \times 100 = \boxed{5\%}$$

Final Answers:

- Maximum error in volume: $\boxed{25\pi \approx 78.54 \text{ cm}^3}$
- Relative error: $\boxed{0.05}$
- Percentage error: $\boxed{5\%}$

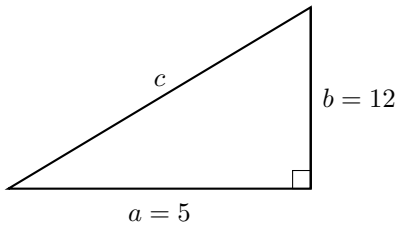
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Question 7

A right triangle has one leg of fixed length 5 cm. The other leg is measured to be 12 cm with a possible measurement error of 0.1 cm.

(a) Use differentials to approximate the maximum error in computing the hypotenuse.

(b) Find the relative and percentage error in the hypotenuse.



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Question 7 - Solution

We are given that $a = 5$ is fixed and $b = 12$ with a possible error $db = 0.1$. The hypotenuse is:

$$c = \sqrt{a^2 + b^2}$$

Differentiate both sides:

$$dc = \frac{1}{2}(a^2 + b^2)^{-1/2} \cdot 2b\, db = \frac{b}{\sqrt{a^2 + b^2}} \cdot db$$

Substitute known values:

$$a = 5, \quad b = 12, \quad db = 0.1$$

$$c = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

$$dc = \frac{12}{13} \cdot 0.1 \approx \boxed{0.0923}$$

(a) Maximum error:

$$\boxed{dc \approx 0.0923 \text{ cm}}$$

(b) Relative and percentage error:

$$\text{Relative error} = \frac{dc}{c} = \frac{0.0923}{13} \approx 0.0071$$

$$\text{Percentage error} = 0.0071 \times 100 \approx \boxed{0.71\%}$$

Final Answers:

- Maximum error in hypotenuse: $\boxed{0.0923 \text{ cm}}$
- Relative error: $\boxed{0.0071}$
- Percentage error: $\boxed{0.71\%}$

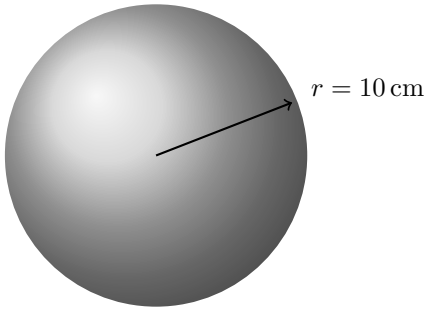
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Question 8

The radius of a spherical balloon is measured to be 10 cm with a possible error of 0.05 cm.

(a) Use differentials to estimate the maximum error in calculating the volume of the balloon.

(b) Estimate the relative and percentage error in the volume.



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Question 8 - Solution

We are given:

$$r = 10 \text{ cm}, \quad dr = 0.05 \text{ cm}$$

The volume of a sphere is:

$$V = \frac{4}{3}\pi r^3$$

Differentiate using differentials:

$$dV = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) \cdot dr = 4\pi r^2 \cdot dr$$

Substitute known values:

$$dV = 4\pi(10)^2 \cdot 0.05 = 4\pi \cdot 100 \cdot 0.05 = 20\pi \approx \boxed{62.83\,\text{cm}^3}$$

(a) Maximum error in volume:

$$\boxed{dV \approx 62.83\,\text{cm}^3}$$

(b) Relative and percentage error:
Actual volume:

$$V = \frac{4}{3}\pi(10)^3 = \frac{4000}{3}\pi \approx 4188.79\,\text{cm}^3$$

$$\text{Relative error} = \frac{dV}{V} \approx \frac{62.83}{4188.79} \approx 0.015$$

$$\text{Percentage error} = 0.015 \times 100 = \boxed{1.5\%}$$

Final Answers:

- Maximum error in volume: $\boxed{62.83\,\text{cm}^3}$
- Relative error: $\boxed{0.015}$
- Percentage error: $\boxed{1.5\%}$

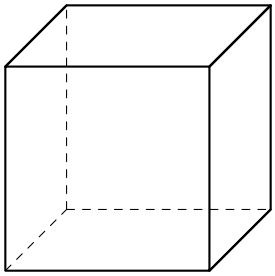
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Question 9

The side of a cube is measured to be 5 cm with a possible error of 0.1 cm.

(a) Use differentials to estimate the maximum error in computing the volume of the cube.

(b) Estimate the relative and percentage error in the volume.



$s = 5\,\text{cm}$

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Question 9 - Solution

We are given:

$$s = 5 \text{ cm}, \quad ds = 0.1 \text{ cm}$$

The volume of a cube is:

$$V = s^3$$

Differentiate using differentials:

$$dV = \frac{d}{ds}(s^3)\, ds = 3s^2\, ds$$

Substitute values:

$$dV = 3(5)^2(0.1) = 3 \cdot 25 \cdot 0.1 = \boxed{7.5\,\text{cm}^3}$$

(a) Maximum error in volume:

$$\boxed{7.5\,\text{cm}^3}$$

(b) Relative and percentage error:

$$V = 5^3 = 125\,\text{cm}^3$$

$$\text{Relative error} = \frac{dV}{V} = \frac{7.5}{125} = 0.06$$

$$\text{Percentage error} = 0.06 \times 100 = \boxed{6\%}$$

Final Answers:

- Maximum error in volume: $\boxed{7.5\,\text{cm}^3}$
- Relative error: $\boxed{0.06}$
- Percentage error: $\boxed{6\%}$

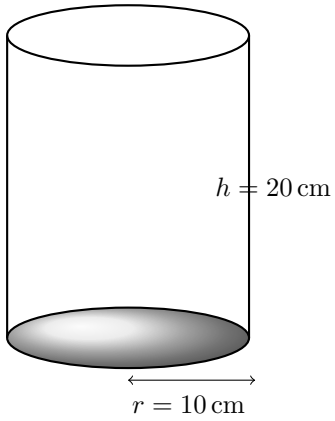
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Question 10

The radius of a right circular cylinder is measured as $r = 10\,\text{cm}$ with a possible error of $0.2\,\text{cm}$, and the height is measured as $h = 20\,\text{cm}$ with a possible error of $0.3\,\text{cm}$.

(a) Use differentials to estimate the maximum error in the volume of the cylinder.

(b) Estimate the percentage error in the volume.



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Question 10 - Solution

We are given:

$$r = 10\,\text{cm}, \quad dr = 0.2\,\text{cm}, \quad h = 20\,\text{cm}, \quad dh = 0.3\,\text{cm}$$

The volume of a cylinder is:

$$V = \pi r^2 h$$

Use differentials:

$$dV = \frac{\partial}{\partial r}dr + \frac{\partial}{\partial h}dh = (2\pi rh) \cdot dr + (\pi r^2) \cdot dh$$

Substitute values:

$$dV = 2\pi(10)(20)(0.2) + \pi(10)^2(0.3)$$

$$= 2\pi(200)(0.2) + \pi(100)(0.3) = \pi(80) + \pi(30) = \pi(110)$$

$$\boxed{dV \approx 345.58\,\text{cm}^3}$$

(a) Estimated maximum error: $\boxed{345.58\,\text{cm}^3}$

(b) Percentage error:

$$V = \pi r^2 h = \pi(100)(20) = 2000\pi \approx 6283.19\,\text{cm}^3$$

$$\text{Percentage error} = \frac{345.58}{6283.19} \times 100 \approx \boxed{5.5\%}$$

Final Answers:

- Maximum error in volume: $\boxed{345.58\,\text{cm}^3}$
- Percentage error: $\boxed{5.5\%}$

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Question 1

Use Newton's Method to approximate a root of the function

$$f(x) = x^3 - x - 1$$

starting with $x_0 = 1.5$, and compute the first two iterations: x_1 and x_2.

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Question 1 - Solution

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)}, \qquad f(x) = x^3 - x - 1, \quad f\prime(x) = 3x^2 - 1.$$

Keeping full precision internally gives

$$x_1 = 1.5000000000 - \frac{0.8750000000}{5.7500000000} \approx \boxed{1.3478}$$

$$x_2 = 1.3478260870 - \frac{0.1006821731}{4.4499054820} \approx \boxed{1.3252}$$

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Question 2

Use Newton's Method to approximate the root of the equation

$$x^3 - 2x - 5 = 0$$

starting with the initial guess $x_1 = 2$. Perform three iterations to compute x_2, x_3, and x_4. Round your answers to four decimal places.

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Question 2 - Solution

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)}, \qquad f(x) = x^3 - 2x - 5, \quad f\prime(x) = 3x^2 - 2.$$

Keeping full precision internally gives

$$x_2 = 2.0000000000 - \frac{-1.0000000000}{10.0000000000} \approx \boxed{2.1000}$$

$$x_3 = 2.1000000000 - \frac{0.0610000000}{11.2300000000} \approx \boxed{2.0946}$$

$$x_4 = 2.0945681211 - \frac{0.0001857232}{11.1616468418} \approx \boxed{2.0946}$$

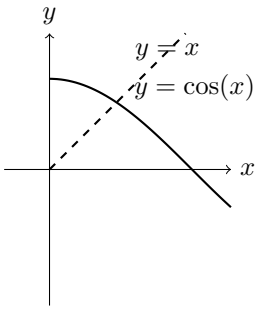
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Question 3

Use Newton's Method to approximate a solution to the equation:

$$\cos(x) = x$$

Start with an initial guess of $x_1 = 0.5$, and compute the next three iterations: x_2, x_3, and x_4, rounding answers to four decimal places.



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Question 3 - Solution

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)}, \qquad f(x) = \cos x - x, \quad f\prime(x) = -\sin x - 1.$$

Keeping full precision internally gives

$$x_2 = 0.5000000000 - \frac{0.3775825619}{-1.4794255386} \approx \boxed{0.7552}$$

$$x_3 = 0.7552224171 - \frac{-0.0271033119}{-1.6854506318} \approx \boxed{0.7391}$$

$$x_4 = 0.7391416661 - \frac{-0.0000946154}{-1.6736538108} \approx \boxed{0.7391}$$

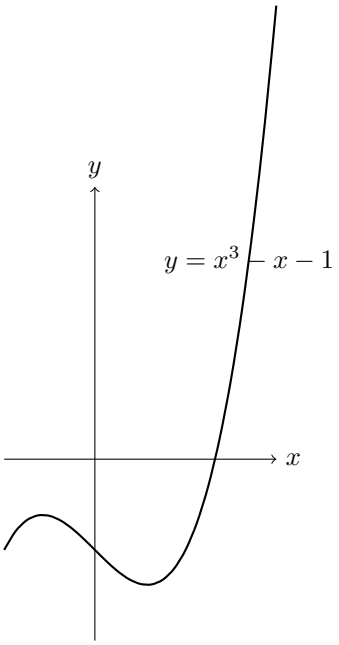
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Question 4

Use Newton's Method to approximate a root of the equation:

$$x^3 - x - 1 = 0$$

Start with an initial guess of $x_1 = 1$, and compute the next three iterations: x_2, x_3, and x_4, rounding answers to four decimal places.



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Question 4 - Solution

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)}, \qquad f(x) = x^3 - x - 1, \quad f\prime(x) = 3x^2 - 1.$$

Keeping full precision internally gives

$$x_2 = 1.0000000000 - \frac{-1.0000000000}{2.0000000000} \approx \boxed{1.5000}$$

$$x_3 = 1.5000000000 - \frac{0.8750000000}{5.7500000000} \approx \boxed{1.3478}$$

$$x_4 = 1.3478260870 - \frac{0.1006821731}{4.4499054820} \approx \boxed{1.3252}$$

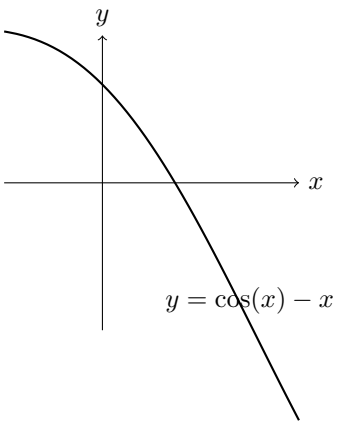
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Question 5

Use Newton's Method to approximate a root of the equation:

$$\cos(x) - x = 0$$

Start with an initial guess of $x_1 = 0.5$, and compute the next three iterations: x_2, x_3, and x_4, rounding to four decimal places.



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Question 5 - Solution

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)}, \qquad f(x) = \cos x - x, \quad f\prime(x) = -\sin x - 1.$$

Keeping full precision internally gives

$$x_2 = 0.5000000000 - \frac{0.3775825619}{-1.4794255386} \approx \boxed{0.7552}$$

$$x_3 = 0.7552224171 - \frac{-0.0271033119}{-1.6854506318} \approx \boxed{0.7391}$$

$$x_4 = 0.7391416661 - \frac{-0.0000946154}{-1.6736538108} \approx \boxed{0.7391}$$

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Question 6

Use Newton's Method to approximate a root of the equation:

$$x^3 - 2x - 5 = 0$$

Start with an initial guess of $x_1 = 2$, and compute the next three approximations: x_2, x_3, and x_4. Round answers to four decimal places.

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Question 6 - Solution

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)}, \qquad f(x) = x^3 - 2x - 5, \quad f\prime(x) = 3x^2 - 2.$$

Keeping full precision internally gives

$$x_2 = 2.0000000000 - \frac{-1.0000000000}{10.0000000000} \approx \boxed{2.1000}$$

$$x_3 = 2.1000000000 - \frac{0.0610000000}{11.2300000000} \approx \boxed{2.0946}$$

$$x_4 = 2.0945681211 - \frac{0.0001857232}{11.1616468418} \approx \boxed{2.0946}$$

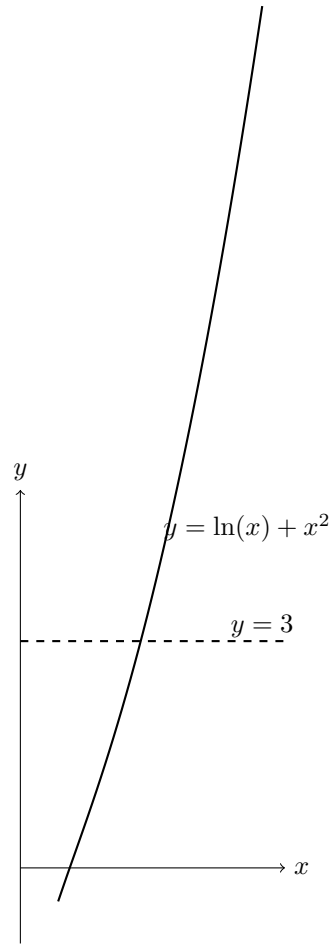
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Question 7

Use Newton's Method to approximate a solution to the equation:

$$\ln(x) + x^2 = 3$$

Start with an initial guess of $x_1 = 1.5$, and compute x_2, x_3, and x_4. Round each to four decimal places.



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Question 7 - Solution

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)}, \qquad f(x) = \ln x + x^2 - 3, \quad f\prime(x) = 1/x + 2x.$$

Keeping full precision internally gives

$$x_2 = 1.5000000000 - \frac{-0.3445348919}{3.6666666667} \approx \boxed{1.5940}$$

$$x_3 = 1.5939640614 - \frac{0.0069454631}{3.8152948397} \approx \boxed{1.5921}$$

$$x_4 = 1.5921436351 - \frac{0.0000026613}{3.8123713061} \approx \boxed{1.5921}$$

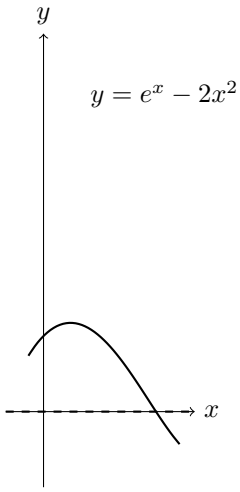
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Question 8

Approximate a root of the equation:

$$e^x - 2x^2 = 0$$

using Newton's Method. Start with $x_1 = 0.5$, and compute x_2, x_3, and x_4 to four decimal places.



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Question 8 - Solution

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)}, \qquad f(x) = e^x - 2x^2, \quad f\prime(x) = e^x - 4x.$$

Keeping full precision internally gives

$$x_2 = 0.5000000000 - \frac{1.1487212707}{-0.3512787293} \approx \boxed{3.7701}$$

$$x_3 = 3.7701133740 - \frac{14.9574735817}{28.3045297918} \approx \boxed{3.2417}$$

$$x_4 = 3.2416653238 - \frac{4.5594909448}{12.6096177926} \approx \boxed{2.8801}$$

These are the requested iterates; three steps do not yet give an accurate root. Continue iterating if a converged root is required.

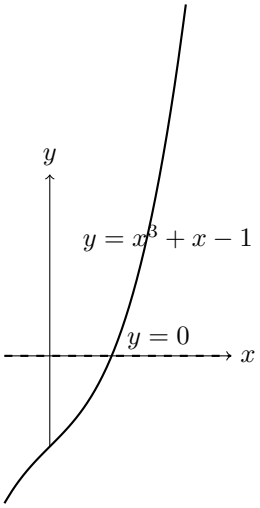
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Question 9

Approximate a solution to the equation:

$$x^3 + x - 1 = 0$$

using Newton's Method. Start with $x_1 = 0.5$, and compute x_2, x_3, and x_4, rounding each to four decimal places.



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Question 9 - Solution

Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)}, \qquad f(x) = x^3 + x - 1, \quad f\prime(x) = 3x^2 + 1.$$

Keeping full precision internally gives

$$x_2 = 0.5000000000 - \frac{-0.3750000000}{1.7500000000} \approx \boxed{0.7143}$$

$$x_3 = 0.7142857143 - \frac{0.0787172012}{2.5306122449} \approx \boxed{0.6832}$$

$$x_4 = 0.6831797235 - \frac{0.0020432938}{2.4002036038} \approx \boxed{0.6823}$$

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Question 10

Consider the equation

$$x^3 - 3x + 1 = 0.$$

(a) Explain why Newton's method fails with the initial guess $x_1 = 1$.

(b) Restart with $x_1 = 0$, and compute x_2, x_3, and x_4. Round the answers to four decimal places.

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Question 10 - Solution

At $x_1 = 1$, $f(1) = -1$ and $f\prime(1) = 0$, so the Newton step is undefined. Restart at $x_1 = 0$. Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)}, \qquad f(x) = x^3 - 3x + 1, \quad f\prime(x) = 3x^2 - 3.$$

Keeping full precision internally gives

$$x_2 = 0.0000000000 - \frac{1.0000000000}{-3.0000000000} \approx \boxed{0.3333}$$

$$x_3 = 0.3333333333 - \frac{0.0370370370}{-2.6666666667} \approx \boxed{0.3472}$$

$$x_4 = 0.3472222222 - \frac{0.0001955804}{-2.6383101852} \approx \boxed{0.3473}$$

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Question 1

A company's revenue (in dollars) from selling x items is given by:

$$R(x) = 50x - 0.2x^2$$

and the cost (in dollars) to produce x items is:

$$C(x) = 10x + 100$$

(a) Find the profit function $P(x)$.
(b) Find the number of items that maximizes profit.
(c) What is the maximum profit?

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Question 1 - Solution

(a) Profit is revenue minus cost:

$$P(x) = R(x) - C(x) = (50x - 0.2x^2) - (10x + 100) = 40x - 0.2x^2 - 100$$

(b) Maximize $P(x) = -0.2x^2 + 40x - 100$.
This is a downward-opening parabola. The maximum occurs at:

$$x = \frac{-b}{2a} = \frac{-40}{2(-0.2)} = \frac{-40}{-0.4} = \boxed{100}$$

(c) Maximum profit:

$$P(100) = 40(100) - 0.2(100)^2 - 100 = 4000 - 2000 - 100 = \boxed{1900}$$

Answer:

$\boxed{\text{Max profit occurs at } x = 100 \text{ items and the maximum profit is \$1900.}}$

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Question 2

A company manufactures and sells x units of a product per week. The revenue (in thousands of dollars) and cost (in thousands of dollars) functions are given by:

$$R(x) = 50x - 0.5x^2, \quad C(x) = 20x + 100$$

(a) Find the profit function $P(x)$.

(b) Determine the number of units x that maximizes the profit.

(c) Compute the maximum profit.

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Question 2 - Solution

(a) Profit Function:

$$P(x) = R(x) - C(x) = (50x - 0.5x^2) - (20x + 100) = -0.5x^2 + 30x - 100$$

(b) Maximize Profit:

$$P'(x) = -x + 30$$

Set $P'(x) = 0$:

$$-x + 30 = 0 \quad \Rightarrow \quad x = 30$$

Second derivative:

$$P''(x) = -1 < 0$$

Hence, profit is maximized at $x = 30$.

(c) Maximum Profit:

$$P(30) = -0.5(30)^2 + 30(30) - 100 = -450 + 900 - 100 = 350$$

Maximum profit occurs at $x = 30$ units, $P_{\max} = 350$ thousand dollars

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Question 3

A company's profit $P(x)$ (in thousands of dollars) depends on the number of units x produced per week:

$$P(x) = -2x^2 + 40x - 120$$

(a) Determine the production level x that maximizes profit.

(b) Compute the maximum profit.

(c) Find the production level x at which the profit is zero.

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Question 3 - Solution

(a) Maximize Profit:

$$P'(x) = \frac{d}{dx}(-2x^2 + 40x - 120) = -4x + 40$$

Set $P'(x) = 0$:

$$-4x + 40 = 0 \quad \Rightarrow \quad x = 10$$

Second derivative:

$$P''(x) = -4 < 0$$

Hence, profit is maximized at $x = 10$.

(b) Maximum Profit:

$$P(10) = -2(10)^2 + 40(10) - 120 = -200 + 400 - 120 = 80$$

(c) Break-even Points (Profit = 0):

$$-2x^2 + 40x - 120 = 0$$

Divide through by 2:

$$x^2 - 20x + 60 = 0$$

Solve using the quadratic formula:

$$x = \frac{20 \pm \sqrt{400 - 240}}{2} = \frac{20 \pm \sqrt{160}}{2} = 10 \pm 2\sqrt{10}$$

Maximum profit occurs at $x = 10$ units, $P_{\max} = 80$ thousand dollars,
Profit is zero at $x = 10 - 2\sqrt{10},\ 10 + 2\sqrt{10}$.

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Question 4

The demand for a product is modeled by

$$p(x) = 100 - 2x,$$

where $p(x)$ is the price per unit in dollars and x is the number of units sold in hundreds. The total cost of producing $100x$ units is

$$C(x) = 20x + 200 \quad \text{dollars.}$$

Require $x \geq 0$ and $p(x) \geq 0$.

(a) Find the revenue and profit functions.

(b) Find the production level and maximum profit in the continuous model.

(c) Find the best whole-number quantity of units and its profit.

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Question 4 - Solution

(a) Revenue and profit

The number sold is $100x$, so revenue is $100x\,p(x)$:

$$R(x) = 10000x - 200x^2, \qquad P(x) = -200x^2 + 9980x - 200.$$

$$P\prime(x) = -400x + 9980, \qquad P\prime\prime(x) = -400 < 0.$$

(b) Continuous maximum

The continuous maximum occurs at

$$\boxed{x = \frac{9980}{400} \approx 24.950000, \quad P_{\max} \approx \$124,300.50.}$$

(c) Whole units

This represents 2495.000000 units. Comparing the adjacent whole-unit counts gives $\boxed{2495}$ units, with profit $\boxed{\$124,300.50}$. The price is nonnegative at these levels.

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Question 5

A company produces and sells x units of a product per week. The weekly revenue and cost functions (in thousands of dollars) are:

$$R(x) = 200x - 2x^2, \quad C(x) = 50x + 300$$

(a) Find the profit function $P(x)$.

(b) Determine the number of units that maximizes the profit.

(c) Compute the maximum profit.

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Question 5 - Solution

$$P(x) = R(x) - C(x) = -2x^2 + 150x - 300.$$

In the continuous model, $P'(x) = -4x + 150 = 0$ gives $x = 37.5$ and $P''(x) = -4 < 0$.
The continuous maximum is 2512.5 thousand dollars.
For whole units, compare $x = 37$ and $x = 38$, the integers nearest the vertex:

$$P(37) = P(38) = 2512.$$

Thus the feasible maximum is

37 or 38 units, 2512 thousand dollars.

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Question 6

The demand for a product is modeled by

$$p(x) = 150 - 3x,$$

where $p(x)$ is the price per unit in dollars and x is the number of units sold in hundreds. The total cost of producing $100x$ units is

$$C(x) = 50x + 300 \quad \text{dollars.}$$

Require $x \geq 0$ and $p(x) \geq 0$.

(a) Find the revenue and profit functions.

(b) Find the production level and maximum profit in the continuous model.

(c) Find the best whole-number quantity of units and its profit.

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Question 6 - Solution

(a) Revenue and profit

The number sold is $100x$, so revenue is $100x\,p(x)$:

$$R(x) = 15000x - 300x^2, \qquad P(x) = -300x^2 + 14950x - 300.$$

$$P\prime(x) = -600x + 14950, \qquad P\prime\prime(x) = -600 < 0.$$

(b) Continuous maximum

The continuous maximum occurs at

$$\boxed{x = \frac{14950}{600} \approx 24.916667, \quad P_{\max} \approx \$185,952.08.}$$

(c) Whole units

This represents 2491.666667 units. Comparing the adjacent whole-unit counts gives $\boxed{2492}$ units, with profit $\boxed{\$185,952.08}$. The price is nonnegative at these levels.

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Question 7

A company's profit $P(x)$ (in dollars) depends on the number of items x produced per week:

$$P(x) = -5x^2 + 300x - 2000$$

(a) Determine the production level x that maximizes profit.

(b) Compute the maximum profit.

(c) Find the production levels x at which the profit is zero.

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Question 7 - Solution

(a) Maximize Profit:

$$P'(x) = \frac{d}{dx}(-5x^2 + 300x - 2000) = -10x + 300$$

Set $P'(x) = 0$:

$$-10x + 300 = 0 \quad \Rightarrow \quad x = 30$$

Second derivative:

$$P''(x) = -10 < 0$$

Hence, profit is maximized at $x = 30$.

(b) Maximum Profit:

$$P(30) = -5(30)^2 + 300(30) - 2000 = -4500 + 9000 - 2000 = 2500$$

(c) Break-even Points (Profit = 0):

$$-5x^2 + 300x - 2000 = 0$$

Divide through by 5:

$$x^2 - 60x + 400 = 0$$

Solve using the quadratic formula:

$$x = \frac{60 \pm \sqrt{3600 - 1600}}{2} = \frac{60 \pm \sqrt{2000}}{2} = 30 \pm 10\sqrt{5}$$

Maximum profit occurs at $x = 30$ units, $P_{\max} = 2500$,
Profit is zero at $x = 30 - 10\sqrt{5},\ 30 + 10\sqrt{5}$.

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Question 8

The weekly revenue (in thousands of dollars) for producing x units is:

$$R(x) = 120x - x^2$$

and the weekly cost is:

$$C(x) = 40x + 200$$

(a) Determine the profit function $P(x)$.

(b) Find the number of units x that maximizes profit.

(c) Compute the maximum profit.

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Question 8 - Solution

(a) Profit Function:

$$P(x) = R(x) - C(x) = (120x - x^2) - (40x + 200) = -x^2 + 80x - 200$$

(b) Maximize Profit:

$$P'(x) = -2x + 80$$

Set $P'(x) = 0$:

$$-2x + 80 = 0 \quad \Rightarrow \quad x = 40$$

Second derivative:

$$P''(x) = -2 < 0$$

Hence, profit is maximized when $x = 40$.

(c) Maximum Profit:

$$P(40) = -(40)^2 + 80(40) - 200 = -1600 + 3200 - 200 = 1400$$

Maximum profit occurs at $x = 40$ units, $P_{\max} = 1400$ thousand dollars

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Question 9

The demand for a product is modeled by

$$p(x) = 80 - x,$$

where $p(x)$ is the price per unit in dollars and x is the number of units sold in hundreds. The total cost of producing $100x$ units is

$$C(x) = 20x + 100 \quad \text{dollars.}$$

Require $x \geq 0$ and $p(x) \geq 0$.

(a) Find the revenue and profit functions.

(b) Find the production level and maximum profit in the continuous model.

(c) Find the best whole-number quantity of units and its profit.

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Question 9 - Solution

(a) Revenue and profit

The number sold is $100x$, so revenue is $100x\,p(x)$:

$$R(x) = 8000x - 100x^2, \qquad P(x) = -100x^2 + 7980x - 100.$$

$$P\prime(x) = -200x + 7980, \qquad P\prime\prime(x) = -200 < 0.$$

(b) Continuous maximum

The continuous maximum occurs at

$$\boxed{x = \frac{7980}{200} \approx 39.900000, \quad P_{\max} \approx \$159,101.00.}$$

(c) Whole units

This represents 3990.000000 units. Comparing the adjacent whole-unit counts gives $\boxed{3990}$ units, with profit $\boxed{\$159,101.00}$. The price is nonnegative at these levels.

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Question 10

A company manufactures and sells a product. The weekly revenue and cost functions (in thousands of dollars) are:

$$R(x) = 500x - 5x^2, \quad C(x) = 200x + 500$$

(a) Determine the profit function $P(x)$.

(b) Find the production level x that maximizes profit.

(c) Compute the maximum profit.

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Question 10 - Solution

(a) Profit Function:

$$P(x) = R(x) - C(x) = (500x - 5x^2) - (200x + 500) = -5x^2 + 300x - 500$$

(b) Maximize Profit:

$$P'(x) = -10x + 300$$

Set $P'(x) = 0$:

$$-10x + 300 = 0 \quad \Rightarrow \quad x = 30$$

Check the second derivative:

$$P''(x) = -10 < 0$$

Thus, profit is maximized when $x = 30$.

(c) Maximum Profit:

$$P(30) = -5(30)^2 + 300(30) - 500 = -4500 + 9000 - 500 = 4000$$

Maximum profit occurs at $x = 30$ units, $P_{\max} = 4000$ thousand dollars

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Question 1

Evaluate the indefinite integral

$$\int \left(x^2 e^{x^3} + \frac{1}{x}\sqrt{1+\ln x} - \frac{3x^2}{1+x^3} \right) dx.$$

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Question 1 - Solution

We integrate term by term.

First term:

$$\int x^2 e^{x^3}\,dx.$$

Let $u = x^3$, so $du = 3x^2\,dx$:

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^{x^3}.$$

Second term:

$$\int \frac{1}{x}\sqrt{1+\ln x}\,dx.$$

Let $u = 1 + \ln x$, so $du = \frac{1}{x}\,dx$:

$$\int \sqrt{u}\,du = \frac{2}{3}u^{3/2} = \frac{2}{3}(1+\ln x)^{3/2}.$$

Third term:

$$\int \frac{3x^2}{1+x^3}\,dx.$$

Let $u = 1 + x^3$, so $du = 3x^2\,dx$:

$$\int \frac{3x^2}{1+x^3}\,dx = \int \frac{du}{u} = \ln(1+x^3).$$

Final Answer:

$$\boxed{\frac{1}{3}e^{x^3} + \frac{2}{3}(1+\ln x)^{3/2} - \ln(1+x^3) + C}$$

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Question 2

Find the most general antiderivative of

$$\int \left(e^{2x} \sin x + \frac{1}{(1-x)^2} \right) dx.$$

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Question 2 - Solution

We evaluate each term separately.

First term:

$$\int e^{2x} \sin x \, dx.$$

Use integration by parts. Let

$$u = \sin x, \qquad dv = e^{2x} \, dx.$$

Then

$$du = \cos x \, dx, \qquad v = \frac{1}{2} e^{2x}.$$

So,

$$\int e^{2x} \sin x \, dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \int e^{2x} \cos x \, dx.$$

Apply integration by parts again to the remaining integral. Let

$$u = \cos x, \qquad dv = e^{2x} \, dx.$$

Then

$$du = -\sin x \, dx, \qquad v = \frac{1}{2} e^{2x}.$$

Thus,

$$\int e^{2x} \cos x \, dx = \frac{1}{2} e^{2x} \cos x + \frac{1}{2} \int e^{2x} \sin x \, dx.$$

Substitute this back:

$$\int e^{2x} \sin x \, dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x - \frac{1}{4} \int e^{2x} \sin x \, dx.$$

Add $\frac{1}{4} \int e^{2x} \sin x \, dx$ to both sides:

$$\frac{5}{4} \int e^{2x} \sin x \, dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x.$$

Solve:

$$\int e^{2x} \sin x \, dx = \frac{e^{2x}}{5} (2 \sin x - \cos x).$$

Second term:

$$\int \frac{1}{(1-x)^2} \, dx.$$

Let

$$u = 1 - x, \qquad du = -dx.$$

Then

$$\int \frac{1}{(1-x)^2} \, dx = -\int u^{-2} \, du = u^{-1} = \frac{1}{1-x}.$$

Final Answer:

$$\boxed{\frac{e^{2x}}{5} (2 \sin x - \cos x) + \frac{1}{1-x} + C}$$

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Question 3

Determine an antiderivative of

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx.$$

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Question 3 - Solution

For $|x| < 1$, set $u = 1 - x^2$, so $du = -2x\,dx$ and $x^2 = 1 - u$.

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int (1-u)u^{-1/2}\,du = -\sqrt{u} + \frac{1}{3}u^{3/2} + C.$$

$$\boxed{-\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2} + C.}$$

Differentiation gives $x/\sqrt{1-x^2} - x\sqrt{1-x^2} = x^3/\sqrt{1-x^2}$.

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Question 4

Evaluate the integral

$$\int \left(\sqrt{x} \ln x + \frac{1}{\sqrt{x}} \right) dx.$$

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Question 4 - Solution

We first rewrite the integrand using exponents:

$$\sqrt{x}\,\ln x + \frac{1}{\sqrt{x}} = x^{1/2}\ln x + x^{-1/2}.$$

For the first term, apply integration by parts with

$$u = \ln x, \qquad dv = x^{1/2}\,dx.$$

Then

$$du = \frac{1}{x}\,dx, \qquad v = \frac{2}{3}x^{3/2}.$$

Thus,

$$\int x^{1/2}\ln x\,dx = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2}\,dx = \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2}.$$

For the second term,

$$\int x^{-1/2}\,dx = 2x^{1/2}.$$

Combining results:

$$\boxed{\frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + 2x^{1/2} + C}$$

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Question 5

Find an antiderivative of

$$\int \left(\frac{1}{x} \arctan x + \frac{x^2}{1+x^2} \right) dx.$$

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Question 5 - Solution

Split the integral:

$$\int \left(\frac{1}{x}\arctan x + \frac{x^2}{1+x^2}\right) dx = \int \frac{\arctan x}{x}\, dx + \int \frac{x^2}{1+x^2}\, dx.$$

1) The rational part

Rewrite:

$$\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}.$$

So

$$\int \frac{x^2}{1+x^2}\, dx = \int \left(1 - \frac{1}{1+x^2}\right) dx = x - \arctan x.$$

2) The non-elementary part $\displaystyle\int \frac{\arctan x}{x}\, dx$

A common attempt is integration by parts, which produces

$$\int \frac{\arctan x}{x}\, dx = \arctan x\, \ln|x| - \int \frac{\ln|x|}{1+x^2}\, dx,$$

and then the substitution $x = \tan\theta$ turns the remaining integral into

$$\int \frac{\ln|x|}{1+x^2}\, dx = \int \ln(\tan\theta)\, d\theta.$$

The key clarification is:

$\displaystyle\int \ln(\tan\theta)\, d\theta$ **is not an elementary antiderivative.** So any step claiming

$$\int \ln(\tan\theta)\, d\theta = \theta \ln(\tan\theta) - \tan\theta + C$$

is incorrect (differentiate it to check).

A correct closed form uses the dilogarithm function Li_2. Start from

$$\arctan x = \frac{1}{2i}\left(\ln(1+ix) - \ln(1-ix)\right).$$

Then

$$\int \frac{\arctan x}{x}\, dx = \frac{1}{2i}\int \frac{\ln(1+ix) - \ln(1-ix)}{x}\, dx.$$

Use the identity

$$\frac{d}{dx}\,\mathrm{Li}_2(u) = -\ln(1-u)\,\frac{u'}{u},$$

so, for real x,

$$\frac{d}{dx}\,\mathrm{Li}_2(-ix) = -\frac{\ln(1+ix)}{x}, \qquad \frac{d}{dx}\,\mathrm{Li}_2(ix) = -\frac{\ln(1-ix)}{x}.$$

Therefore,

$$\int \frac{\ln(1+ix)}{x}\, dx = -\,\mathrm{Li}_2(-ix) + C, \qquad \int \frac{\ln(1-ix)}{x}\, dx = -\,\mathrm{Li}_2(ix) + C,$$

and hence

$$\int \frac{\arctan x}{x}\, dx = \frac{1}{2i}\Big(\mathrm{Li}_2(ix) - \mathrm{Li}_2(-ix)\Big) + C.$$

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Final Answer

Combine both parts:

$$\int \left(\frac{1}{x} \arctan x + \frac{x^2}{1+x^2} \right) \, dx = (x - \arctan x) + \frac{1}{2i} \Big(\mathrm{Li}_2(ix) - \mathrm{Li}_2(-ix) \Big) + C.$$

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Question 6

Find the general antiderivative of

$$\int \left(xe^{-x^2} + \ln(1+x) \right) \, dx.$$

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Question 6 - Solution

We evaluate each term separately.

First term:

$$\int xe^{-x^2}\,dx.$$

Let $u = -x^2$, so $du = -2x\,dx$:

$$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^{-x^2}.$$

Second term:

$$\int \ln(1+x)\,dx.$$

Use integration by parts with

$$u = \ln(1+x), \qquad dv = dx.$$

Then

$$du = \frac{1}{1+x}\,dx, \qquad v = x.$$

Thus,

$$\int \ln(1+x)\,dx = x\ln(1+x) - \int \frac{x}{1+x}\,dx.$$

Rewrite the integrand:

$$\frac{x}{1+x} = 1 - \frac{1}{1+x}.$$

Hence,

$$\int \frac{x}{1+x}\,dx = x - \ln(1+x).$$

Combining results:

$$\boxed{-\frac{1}{2}e^{-x^2} + x\ln(1+x) - x + \ln(1+x) + C}$$

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Question 7

Find an antiderivative of

$$\int \left(\frac{\sqrt{1+2x}}{x} - \frac{1}{x\sqrt{1+2x}} \right) dx.$$

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Question 7 - Solution

Combine the terms into a single fraction:

$$\frac{\sqrt{1+2x}}{x} - \frac{1}{x\sqrt{1+2x}} = \frac{(1+2x)-1}{x\sqrt{1+2x}} = \frac{2x}{x\sqrt{1+2x}} = \frac{2}{\sqrt{1+2x}}.$$

Thus the integral simplifies to

$$\int \frac{2}{\sqrt{1+2x}}\,dx.$$

Let

$$u = 1+2x, \qquad du = 2\,dx.$$

Then

$$\int \frac{2}{\sqrt{1+2x}}\,dx = \int u^{-1/2}\,du = 2u^{1/2}.$$

Substitute back:

$$\boxed{2\sqrt{1+2x}+C}$$

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Question 8

Find the general antiderivative of

$$\int \left(\ln\left(x + \sqrt{1+x^2}\right) + \frac{x^2}{\sqrt{1+x^2}} \right) dx.$$

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Question 8 - Solution

Split the integral:

$$\int \ln\left(x+\sqrt{1+x^2}\right)\,dx \;+\; \int \frac{x^2}{\sqrt{1+x^2}}\,dx.$$

Part A: $\displaystyle\int \ln\left(x+\sqrt{1+x^2}\right)\,dx$

Use integration by parts with

$$u=\ln\left(x+\sqrt{1+x^2}\right), \qquad dv=dx, \qquad v=x.$$

Let

$$g(x)=x+\sqrt{1+x^2}.$$

Then

$$du=\frac{g'(x)}{g(x)}\,dx, \qquad g'(x)=1+\frac{x}{\sqrt{1+x^2}}.$$

Thus

$$du=\frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}\,dx=\frac{1}{\sqrt{1+x^2}}\,dx.$$

Therefore,

$$\int \ln\left(x+\sqrt{1+x^2}\right)\,dx = x\ln\left(x+\sqrt{1+x^2}\right)-\int \frac{x}{\sqrt{1+x^2}}\,dx.$$

Let $w=1+x^2$, $dw=2x\,dx$:

$$\int \frac{x}{\sqrt{1+x^2}}\,dx=\frac{1}{2}\int w^{-1/2}\,dw=\sqrt{1+x^2}.$$

Hence,

$$\int \ln\left(x+\sqrt{1+x^2}\right)\,dx = x\ln\left(x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}.$$

Part B: $\displaystyle\int \frac{x^2}{\sqrt{1+x^2}}\,dx$ (use an easier trigonometric substitution)

Let

$$x=\tan\theta, \qquad dx=\sec^2\theta\,d\theta, \qquad \sqrt{1+x^2}=\sec\theta.$$

Then

$$\int \frac{x^2}{\sqrt{1+x^2}}\,dx=\int \frac{\tan^2\theta}{\sec\theta}\,\sec^2\theta\,d\theta=\int \tan^2\theta\,\sec\theta\,d\theta.$$

Use $\tan^2\theta=\sec^2\theta-1$:

$$\int \tan^2\theta\,\sec\theta\,d\theta=\int(\sec^3\theta-\sec\theta)\,d\theta.$$

Evaluate each term directly:

$$\int \sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|,$$

and for $\int \sec^3\theta\,d\theta$, integrate by parts:

$$\int \sec^3\theta\,d\theta=\frac{1}{2}\left(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right).$$

Thus,

$$\int(\sec^3\theta-\sec\theta)\,d\theta=\frac{1}{2}\sec\theta\tan\theta-\frac{1}{2}\ln|\sec\theta+\tan\theta|.$$

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Return to x using $\tan\theta = x$, $\sec\theta = \sqrt{1+x^2}$:

$$\int \frac{x^2}{\sqrt{1+x^2}}\,dx = \frac{1}{2}x\sqrt{1+x^2} - \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right).$$

Combine Parts A and B:

$$\boxed{x\ln\left(x+\sqrt{1+x^2}\right) - \sqrt{1+x^2} + \frac{1}{2}x\sqrt{1+x^2} - \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right) + C}$$

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Question 9

Find the general antiderivative of

$$\int \frac{\arctan x}{x^2}\, dx.$$

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Question 9 - Solution

We use integration by parts.

Let

$$u = \arctan x, \qquad dv = \frac{1}{x^2}\,dx.$$

Then

$$du = \frac{1}{1+x^2}\,dx, \qquad v = -\frac{1}{x}.$$

Applying integration by parts,

$$\int \frac{\arctan x}{x^2}\,dx = -\frac{\arctan x}{x} + \int \frac{1}{x(1+x^2)}\,dx.$$

Now decompose the integrand:

$$\frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}.$$

Thus,

$$\int \frac{1}{x(1+x^2)}\,dx = \ln|x| - \frac{1}{2}\ln(1+x^2).$$

Final Answer:

$$\boxed{-\frac{\arctan x}{x} + \ln|x| - \frac{1}{2}\ln(1+x^2) + C}$$

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Question 10

Find an antiderivative of

$$\int \frac{x \ln(1+x^2)}{1+x^2}\, dx.$$

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Question 10 - Solution

Use the substitution

$$u = \ln(1 + x^2).$$

Then

$$du = \frac{2x}{1 + x^2}\,dx, \qquad \frac{1}{2}du = \frac{x}{1 + x^2}\,dx.$$

Rewrite the integral:

$$\int \frac{x\,\ln(1 + x^2)}{1 + x^2}\,dx = \frac{1}{2}\int u\,du.$$

Integrate:

$$\frac{1}{2}\int u\,du = \frac{1}{4}u^2.$$

Substitute back:

$$\boxed{\frac{1}{4}\big(\ln(1 + x^2)\big)^2 + C}$$

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Question 1

Compute the indefinite integral

$$\int x^2 e^x \, dx.$$

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Question 1 - Solution

Use integration by parts with

$$u = x^2, \qquad dv = e^x\,dx.$$

Then

$$du = 2x\,dx, \qquad v = e^x.$$

So

$$\int x^2 e^x\,dx = x^2 e^x - \int 2xe^x\,dx.$$

Apply integration by parts again to $\int 2xe^x\,dx$:

$$u = 2x, \quad dv = e^x\,dx \Rightarrow du = 2\,dx,\ v = e^x.$$

Thus,

$$\int 2xe^x\,dx = 2xe^x - 2\int e^x\,dx = 2xe^x - 2e^x.$$

Hence,

$$\int x^2 e^x\,dx = e^x(x^2 - 2x + 2) + C.$$

Final Answer:

$$\boxed{\int x^2 e^x\,dx = e^x(x^2 - 2x + 2) + C}$$

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Question 2

Evaluate the integral

$$\int x^2 \ln(x^2 + 1)\, dx.$$

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Question 2 - Solution

Use integration by parts. Let

$$u = \ln(x^2+1), \qquad dv = x^2\,dx.$$

Then

$$du = \frac{2x}{x^2+1}\,dx, \qquad v = \frac{x^3}{3}.$$

Apply integration by parts:

$$\int x^2 \ln(x^2+1)\,dx = \frac{x^3}{3}\ln(x^2+1) - \frac{2}{3}\int \frac{x^4}{x^2+1}\,dx.$$

Simplify the integrand by division:

$$\frac{x^4}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1}.$$

Thus,

$$\int \frac{x^4}{x^2+1}\,dx = \int (x^2-1)\,dx + \int \frac{1}{x^2+1}\,dx = \frac{x^3}{3} - x + \arctan x.$$

Substitute back:

$$\int x^2 \ln(x^2+1)\,dx = \frac{x^3}{3}\ln(x^2+1) - \frac{2}{3}\left(\frac{x^3}{3} - x + \arctan x\right).$$

Final Answer:

$$\boxed{\frac{x^3}{3}\ln(x^2+1) - \frac{2x^3}{9} + \frac{2x}{3} - \frac{2}{3}\arctan x + C}$$

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Question 3

Evaluate the integral

$$\int \frac{x^3}{(x^2+1)^2}\,dx.$$

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Question 3 - Solution

Rewrite the integrand by separating powers of x:

$$\frac{x^3}{(x^2+1)^2} = \frac{x(x^2+1-1)}{(x^2+1)^2} = \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}.$$

Split the integral:

$$\int \frac{x}{x^2+1}\,dx - \int \frac{x}{(x^2+1)^2}\,dx.$$

For the first integral, let $u = x^2+1$, $du = 2x\,dx$:

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(x^2+1).$$

For the second integral, use the same substitution:

$$\int \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\int u^{-2}\,du = -\frac{1}{2u} = -\frac{1}{2(x^2+1)}.$$

Combine results:

$$\boxed{\frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C}$$

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Question 4

Evaluate the integral

$$\int x\sqrt{x^2+4}\,\ln(x^2+4)\,dx.$$

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Question 4 - Solution

Use substitution. Let

$$u = x^2 + 4, \qquad du = 2x\,dx.$$

Then

$$\int x\sqrt{x^2+4}\,\ln(x^2+4)\,dx = \frac{1}{2}\int u^{1/2}\ln u\,du.$$

Apply integration by parts with

$$w = \ln u, \qquad dv = u^{1/2}\,du.$$

Then

$$dw = \frac{1}{u}\,du, \qquad v = \frac{2}{3}u^{3/2}.$$

Thus,

$$\int u^{1/2}\ln u\,du = \frac{2}{3}u^{3/2}\ln u - \frac{2}{3}\int u^{1/2}\,du = \frac{2}{3}u^{3/2}\ln u - \frac{4}{9}u^{3/2}.$$

Multiply by $\frac{1}{2}$ and substitute back:

$$\boxed{\frac{1}{3}(x^2+4)^{3/2}\ln(x^2+4) - \frac{2}{9}(x^2+4)^{3/2} + C}$$

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Question 5

Evaluate

$$\int \frac{x^2}{\sqrt{x^2+1}}\,dx.$$

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Question 5 - Solution

Step 1: Rewrite the integrand

Write $x^2 = (x^2+1) - 1$:

$$\frac{x^2}{\sqrt{x^2+1}} = \frac{(x^2+1)-1}{\sqrt{x^2+1}} = \sqrt{x^2+1} - \frac{1}{\sqrt{x^2+1}}.$$

Thus,

$$\int \frac{x^2}{\sqrt{x^2+1}}\,dx = \int \sqrt{x^2+1}\,dx - \int \frac{1}{\sqrt{x^2+1}}\,dx.$$

Step 2: Trigonometric substitution

Let

$$x = \tan\theta, \qquad dx = \sec^2\theta\,d\theta.$$

Then

$$\sqrt{x^2+1} = \sqrt{\tan^2\theta + 1} = \sec\theta.$$

Step 3: Evaluate $\int \sqrt{x^2+1}\,dx$

Substitute:

$$\int \sqrt{x^2+1}\,dx = \int \sec\theta\,(\sec^2\theta\,d\theta) = \int \sec^3\theta\,d\theta.$$

To evaluate $\int \sec^3\theta\,d\theta$, write

$$\sec^3\theta = \sec\theta\sec^2\theta.$$

Let

$$u = \sec\theta, \qquad dv = \sec^2\theta\,d\theta.$$

Then

$$du = \sec\theta\tan\theta\,d\theta, \qquad v = \tan\theta.$$

By integration by parts,

$$\int \sec^3\theta\,d\theta = \sec\theta\tan\theta - \int \tan\theta(\sec\theta\tan\theta)\,d\theta.$$

Simplify the integrand:

$$\tan^2\theta = \sec^2\theta - 1,$$

so

$$\int \sec^3\theta\,d\theta = \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\,d\theta.$$

Distribute:

$$= \sec\theta\tan\theta - \int \sec^3\theta\,d\theta + \int \sec\theta\,d\theta.$$

Move the repeated integral to the left:

$$2\int \sec^3\theta\,d\theta = \sec\theta\tan\theta + \int \sec\theta\,d\theta.$$

Now compute $\int \sec\theta\,d\theta$:

$$\int \sec\theta\,d\theta = \int \frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}\,d\theta.$$

Let

$$u = \sec\theta + \tan\theta, \qquad du = (\sec\theta\tan\theta + \sec^2\theta)\,d\theta = \sec\theta(\sec\theta+\tan\theta)\,d\theta.$$

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Hence,

$$\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C.$$

Substitute back:

$$2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|.$$

Therefore,

$$\int \sec^3\theta\, d\theta = \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C.$$

Returning to x:

$$\sec\theta = \sqrt{x^2+1}, \qquad \tan\theta = x.$$

Thus,

$$\int \sqrt{x^2+1}\, dx = \frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left(x + \sqrt{x^2+1}\right)\right) + C.$$

Step 4: Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\, dx$

Substitute again:

$$\int \frac{1}{\sqrt{x^2+1}}\, dx = \int \frac{1}{\sec\theta}(\sec^2\theta\, d\theta) = \int \sec\theta\, d\theta.$$

From above,

$$\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left(x + \sqrt{x^2+1}\right) + C.$$

Step 5: Combine results

$$\int \frac{x^2}{\sqrt{x^2+1}}\, dx = \frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left(x + \sqrt{x^2+1}\right)\right) - \ln\left(x + \sqrt{x^2+1}\right) + C.$$

Combine logarithms:

$$= \frac{x}{2}\sqrt{x^2+1} - \frac{1}{2}\ln\left(x + \sqrt{x^2+1}\right) + C.$$

Final Answer:

$$\boxed{\int \frac{x^2}{\sqrt{x^2+1}}\, dx = \frac{x}{2}\sqrt{x^2+1} - \frac{1}{2}\ln\left(x + \sqrt{x^2+1}\right) + C}$$

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Question 6

Evaluate the integral

$$\int \frac{x^3 \ln x}{1+x^2}\, dx.$$

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Question 6 - Solution

Rewrite the integrand by splitting the rational expression:

$$\frac{x^3}{1+x^2} = x - \frac{x}{1+x^2}.$$

Thus,

$$\int \frac{x^3 \ln x}{1+x^2}\,dx = \int x \ln x\,dx - \int \frac{x \ln x}{1+x^2}\,dx.$$

First integral: Use integration by parts with

$$u = \ln x, \qquad dv = x\,dx.$$

Then

$$du = \frac{1}{x}\,dx, \qquad v = \frac{x^2}{2}.$$

So

$$\int x \ln x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4}.$$

Second integral: Use integration by parts with

$$u = \ln x, \qquad dv = \frac{x}{1+x^2}\,dx.$$

Then

$$du = \frac{1}{x}\,dx, \qquad v = \frac{1}{2}\ln(1+x^2).$$

Hence,

$$\int \frac{x \ln x}{1+x^2}\,dx = \frac{1}{2}\ln x\,\ln(1+x^2) - \frac{1}{2}\int \frac{\ln(1+x^2)}{x}\,dx.$$

The remaining integral is non-elementary and is expressed using the dilogarithm function:

$$\int \frac{\ln(1+x^2)}{x}\,dx = -\frac{1}{2}\,\mathrm{Li}_2(-x^2).$$

Therefore,

$$\int \frac{x \ln x}{1+x^2}\,dx = \frac{1}{2}\ln x\,\ln(1+x^2) + \frac{1}{4}\,\mathrm{Li}_2(-x^2).$$

Final Answer:

$$\boxed{\frac{x^2}{2}\ln x - \frac{x^2}{4} - \frac{1}{2}\ln x\,\ln(1+x^2) - \frac{1}{4}\,\mathrm{Li}_2(-x^2) + C}$$

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Question 7

Evaluate the integral

$$\int \frac{\sqrt{x^2+1}}{x}\,dx.$$

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Question 7 - Solution

For $x \neq 0$, write $s = \sqrt{1+x^2}$, so $ds = (x/s)\,dx$ and $x^2 = s^2 - 1$.

$$\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{s^2}{s^2-1}\,ds = s + \frac{1}{2}\ln\left|\frac{s-1}{s+1}\right| + C.$$

Since $(s-1)(s+1) = x^2$, this becomes

$$\boxed{\sqrt{1+x^2} + \ln\left|\frac{x}{1+\sqrt{1+x^2}}\right| + C.}$$

The formula is valid on either interval $(-\infty, 0)$ or $(0, \infty)$; constants may differ between intervals.

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Question 8

Evaluate the integral

$$\int (x^2+1)\ln(x^2+1)\,dx.$$

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Question 8 - Solution

Using substitution does not work!

$$u = x^2 + 1, \qquad du = 2x\,dx.$$

Rewrite the integrand:

$$(x^2+1)\ln(x^2+1)\,dx = u\ln u \cdot \frac{du}{2x}.$$

Instead, apply integration by parts directly.
Let

$$u = \ln(x^2+1), \qquad dv = (x^2+1)\,dx.$$

Then

$$du = \frac{2x}{x^2+1}\,dx, \qquad v = \frac{x^3}{3} + x.$$

Apply integration by parts:

$$\int (x^2+1)\ln(x^2+1)\,dx = \left(\frac{x^3}{3}+x\right)\ln(x^2+1) - \int \left(\frac{x^3}{3}+x\right)\frac{2x}{x^2+1}\,dx.$$

Simplify the remaining integral:

$$\left(\frac{x^3}{3}+x\right)\frac{2x}{x^2+1} = \frac{2x^4}{3(x^2+1)} + \frac{2x^2}{x^2+1}.$$

Use algebraic division:

$$\frac{x^4}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1}.$$

Thus,

$$\int (x^2+1)\ln(x^2+1)\,dx = \left(\frac{x^3}{3}+x\right)\ln(x^2+1) - \int \left(\frac{2}{3}(x^2-1) + \frac{2}{3(x^2+1)} + \frac{2x^2}{x^2+1}\right)dx.$$

Compute term by term:

$$\int (x^2-1)\,dx = \frac{x^3}{3} - x,$$

$$\int \frac{1}{x^2+1}\,dx = \arctan x, \qquad \int \frac{x^2}{x^2+1}\,dx = x - \arctan x.$$

Putting everything together,

$$\boxed{\int (x^2+1)\ln(x^2+1)\,dx = \left(\frac{x^3}{3}+x\right)\ln(x^2+1) - \frac{2x^3}{9} - \frac{4x}{3} + \frac{4}{3}\arctan x + C}$$

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Question 9

Evaluate the integral

$$\int \frac{x^4}{(1+x^2)^{3/2}}\,dx.$$

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Question 9 - Solution

Use the trigonometric substitution

$$x = \tan\theta, \qquad dx = \sec^2\theta\, d\theta.$$

Then

$$1 + x^2 = \sec^2\theta, \qquad (1+x^2)^{3/2} = \sec^3\theta.$$

Substitute into the integral:

$$\int \frac{\tan^4\theta}{\sec^3\theta}\sec^2\theta\, d\theta = \int \tan^4\theta\cos\theta\, d\theta.$$

Rewrite $\tan^4\theta$:

$$\tan^4\theta = (\sec^2\theta - 1)^2 = \sec^4\theta - 2\sec^2\theta + 1.$$

Thus,

$$\int (\sec^3\theta - 2\sec\theta + \cos\theta)\, d\theta.$$

Integrate term by term:

$$\int \sec^3\theta\, d\theta = \tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln(\sec\theta + \tan\theta),$$

$$\int \sec\theta\, d\theta = \ln(\sec\theta + \tan\theta), \qquad \int \cos\theta\, d\theta = \sin\theta.$$

Combining,

$$\frac{1}{2}\sec\theta\tan\theta - \frac{3}{2}\ln(\sec\theta + \tan\theta) + \sin\theta + C.$$

Substitute back:

$$\sec\theta = \sqrt{1+x^2}, \qquad \tan\theta = x, \qquad \sin\theta = \frac{x}{\sqrt{1+x^2}}.$$

Therefore,

$$\boxed{\frac{x}{2}\sqrt{1+x^2} - \frac{3}{2}\ln\left(x + \sqrt{1+x^2}\right) + \frac{x}{\sqrt{1+x^2}} + C}$$

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Question 10

Evaluate the indefinite integral:

$$\int \frac{x^2 + 2x + 2}{\sqrt{x^2 + 2x + 5}}\, dx.$$

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Question 10 - Solution

We are given:

$$\int \frac{x^2+2x+2}{\sqrt{x^2+2x+5}}\,dx$$

Rewrite the numerator:

$$x^2+2x+2=(x^2+2x+5)-3.$$

Then

$$\int \frac{x^2+2x+2}{\sqrt{x^2+2x+5}}\,dx=\int \sqrt{x^2+2x+5}\,dx-3\int \frac{dx}{\sqrt{x^2+2x+5}}.$$

Complete the square:

$$x^2+2x+5=(x+1)^2+4.$$

Let

$$t=x+1.$$

Then

$$x^2+2x+5=t^2+4$$

and

$$dt=dx.$$

So the integral becomes:

$$\int \sqrt{t^2+4}\,dt-3\int \frac{dt}{\sqrt{t^2+4}}.$$

Using the formulas

$$\int \sqrt{t^2+a^2}\,dt=\frac{t}{2}\sqrt{t^2+a^2}+\frac{a^2}{2}\ln\left|t+\sqrt{t^2+a^2}\right|$$

and

$$\int \frac{dt}{\sqrt{t^2+a^2}}=\ln\left|t+\sqrt{t^2+a^2}\right|,$$

with $a=2$, we get:

$$\int \sqrt{t^2+4}\,dt=\frac{t}{2}\sqrt{t^2+4}+2\ln\left|t+\sqrt{t^2+4}\right|$$

and

$$3\int \frac{dt}{\sqrt{t^2+4}}=3\ln\left|t+\sqrt{t^2+4}\right|.$$

Therefore,

$$\int \frac{x^2+2x+2}{\sqrt{x^2+2x+5}}\,dx=\frac{t}{2}\sqrt{t^2+4}+2\ln\left|t+\sqrt{t^2+4}\right|-3\ln\left|t+\sqrt{t^2+4}\right|.$$

Simplify:

$$=\frac{t}{2}\sqrt{t^2+4}-\ln\left|t+\sqrt{t^2+4}\right|+C.$$

Substitute back $t=x+1$:

$$\boxed{\frac{x+1}{2}\sqrt{x^2+2x+5}-\ln\left|x+1+\sqrt{x^2+2x+5}\right|+C}$$

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Question 1

Evaluate the integral

$$\int \frac{x\, e^{\sqrt{1+x^2}}}{\sqrt{1+x^2}}\, dx.$$

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Question 1 - Solution

Use substitution. Let

$$u = \sqrt{1+x^2}.$$

Then

$$u^2 = 1 + x^2 \quad \Rightarrow \quad 2u\,du = 2x\,dx \quad \Rightarrow \quad x\,dx = u\,du.$$

Substitute into the integral:

$$\int \frac{x\,e^{\sqrt{1+x^2}}}{\sqrt{1+x^2}}\,dx = \int \frac{u\,e^u}{u}\,du = \int e^u\,du.$$

Integrate:

$$\int e^u\,du = e^u.$$

Substitute back:

$$\boxed{e^{\sqrt{1+x^2}} + C}$$

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Question 2

Evaluate the integral

$$\int \frac{\ln(x^2+1)}{x^2+1}\, x\, dx.$$

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Question 2 - Solution

Use substitution. Let

$$u = \ln(x^2 + 1).$$

Then

$$du = \frac{2x}{x^2+1}\,dx \quad \Rightarrow \quad \frac{x}{x^2+1}\,dx = \frac{1}{2}\,du.$$

Substitute:

$$\int \frac{\ln(x^2+1)}{x^2+1}\,x\,dx = \frac{1}{2}\int u\,du.$$

Integrate:

$$\frac{1}{2}\int u\,du = \frac{1}{4}u^2.$$

Substitute back:

$$\boxed{\frac{1}{4}\big(\ln(x^2+1)\big)^2 + C}$$

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Question 3

Evaluate the integral

$$\int \frac{x^3}{\sqrt{1+x^4}}\,dx.$$

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Question 3 - Solution

Use substitution. Let

$$u = 1 + x^4.$$

Then

$$du = 4x^3\,dx \quad \Rightarrow \quad x^3\,dx = \frac{1}{4}\,du.$$

Substitute into the integral:

$$\int \frac{x^3}{\sqrt{1+x^4}}\,dx = \frac{1}{4}\int u^{-1/2}\,du.$$

Integrate:

$$\frac{1}{4}\int u^{-1/2}\,du = \frac{1}{4}\cdot 2u^{1/2} = \frac{1}{2}\sqrt{u}.$$

Substitute back:

$$\boxed{\frac{1}{2}\sqrt{1+x^4}+C}$$

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Question 4

Evaluate the integral

$$\int \frac{x \sin(x^2)}{\sqrt{1+\cos(x^2)}}\, dx.$$

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Question 4 - Solution

Use substitution. Let

$$u = \cos(x^2).$$

Then

$$du = -\sin(x^2) \cdot 2x\,dx \quad \Rightarrow \quad -\frac{1}{2}\,du = x\sin(x^2)\,dx.$$

Substitute into the integral:

$$\int \frac{x\,\sin(x^2)}{\sqrt{1+\cos(x^2)}}\,dx = -\frac{1}{2}\int \frac{du}{\sqrt{1+u}}.$$

Integrate:

$$-\frac{1}{2}\int (1+u)^{-1/2}\,du = -\frac{1}{2}\cdot 2(1+u)^{1/2} = -\sqrt{1+u}.$$

Substitute back:

$$\boxed{-\sqrt{1+\cos(x^2)}+C}$$

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Question 5

Evaluate the integral

$$\int \frac{2x}{1+\sqrt{1+x^2}}\,dx.$$

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Question 5 - Solution

Use substitution. Let

$$u = \sqrt{1+x^2}.$$

Then

$$u^2 = 1 + x^2 \quad \Rightarrow \quad 2u\,du = 2x\,dx \quad \Rightarrow \quad 2x\,dx = 2u\,du.$$

Substitute into the integral:

$$\int \frac{2x}{1+\sqrt{1+x^2}}\,dx = \int \frac{2u}{1+u}\,du.$$

Rewrite the integrand:

$$\frac{2u}{1+u} = 2\left(1 - \frac{1}{1+u}\right).$$

Integrate:

$$\int 2\left(1 - \frac{1}{1+u}\right)\,du = 2u - 2\ln(1+u).$$

Substitute back:

$$\boxed{2\sqrt{1+x^2} - 2\ln\left(1+\sqrt{1+x^2}\right) + C}$$

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Question 6

Evaluate the integral

$$\int \frac{x\, e^{1/(1+x^2)}}{(1+x^2)^2}\, dx.$$

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Question 6 - Solution

Use substitution. Let

$$u = \frac{1}{1+x^2}.$$

Then

$$du = -\frac{2x}{(1+x^2)^2}\,dx \quad \Rightarrow \quad -\frac{1}{2}\,du = \frac{x}{(1+x^2)^2}\,dx.$$

Substitute into the integral:

$$\int \frac{x\,e^{1/(1+x^2)}}{(1+x^2)^2}\,dx = -\frac{1}{2}\int e^u\,du.$$

Integrate:

$$-\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u.$$

Substitute back:

$$\boxed{-\frac{1}{2}e^{1/(1+x^2)} + C}$$

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Question 7

Evaluate

$$\int \frac{x}{\sqrt{1-x^2}} \ln\left(1+\sqrt{1-x^2}\right) dx.$$

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Question 7 - Solution

Let

$$u = \sqrt{1 - x^2}.$$

Then

$$du = \frac{-x}{\sqrt{1 - x^2}}\,dx \qquad \Rightarrow \qquad \frac{x}{\sqrt{1 - x^2}}\,dx = -du.$$

So the integral becomes

$$-\int \ln(1 + u)\,du.$$

Now integrate by parts. Take

$$w = \ln(1 + u), \qquad dv = du,$$

so

$$dw = \frac{1}{1 + u}\,du, \qquad v = u.$$

Then

$$-\int \ln(1 + u)\,du = -\left(u \ln(1 + u) - \int \frac{u}{1 + u}\,du\right).$$

Simplify

$$\frac{u}{1 + u} = 1 - \frac{1}{1 + u},$$

hence

$$\int \frac{u}{1 + u}\,du = \int \left(1 - \frac{1}{1 + u}\right) du = u - \ln(1 + u).$$

Therefore,

$$-\left(u \ln(1 + u) - u + \ln(1 + u)\right) = -u \ln(1 + u) + u - \ln(1 + u) + C.$$

Substitute back $u = \sqrt{1 - x^2}$:

$$\boxed{-\sqrt{1 - x^2}\,\ln\!\left(1 + \sqrt{1 - x^2}\right) + \sqrt{1 - x^2} - \ln\!\left(1 + \sqrt{1 - x^2}\right) + C}$$

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Question 8

Evaluate the integral

$$\int \frac{3x^2}{1+x^3}\sqrt{\ln(1+x^3)}\,dx.$$

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Question 8 - Solution

Use substitution. Let

$$u = \ln(1 + x^3).$$

Then

$$du = \frac{3x^2}{1 + x^3}\, dx.$$

Substitute into the integral:

$$\int \frac{3x^2}{1 + x^3} \sqrt{\ln(1 + x^3)}\, dx = \int \sqrt{u}\, du.$$

Integrate:

$$\int u^{1/2}\, du = \frac{2}{3} u^{3/2}.$$

Substitute back:

$$\boxed{\frac{2}{3}\big(\ln(1 + x^3)\big)^{3/2} + C}$$

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Question 9

Evaluate the integral

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx.$$

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Question 9 - Solution

Rewrite the integrand to reveal a derivative pattern:

$$2\cos(\sqrt{x}) \cdot \frac{d}{dx}(\sqrt{x}).$$

Let

$$u = \sqrt{x}.$$

Then

$$du = \frac{1}{2\sqrt{x}}\,dx \quad \Rightarrow \quad dx = 2\sqrt{x}\,du.$$

Substitute into the integral:

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx = \int \frac{\cos(u)}{\sqrt{x}} \cdot 2\sqrt{x}\,du = 2\int \cos u\,du.$$

Integrate:

$$2\int \cos u\,du = 2\sin u.$$

Substitute back:

$$\boxed{2\sin(\sqrt{x}) + C}$$

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Question 10

Evaluate the integral

$$\int x \ln(1 - x^2)\, dx.$$

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Question 10 - Solution

Let

$$u = 1 - x^2.$$

Then

$$du = -2x\,dx.$$

Rewrite the integral:

$$\int x \ln(1 - x^2)\,dx = -\frac{1}{2}\int \ln u\,du.$$

Now integrate:

$$\int \ln u\,du = u \ln u - u.$$

Therefore,

$$-\frac{1}{2}(u \ln u - u) + C.$$

Substitute back:

$$\boxed{-\frac{1}{2}\left[(1 - x^2)\ln(1 - x^2) - (1 - x^2)\right] + C}$$

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Question 1

Evaluate the integral

$$\int \frac{x}{1+x^2}\sqrt{1+\ln(1+x^2)}\,dx.$$

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Question 1 - Solution

Step 1: Choose a substitution

Set $u = 1 + \ln(1 + x^2)$, so $du = 2x/(1 + x^2)\,dx$.

$$\int \frac{x}{1+x^2}\sqrt{1+\ln(1+x^2)}\,dx = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{3}u^{3/2} + C.$$

$$\boxed{\frac{1}{3}\big(1+\ln(1+x^2)\big)^{3/2} + C.}$$

This is valid for every real x.

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Question 2

Evaluate the integral

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx.$$

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Question 2 - Solution

Because the integrand contains $\sqrt{4-x^2}$, a trigonometric substitution is natural.

Let

$$x = 2\sin\theta.$$

Then

$$dx = 2\cos\theta\,d\theta, \qquad \sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta.$$

Substitute into the integral:

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx = \int \frac{4\sin^2\theta}{2\cos\theta}\cdot 2\cos\theta\,d\theta = \int 4\sin^2\theta\,d\theta.$$

Use the identity

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}.$$

Then

$$\int 4\sin^2\theta\,d\theta = 4\int \frac{1-\cos(2\theta)}{2}\,d\theta = 2\int (1-\cos(2\theta))\,d\theta.$$

Integrate:

$$2\left(\theta - \frac{1}{2}\sin(2\theta)\right) = 2\theta - \sin(2\theta).$$

Return to x using

$$\theta = \arcsin\left(\frac{x}{2}\right), \qquad \sin(2\theta) = 2\sin\theta\cos\theta = \frac{x}{2}\sqrt{4-x^2}.$$

Thus,

$$\boxed{2\arcsin\left(\frac{x}{2}\right) - \frac{x}{2}\sqrt{4-x^2} + C}$$

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Question 3

Evaluate the integral

$$\int \frac{x^2}{\sqrt{x^2-9}}\,dx, \qquad x > 3.$$

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Question 3 - Solution

Because the integrand involves $\sqrt{x^2-a^2}$, a *hyperbolic substitution* is natural.

Let

$$x = 3\cosh t.$$

Then

$$dx = 3\sinh t\,dt, \qquad \sqrt{x^2-9} = \sqrt{9\cosh^2 t - 9} = 3\sinh t.$$

Substitute into the integral:

$$\int \frac{x^2}{\sqrt{x^2-9}}\,dx = \int \frac{9\cosh^2 t}{3\sinh t}\cdot 3\sinh t\,dt = \int 9\cosh^2 t\,dt.$$

Use the identity

$$\cosh^2 t = \frac{1+\cosh(2t)}{2}.$$

Then

$$\int 9\cosh^2 t\,dt = \frac{9}{2}\int \big(1+\cosh(2t)\big)\,dt = \frac{9}{2}\left(t + \frac{1}{2}\sinh(2t)\right).$$

Simplify:

$$\frac{9}{2}t + \frac{9}{4}\sinh(2t).$$

Return to x. Since

$$x = 3\cosh t \quad \Rightarrow \quad t = \operatorname{arcosh}\left(\frac{x}{3}\right),$$

and

$$\sinh(2t) = 2\sinh t\cosh t = 2\left(\frac{\sqrt{x^2-9}}{3}\right)\left(\frac{x}{3}\right) = \frac{2x\sqrt{x^2-9}}{9}.$$

Substitute back:

$$\boxed{\frac{9}{2}\operatorname{arcosh}\left(\frac{x}{3}\right) + \frac{x}{2}\sqrt{x^2-9} + C}$$

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Question 4

Evaluate the integral

$$\int \frac{x}{\sqrt{x^2+2x+5}}\,dx.$$

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Question 4 - Solution

Complete the square:

$$x^2 + 2x + 5 = (x+1)^2 + 4.$$

Let

$$u = x + 1, \qquad du = dx, \qquad x = u - 1.$$

Then the integral becomes

$$\int \frac{u-1}{\sqrt{u^2+4}}\,du = \int \frac{u}{\sqrt{u^2+4}}\,du - \int \frac{1}{\sqrt{u^2+4}}\,du.$$

First integral: Let $w = u^2 + 4$, so $dw = 2u\,du$:

$$\int \frac{u}{\sqrt{u^2+4}}\,du = \frac{1}{2}\int w^{-1/2}\,dw = \sqrt{u^2+4}.$$

Second integral (expanded):

$$\int \frac{1}{\sqrt{u^2+4}}\,du.$$

Use a trigonometric substitution. Let

$$u = 2\tan\theta.$$

Then

$$du = 2\sec^2\theta\,d\theta, \qquad \sqrt{u^2+4} = \sqrt{4\tan^2\theta + 4} = 2\sec\theta.$$

Substitute:

$$\int \frac{1}{\sqrt{u^2+4}}\,du = \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta = \int \sec\theta\,d\theta.$$

Recall:

$$\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|.$$

Return to u:

$$\tan\theta = \frac{u}{2}, \qquad \sec\theta = \frac{\sqrt{u^2+4}}{2}.$$

Thus,

$$\int \frac{1}{\sqrt{u^2+4}}\,du = \ln\left(u + \sqrt{u^2+4}\right).$$

Final Answer:

$$\boxed{\sqrt{x^2+2x+5} - \ln\left(x+1+\sqrt{x^2+2x+5}\right) + C}$$

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Question 5

Evaluate the integral

$$\int \frac{dx}{1+\sin x}.$$

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Question 5 - Solution

The integrand is defined wherever $1 + \sin x \neq 0$.

A primitive valid on each such interval is

$$\boxed{-\frac{\cos x}{1+\sin x} + C.}$$

Indeed, the quotient rule gives

$$\frac{d}{dx}\left(-\frac{\cos x}{1+\sin x}\right) = \frac{\sin x(1+\sin x) + \cos^2 x}{(1+\sin x)^2} = \frac{1}{1+\sin x}.$$

This expression introduces no extra excluded points.

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Question 6

Evaluate the integral

$$\int \sqrt{x^2 + x}\, dx.$$

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Question 6 - Solution

Because the integrand involves a square root of a quadratic expression, an *Euler substitution* is effective.

Complete the square:

$$x^2 + x = \left(x + \tfrac{1}{2}\right)^2 - \tfrac{1}{4}.$$

Use the substitution

$$x + \tfrac{1}{2} = \frac{1}{2}\left(t - \frac{1}{t}\right), \qquad t > 0.$$

Then

$$dx = \frac{1}{2}\left(1 + \frac{1}{t^2}\right) dt,$$

and

$$\sqrt{x^2 + x} = \sqrt{\frac{1}{4}\left(t - \frac{1}{t}\right)^2 - \frac{1}{4}} = \frac{1}{2}\left(t - \frac{1}{t}\right).$$

Substitute into the integral:

$$\int \sqrt{x^2 + x}\, dx = \int \frac{1}{2}\left(t - \frac{1}{t}\right) \cdot \frac{1}{2}\left(1 + \frac{1}{t^2}\right) dt.$$

Simplify:

$$\frac{1}{4}\int \left(t - \frac{1}{t^3}\right) dt.$$

Integrate:

$$\frac{1}{4}\left(\frac{t^2}{2} + \frac{1}{2t^2}\right) = \frac{1}{8}\left(t^2 + \frac{1}{t^2}\right).$$

Substitute back using

$$t - \frac{1}{t} = 2\left(x + \tfrac{1}{2}\right).$$

Thus,

$$\boxed{\frac{1}{2}\left(x + \tfrac{1}{2}\right)\sqrt{x^2 + x} - \frac{1}{8}\ln\left|2x + 1 + 2\sqrt{x^2 + x}\right| + C}$$

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Question 7

Evaluate the integral

$$\int \sqrt{x^2 + x}\, dx.$$

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Question 7 - Solution

Begin by completing the square:

$$x^2 + x = \left(x + \tfrac{1}{2}\right)^2 - \tfrac{1}{4}.$$

Let

$$u = x + \tfrac{1}{2}.$$

Then

$$du = dx, \qquad \sqrt{x^2 + x} = \sqrt{u^2 - \tfrac{1}{4}}.$$

The integral becomes

$$\int \sqrt{u^2 - \tfrac{1}{4}}\, du.$$

Now use a standard substitution for expressions of the form $\sqrt{u^2 - a^2}$. Let

$$u = \tfrac{1}{2}\sec\theta.$$

Then

$$du = \tfrac{1}{2}\sec\theta\tan\theta\, d\theta, \qquad \sqrt{u^2 - \tfrac{1}{4}} = \tfrac{1}{2}\tan\theta.$$

Substitute:

$$\int \sqrt{u^2 - \tfrac{1}{4}}\, du = \int \tfrac{1}{2}\tan\theta \cdot \tfrac{1}{2}\sec\theta\tan\theta\, d\theta = \frac{1}{4}\int \tan^2\theta\sec\theta\, d\theta.$$

Use the identity $\tan^2\theta = \sec^2\theta - 1$:

$$\frac{1}{4}\int (\sec^3\theta - \sec\theta)\, d\theta.$$

Integrate:

$$\int \sec^3\theta\, d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|),$$

$$\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta|.$$

Thus,

$$\frac{1}{4}\left[\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta|\right].$$

Return to x using

$$\sec\theta = 2u = 2x + 1, \qquad \tan\theta = 2\sqrt{x^2 + x}.$$

Final Answer:

$$\boxed{\frac{1}{2}\left(x + \tfrac{1}{2}\right)\sqrt{x^2 + x} - \frac{1}{8}\ln\left|2x + 1 + 2\sqrt{x^2 + x}\right| + C}$$

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Question 8

Evaluate the integral

$$\int \frac{x+2}{\sqrt{x^2+4x+8}}\,dx.$$

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Question 8 - Solution

Complete the square in the denominator:

$$x^2 + 4x + 8 = (x+2)^2 + 4.$$

Let

$$u = x + 2.$$

Then

$$du = dx, \qquad x + 2 = u.$$

Substitute into the integral:

$$\int \frac{x+2}{\sqrt{x^2+4x+8}}\,dx = \int \frac{u}{\sqrt{u^2+4}}\,du.$$

Now use a direct substitution. Let

$$w = u^2 + 4, \qquad dw = 2u\,du.$$

Then

$$\int \frac{u}{\sqrt{u^2+4}}\,du = \frac{1}{2}\int w^{-1/2}\,dw = \sqrt{w}.$$

Substitute back:

$$\boxed{\sqrt{x^2+4x+8} + C}$$

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Question 9

Evaluate the integral

$$\int \frac{x+1}{(x^2+2x+3)^{3/2}}\,dx.$$

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Question 9 - Solution

Begin by completing the square in the denominator:

$$x^2 + 2x + 3 = (x+1)^2 + 2.$$

Let

$$u = x + 1.$$

Then

$$du = dx, \qquad (x^2 + 2x + 3)^{3/2} = (u^2 + 2)^{3/2}.$$

Substitute into the integral:

$$\int \frac{u}{(u^2+2)^{3/2}}\, du.$$

Now use a direct substitution. Let

$$w = u^2 + 2, \qquad dw = 2u\, du.$$

Then

$$\int \frac{u}{(u^2+2)^{3/2}}\, du = \frac{1}{2}\int w^{-3/2}\, dw.$$

Integrate:

$$\frac{1}{2}\int w^{-3/2}\, dw = \frac{1}{2}\left(-2w^{-1/2}\right) = -\frac{1}{\sqrt{w}}.$$

Substitute back:

$$\boxed{-\frac{1}{\sqrt{x^2+2x+3}} + C}$$

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Question 10

Evaluate the integral

$$\int \frac{e^x}{\sqrt{1+e^{2x}}}\,dx.$$

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Question 10 - Solution

Begin with an exponential substitution. Let

$$u = e^x.$$

Then

$$du = e^x\,dx.$$

Substitute into the integral:

$$\int \frac{e^x}{\sqrt{1+e^{2x}}}\,dx = \int \frac{1}{\sqrt{1+u^2}}\,du.$$

Inverse trigonometric step (expanded):

We recognize the integrand as the derivative of $\operatorname{arsinh} u$ or $\arctan u$–type expressions. Use a trigonometric substitution to justify the result.

Let

$$u = \tan\theta.$$

Then

$$du = \sec^2\theta\,d\theta, \qquad \sqrt{1+u^2} = \sqrt{1+\tan^2\theta} = \sec\theta.$$

Substitute:

$$\int \frac{1}{\sqrt{1+u^2}}\,du = \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int \sec\theta\,d\theta.$$

Recall the standard integral:

$$\int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta|.$$

Substitute back using

$$\tan\theta = u, \qquad \sec\theta = \sqrt{1+u^2}.$$

Thus,

$$\int \frac{1}{\sqrt{1+u^2}}\,du = \ln\left(u+\sqrt{1+u^2}\right).$$

Substitute back $u = e^x$:

$$\boxed{\ln\left(e^x+\sqrt{1+e^{2x}}\right)+C}$$

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Question 1

Find the exact area of the region enclosed by the curves

$$y = x\sqrt{4 - x^2} \quad \text{and} \quad y = x^2.$$

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Question 1 - Solution

First find the points of intersection:

$$x\sqrt{4-x^2} = x^2.$$

Factor:

$$x\big(\sqrt{4-x^2} - x\big) = 0.$$

Thus,

$$x = 0 \quad \text{or} \quad \sqrt{4-x^2} = x.$$

Square the second equation:

$$4 - x^2 = x^2 \quad \Rightarrow \quad x^2 = 2 \quad \Rightarrow \quad x = \sqrt{2}.$$

On the interval $[0, \sqrt{2}]$, we have

$$x\sqrt{4-x^2} \geq x^2.$$

Therefore, the area is

$$A = \int_0^{\sqrt{2}} \left(x\sqrt{4-x^2} - x^2\right)\, dx.$$

Split the integral:

$$A = \int_0^{\sqrt{2}} x\sqrt{4-x^2}\, dx - \int_0^{\sqrt{2}} x^2\, dx.$$

For the first integral, use substitution. Let

$$u = 4 - x^2, \qquad du = -2x\, dx.$$

Then

$$\int x\sqrt{4-x^2}\, dx = -\frac{1}{2}\int u^{1/2}\, du = -\frac{1}{3}u^{3/2} = -\frac{1}{3}(4-x^2)^{3/2}.$$

Evaluate from 0 to $\sqrt{2}$:

$$\left[-\frac{1}{3}(4-x^2)^{3/2}\right]_0^{\sqrt{2}} = -\frac{1}{3}(2\sqrt{2}) + \frac{1}{3}(8) = \frac{8-2\sqrt{2}}{3}.$$

For the second integral:

$$\int_0^{\sqrt{2}} x^2\, dx = \left[\frac{x^3}{3}\right]_0^{\sqrt{2}} = \frac{2\sqrt{2}}{3}.$$

Subtract:

$$A = \frac{8-2\sqrt{2}}{3} - \frac{2\sqrt{2}}{3} = \frac{8-4\sqrt{2}}{3}.$$

$$\boxed{\frac{8-4\sqrt{2}}{3}}$$

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Question 2

Find the exact area of the region enclosed by the curves

$$y = \ln x \quad \text{and} \quad y = x - 1,$$

restricted to the interval $0 < x \leq 1$.

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Question 2 - Solution

On the interval $0 < x \leq 1$, the curves intersect where

$$\ln x = x - 1,$$

which occurs at

$$x = 1.$$

For $0 < x < 1$, we have

$$x - 1 \geq \ln x,$$

so the area between the curves is

$$A = \int_0^1 \big((x-1) - \ln x\big)\, dx.$$

Compute each term:

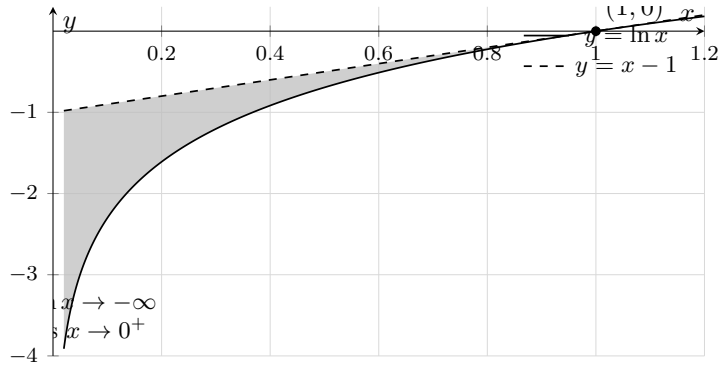
$$\int_0^1 (x-1)\, dx = \left[\frac{x^2}{2} - x\right]_0^1 = -\frac{1}{2},$$

$$\int_0^1 \ln x\, dx = [x \ln x - x]_0^1 = -1.$$

Therefore,

$$A = -\frac{1}{2} - (-1) = \frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$



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Question 3

Find the exact area of the region enclosed by the curves

$$y = \sqrt{x} \quad \text{and} \quad y = x^2,$$

restricted to the interval $0 \leq x \leq 1$.

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Question 3 - Solution

First determine which curve lies above the other on $[0, 1]$. For $0 < x < 1$,

$$\sqrt{x} \geq x^2.$$

Thus, the enclosed area is

$$A = \int_0^1 \left(\sqrt{x} - x^2\right) dx.$$

Evaluate each term:

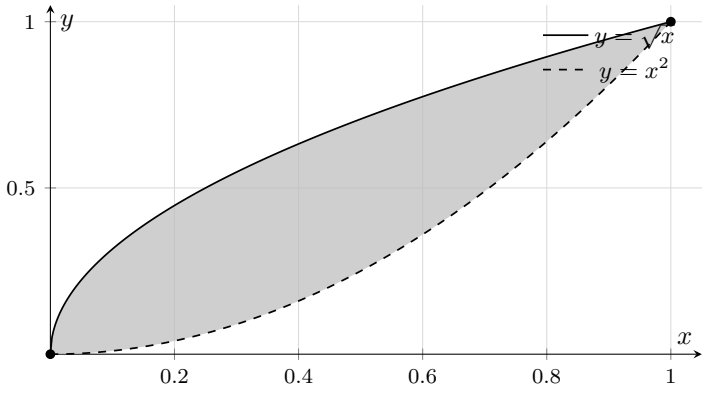
$$\int_0^1 \sqrt{x}\, dx = \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3},$$

$$\int_0^1 x^2\, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

Subtract:

$$A = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

$$\boxed{\frac{1}{3}}$$



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Question 4

Find the exact area of the region enclosed by the curves

$$y = x\sqrt{2-x} \quad \text{and} \quad y = x.$$

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Question 4 - Solution

The intersections satisfy $x\sqrt{2-x} = x$, giving $x = 0, 1$.

For $0 < x < 1$, the square-root curve is above the line, hence

$$A = \int_0^1 \left(x\sqrt{2-x} - x\right) dx.$$

With $u = 2 - x$,

$$\int_0^1 x\sqrt{2-x}\, dx = \int_1^2 (2-u)\sqrt{u}\, du = \left[\frac{4}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_1^2 = \frac{16\sqrt{2}-14}{15}.$$

$$\boxed{A = \frac{32\sqrt{2}-43}{30} > 0.}$$

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Question 5

Find the exact area of the region enclosed by the curves

$$y = |x| \quad \text{and} \quad y = x^2.$$

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Question 5 - Solution

First find the points of intersection by solving

$$|x| = x^2.$$

For $x \geq 0$, this becomes

$$x = x^2 \quad \Rightarrow \quad x = 0 \text{ or } x = 1.$$

For $x < 0$, we have

$$-x = x^2 \quad \Rightarrow \quad x = 0 \text{ or } x = -1.$$

Thus, the curves intersect at

$$x = -1,\ 0,\ 1.$$

On $[0, 1]$, we have $|x| = x \geq x^2$, and by symmetry the same holds on $[-1, 0]$. Therefore, the total area is twice the area on $[0, 1]$:

$$A = 2\int_0^1 (x - x^2)\, dx.$$

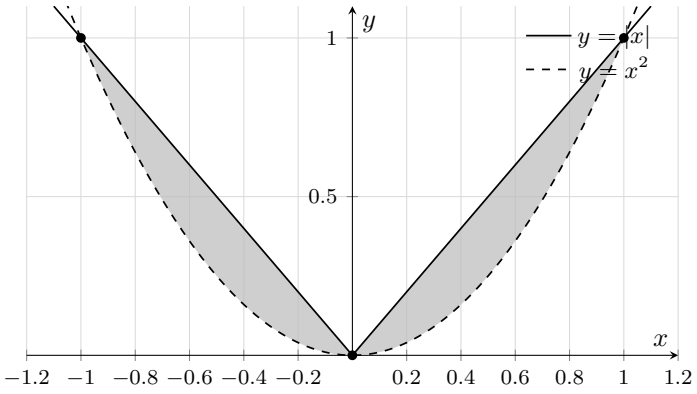
Compute the integral:

$$\int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Thus,

$$A = 2 \cdot \frac{1}{6} = \frac{1}{3}.$$

$$\boxed{\frac{1}{3}}$$



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Question 6

Find the exact area of the region enclosed by the curves

$$y = x \quad \text{and} \quad y = x^3.$$

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Question 6 - Solution

First find the points of intersection:

$$x = x^3 \quad \Rightarrow \quad x(x^2 - 1) = 0.$$

Thus,

$$x = -1,\ 0,\ 1.$$

On the interval $(0, 1)$, we have

$$x \geq x^3,$$

and by symmetry, the same configuration occurs on $(-1, 0)$.

Therefore, the total enclosed area is twice the area on $[0, 1]$:

$$A = 2\int_0^1 (x - x^3)\, dx.$$

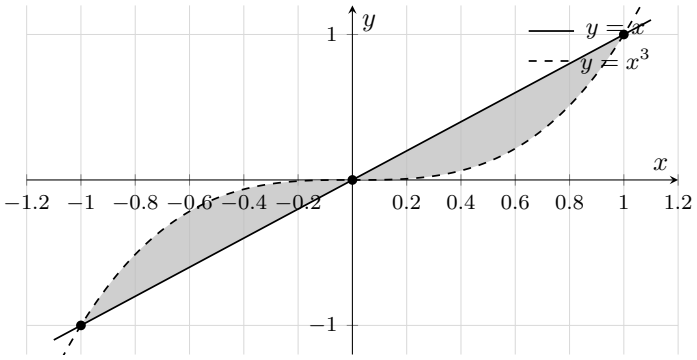
Evaluate the integral:

$$\int_0^1 (x - x^3)\, dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Thus,

$$A = 2 \cdot \frac{1}{4} = \frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$



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Question 7

Find the exact area of the region enclosed by the curves

$$y = \sin x \quad \text{and} \quad y = \cos x$$

on the interval

$$\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}.$$

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Question 7 - Solution

First find the points of intersection by solving

$$\sin x = \cos x.$$

This gives

$$\tan x = 1 \quad \Rightarrow \quad x = \frac{\pi}{4} + k\pi.$$

The smallest two consecutive intersection points occur at

$$x = \frac{\pi}{4} \quad \text{and} \quad x = \frac{5\pi}{4}.$$

On the interval between these points, we have

$$\sin x \geq \cos x.$$

Thus the enclosed area is

$$A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx.$$

Compute:

$$\int \sin x\, dx = -\cos x, \qquad \int \cos x\, dx = \sin x.$$

So

$$A = \left[-\cos x - \sin x\right]_{\pi/4}^{5\pi/4}.$$

Evaluate:

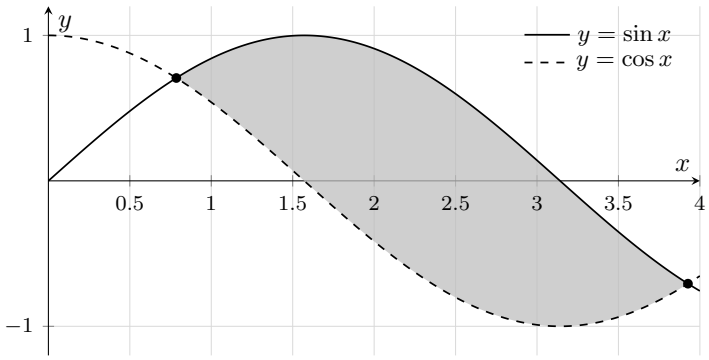
$$-\cos\left(\frac{5\pi}{4}\right) - \sin\left(\frac{5\pi}{4}\right) = \sqrt{2},$$

$$-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right) = -\sqrt{2}.$$

Thus,

$$A = \sqrt{2} - (-\sqrt{2}) = 2\sqrt{2}.$$

$$\boxed{2\sqrt{2}}$$



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Question 8

Find the exact area of the region enclosed by the curves

$$y = e^{-x} \quad \text{and} \quad y = 1 + \left(\frac{1}{e} - 1\right) x.$$

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Question 8 - Solution

First find the intersection points:

$$e^{-x} = 1 + \left(\frac{1}{e} - 1\right) x.$$

At $x = 0$,

$$e^0 = 1 = 1 + \left(\frac{1}{e} - 1\right) \cdot 0.$$

At $x = 1$,

$$e^{-1} = \frac{1}{e} = 1 + \left(\frac{1}{e} - 1\right) \cdot 1.$$

So the curves intersect at $x = 0$ and $x = 1$.

Since e^{-x} is convex, the chord (the line) lies above e^{-x} on $(0, 1)$, so the area is

$$A = \int_0^1 \left(1 + \left(\frac{1}{e} - 1\right) x - e^{-x}\right) dx.$$

Compute each piece:

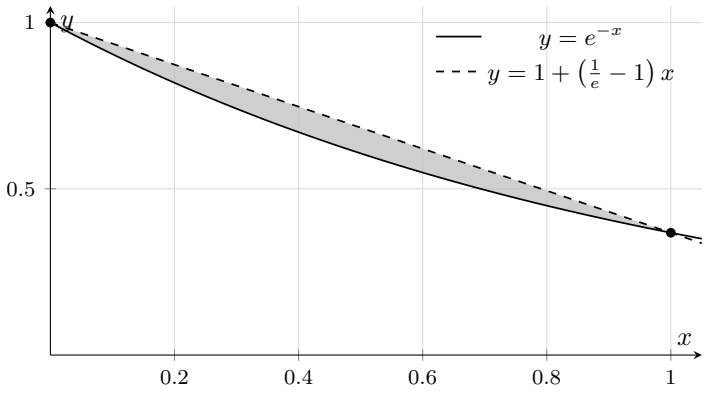
$$\int_0^1 \left(1 + \left(\frac{1}{e} - 1\right) x\right) dx = \left[x + \left(\frac{1}{e} - 1\right) \frac{x^2}{2}\right]_0^1 = 1 + \frac{1}{2}\left(\frac{1}{e} - 1\right) = \frac{1}{2} + \frac{1}{2e},$$

$$\int_0^1 e^{-x}\, dx = \left[-e^{-x}\right]_0^1 = 1 - \frac{1}{e}.$$

Thus

$$A = \left(\frac{1}{2} + \frac{1}{2e}\right) - \left(1 - \frac{1}{e}\right) = -\frac{1}{2} + \frac{3}{2e} = \frac{1}{2}\left(\frac{3}{e} - 1\right).$$

$$\boxed{\frac{1}{2}\left(\frac{3}{e} - 1\right)}$$



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Question 9

Find the exact area of the region enclosed by the curves

$$y = \ln(1+x) \quad \text{and} \quad y = \frac{x}{1+x},$$

between $x = 0$ *and* $x = 1$.

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Question 9 - Solution

On the interval $0 \le x \le 1$, we compare the curves. At $x = 0$,

$$\ln(1) = 0, \qquad \frac{0}{1} = 0.$$

At $x = 1$,

$$\ln 2 > \frac{1}{2},$$

so

$$\ln(1+x) \ge \frac{x}{1+x} \quad \text{on } [0,1].$$

Thus the enclosed area is

$$A = \int_0^1 \left(\ln(1+x) - \frac{x}{1+x} \right) dx.$$

Split the integral:

$$A = \int_0^1 \ln(1+x)\,dx - \int_0^1 \frac{x}{1+x}\,dx.$$

Evaluate the first integral:

$$\int \ln(1+x)\,dx = (1+x)\ln(1+x) - x,$$

so

$$\int_0^1 \ln(1+x)\,dx = \big[(1+x)\ln(1+x) - x\big]_0^1 = 2\ln 2 - 1.$$

For the second integral, rewrite:

$$\frac{x}{1+x} = 1 - \frac{1}{1+x}.$$

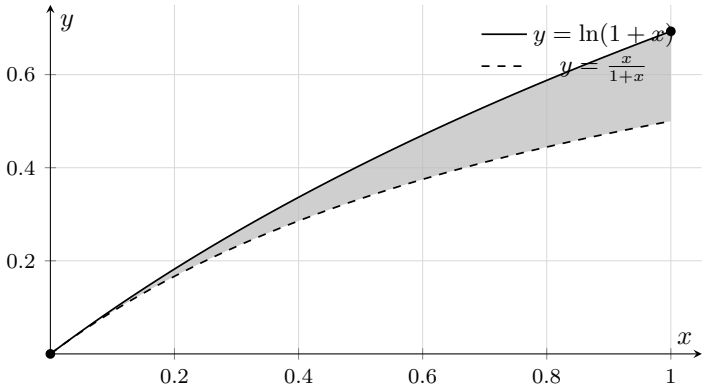
Then

$$\int_0^1 \frac{x}{1+x}\,dx = \int_0^1 \left(1 - \frac{1}{1+x}\right) dx = 1 - \ln 2.$$

Subtract:

$$A = (2\ln 2 - 1) - (1 - \ln 2) = 3\ln 2 - 2.$$

$$\boxed{3\ln 2 - 2}$$



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Question 10

Find the exact area of the region bounded by the curves

$$y = \sqrt{x} \quad \text{and} \quad y = \ln x,$$

between $x = 1$ and $x = e^2$.

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Question 10 - Solution

On $1 \le x \le e^2$, we have $\sqrt{x} \ge \ln x$, so

$$A = \int_1^{e^2} \left(\sqrt{x} - \ln x\right) dx.$$

Compute:

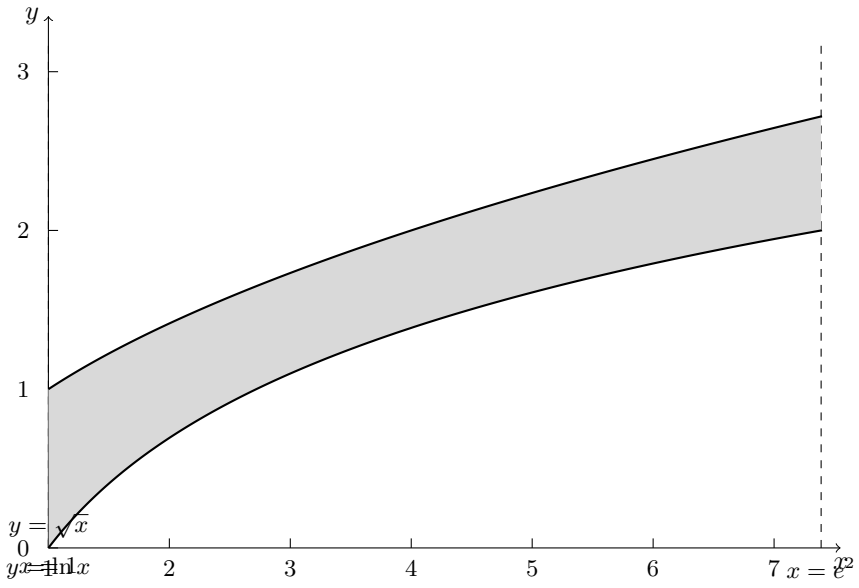
$$\int_1^{e^2} \sqrt{x}\, dx = \left[\frac{2}{3}x^{3/2}\right]_1^{e^2} = \frac{2}{3}(e^3 - 1),$$

$$\int_1^{e^2} \ln x\, dx = \left[x \ln x - x\right]_1^{e^2} = e^2 + 1.$$

Thus

$$A = \frac{2}{3}(e^3 - 1) - (e^2 + 1) = \frac{2}{3}e^3 - e^2 - \frac{5}{3}.$$

$$\boxed{\frac{2}{3}e^3 - e^2 - \frac{5}{3}}$$



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Question 1

Consider the function

$$f(x) = 2x + 1$$

on the interval $[0, 2]$.

Using the definition of the definite integral, express

$$\int_0^2 (2x + 1)\, dx$$

as the limit of a Riemann sum with n equal subintervals.

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Question 1 - Solution

Divide the interval $[0, 2]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{2}{n}.$$

Using right endpoints, the ith sample point is

$$x_i = 0 + i\Delta x = \frac{2i}{n}.$$

Evaluate the function at x_i:

$$f(x_i) = 2\left(\frac{2i}{n}\right) + 1 = \frac{4i}{n} + 1.$$

The Riemann sum is

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \left(\frac{4i}{n} + 1\right)\frac{2}{n}.$$

Simplify:

$$\frac{2}{n}\sum_{i=1}^{n}\left(\frac{4i}{n} + 1\right) = \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{2}{n}\sum_{i=1}^{n} 1.$$

Using

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} 1 = n,$$

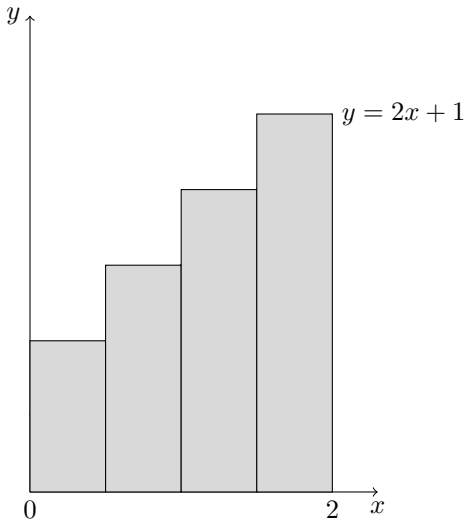
we obtain

$$\frac{8}{n^2}\cdot\frac{n(n+1)}{2} + \frac{2}{n}\cdot n = 4\left(1 + \frac{1}{n}\right) + 2.$$

Thus,

$$\int_0^2 (2x+1)\,dx = \lim_{n\to\infty}\left(6 + \frac{4}{n}\right) = 6.$$

$$\boxed{6}$$



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Question 2

Let

$$f(x) = x^2$$

on the interval $[1, 3]$.

Using the definition of the definite integral, write

$$\int_1^3 x^2 \, dx$$

as the limit of a Riemann sum using midpoints.

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Question 2 - Solution

Divide $[1,3]$ into n equal subintervals of width $\Delta x = 2/n$.

The midpoints are $x_i^* = 1 + (2i-1)/n$.

$$M_n = \frac{2}{n}\sum_{i=1}^{n}\left(1 + \frac{2i-1}{n}\right)^2.$$

Use $\sum(2i-1) = n^2$ and $\sum(2i-1)^2 = n(4n^2-1)/3$:

$$M_n = 2 + 4 + \frac{2(4n^2-1)}{3n^2} = \frac{26}{3} - \frac{2}{3n^2}.$$

$$\boxed{\int_1^3 x^2\,dx = \lim_{n\to\infty} M_n = \frac{26}{3}.}$$

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Question 3

Let

$$f(x) = \sqrt{x}$$

on the interval $[0, 4]$.

Using the definition of the definite integral, write

$$\int_0^4 \sqrt{x}\, dx$$

as the limit of a Riemann sum using left endpoints, and evaluate the limit.

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Question 3 - Solution

Divide the interval $[0, 4]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{4}{n}.$$

Using left endpoints, the ith sample point is

$$x_{i-1} = (i-1)\Delta x = \frac{4(i-1)}{n}.$$

Evaluate the function:

$$f(x_{i-1}) = \sqrt{\frac{4(i-1)}{n}} = 2\sqrt{\frac{i-1}{n}}.$$

The Riemann sum is

$$\sum_{i=1}^{n} f(x_{i-1})\,\Delta x = \sum_{i=1}^{n} 2\sqrt{\frac{i-1}{n}} \cdot \frac{4}{n} = \frac{8}{n^{3/2}} \sum_{i=1}^{n} \sqrt{i-1}.$$

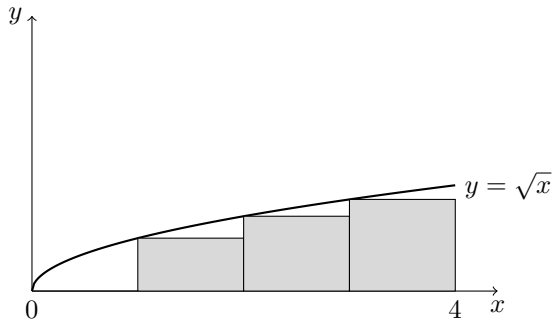
Thus,

$$\int_0^4 \sqrt{x}\,dx = \lim_{n\to\infty} \frac{8}{n^{3/2}} \sum_{i=1}^{n} \sqrt{i-1}.$$

As $n \to \infty$, this Riemann sum converges to the definite integral:

$$\int_0^4 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{16}{3}.$$

$$\boxed{\frac{16}{3}}$$



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Question 4

Let

$$f(x) = \frac{1}{1+x^2}$$

on the interval $[-1, 1]$.

Using the definition of the definite integral, express

$$\int_{-1}^{1} \frac{1}{1+x^2}\, dx$$

as the limit of a Riemann sum using right endpoints, and evaluate the limit.

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Question 4 - Solution

Divide the interval $[-1, 1]$ into n equal subintervals. The width of each subinterval is

$$\Delta x = \frac{1-(-1)}{n} = \frac{2}{n}.$$

Using right endpoints, the ith sample point is

$$x_i = -1 + i\Delta x = -1 + \frac{2i}{n}.$$

Evaluate the function:

$$f(x_i) = \frac{1}{1+\left(-1+\frac{2i}{n}\right)^2}.$$

The Riemann sum is

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = \frac{2}{n}\sum_{i=1}^{n} \frac{1}{1+\left(-1+\frac{2i}{n}\right)^2}.$$

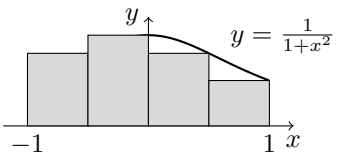
Thus,

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} \frac{1}{1+\left(-1+\frac{2i}{n}\right)^2}.$$

Evaluating the definite integral,

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = \left[\arctan x\right]_{-1}^{1} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$$

$$\boxed{\frac{\pi}{2}}$$



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Question 5

Let

$$f(x) = e^x$$

on the interval $[0, 1]$.

Using the definition of the definite integral, express

$$\int_0^1 e^x \, dx$$

as the limit of a Riemann sum using right endpoints, and evaluate the limit.

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Question 5- Solution

Partition the interval $[0, 1]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{1}{n}.$$

Using right endpoints, the ith sample point is

$$x_i = i\Delta x = \frac{i}{n}.$$

Evaluate the function at each sample point:

$$f(x_i) = e^{i/n}.$$

The Riemann sum is

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = \frac{1}{n}\sum_{i=1}^{n} e^{i/n}.$$

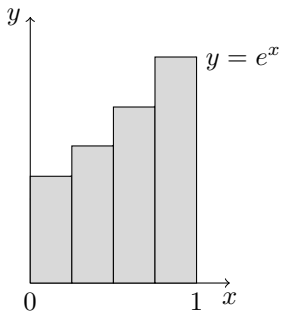
Thus,

$$\int_0^1 e^x\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} e^{i/n}.$$

Evaluating the definite integral,

$$\int_0^1 e^x\,dx = \left[e^x\right]_0^1 = e - 1.$$

$$\boxed{e-1}$$



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Question 6

Let

$$f(x) = |x|$$

on the interval $[-2, 2]$.

Using the definition of the definite integral, express

$$\int_{-2}^{2} |x|\, dx$$

as the limit of a Riemann sum using equal subintervals, and evaluate the limit.

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Question 6 - Solution

Divide the interval $[-2, 2]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{2-(-2)}{n} = \frac{4}{n}.$$

Using right endpoints, the ith sample point is

$$x_i = -2 + i\Delta x = -2 + \frac{4i}{n}.$$

Evaluate the function:

$$f(x_i) = \left|-2 + \frac{4i}{n}\right|.$$

The Riemann sum is

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = \frac{4}{n}\sum_{i=1}^{n}\left|-2 + \frac{4i}{n}\right|.$$

Thus,

$$\int_{-2}^{2} |x|\,dx = \lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^{n}\left|-2 + \frac{4i}{n}\right|.$$

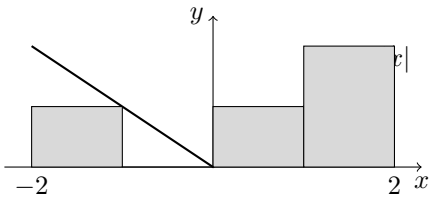
To evaluate the integral, use symmetry:

$$\int_{-2}^{2} |x|\,dx = 2\int_{0}^{2} x\,dx.$$

Compute:

$$2\left[\frac{x^2}{2}\right]_0^2 = 4.$$

$$\boxed{4}$$



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Question 7

Let

$$f(x) = 3 - 2x$$

on the interval $[0, 1]$.

Using the definition of the definite integral, express

$$\int_0^1 (3 - 2x)\, dx$$

as the limit of a Riemann sum using left endpoints, and evaluate the limit.

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Question 7 - Solution

Partition the interval $[0,1]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{1}{n}.$$

Using left endpoints, the ith sample point is

$$x_{i-1} = (i-1)\Delta x = \frac{i-1}{n}.$$

Evaluate the function at each sample point:

$$f(x_{i-1}) = 3 - 2\left(\frac{i-1}{n}\right).$$

The Riemann sum is

$$\sum_{i=1}^{n} f(x_{i-1})\,\Delta x = \frac{1}{n}\sum_{i=1}^{n}\left(3 - \frac{2(i-1)}{n}\right).$$

Separate the sum:

$$\frac{1}{n}\left(\sum_{i=1}^{n} 3 - \frac{2}{n}\sum_{i=1}^{n}(i-1)\right).$$

Using

$$\sum_{i=1}^{n} 1 = n, \qquad \sum_{i=1}^{n}(i-1) = \frac{n(n-1)}{2},$$

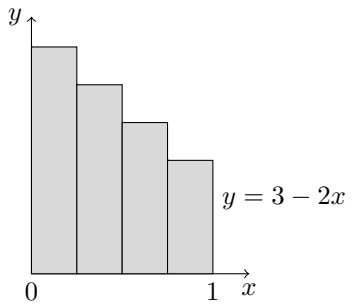
we obtain

$$\frac{1}{n}\left(3n - \frac{2}{n}\cdot\frac{n(n-1)}{2}\right) = 3 - (n-1)\frac{1}{n}.$$

Thus,

$$\int_0^1 (3-2x)\,dx = \lim_{n\to\infty}\left(3 - \frac{n-1}{n}\right) = 2.$$

$$\boxed{2}$$



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Question 8

Let

$$f(x) = \frac{1}{x}$$

on the interval $[1, e]$.

Using the definition of the definite integral, express

$$\int_1^e \frac{1}{x}\, dx$$

as the limit of a Riemann sum using right endpoints, and evaluate the limit.

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Question 8 - Solution

Partition the interval $[1, e]$ into n equal subintervals. The width of each subinterval is

$$\Delta x = \frac{e-1}{n}.$$

Using right endpoints, the ith sample point is

$$x_i = 1 + i\Delta x = 1 + i\frac{e-1}{n}.$$

Evaluate the function at each sample point:

$$f(x_i) = \frac{1}{1 + i\frac{e-1}{n}}.$$

The Riemann sum is

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = \frac{e-1}{n}\sum_{i=1}^{n} \frac{1}{1 + i\frac{e-1}{n}}.$$

Thus,

$$\int_1^e \frac{1}{x}\,dx = \lim_{n\to\infty} \frac{e-1}{n}\sum_{i=1}^{n} \frac{1}{1 + i\frac{e-1}{n}}.$$

Evaluating the definite integral,

$$\int_1^e \frac{1}{x}\,dx = \big[\ln x\big]_1^e = 1.$$

$$\boxed{1}$$

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Question 9

Let

$$f(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 2, & 1 < x \le 2. \end{cases}$$

Using the definition of the definite integral, express

$$\int_0^2 f(x)\,dx$$

as the limit of a Riemann sum, and evaluate the limit.

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Question 9 - Solution

Partition the interval $[0, 2]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{2}{n}.$$

Using right endpoints, the ith sample point is

$$x_i = \frac{2i}{n}.$$

The Riemann sum is

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = \frac{2}{n}\sum_{i=1}^{n} f\left(\frac{2i}{n}\right).$$

Split the sum according to where the function changes behavior. The condition $x_i \le 1$ corresponds to

$$\frac{2i}{n} \le 1 \quad \Rightarrow \quad i \le \frac{n}{2}.$$

Thus,

$$\sum_{i=1}^{n} f\left(\frac{2i}{n}\right) = \sum_{i=1}^{n/2} 2\left(\frac{2i}{n}\right) + \sum_{i=n/2+1}^{n} 2.$$

Compute each sum:

$$\sum_{i=1}^{n/2} \frac{4i}{n} = \frac{4}{n}\cdot\frac{(n/2)(n/2+1)}{2} = \frac{n}{2} + 1,$$

$$\sum_{i=n/2+1}^{n} 2 = 2\cdot\frac{n}{2} = n.$$

Therefore,

$$\sum_{i=1}^{n} f\left(\frac{2i}{n}\right) = \frac{n}{2} + 1 + n = \frac{3n}{2} + 1.$$

Multiply by Δx:

$$\frac{2}{n}\left(\frac{3n}{2} + 1\right) = 3 + \frac{2}{n}.$$

Taking the limit,

$$\int_0^2 f(x)\,dx = \lim_{n\to\infty}\left(3 + \frac{2}{n}\right) = 3.$$

$$\boxed{3}$$

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Question 10

Let

$$f(x) = \sin x$$

on the interval $[0, \pi]$.

Using the definition of the definite integral, express

$$\int_0^{\pi} \sin x \, dx$$

as the limit of a Riemann sum using midpoints, and evaluate the limit.

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Question 10 - Solution

Partition the interval $[0, \pi]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{\pi}{n}.$$

The midpoint of the ith subinterval is

$$x_i^* = \left(i - \frac{1}{2}\right)\Delta x = \left(i - \frac{1}{2}\right)\frac{\pi}{n}.$$

Evaluate the function at each midpoint:

$$f(x_i^*) = \sin\left(\left(i - \frac{1}{2}\right)\frac{\pi}{n}\right).$$

The Riemann sum is

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x = \frac{\pi}{n}\sum_{i=1}^{n} \sin\left(\left(i - \frac{1}{2}\right)\frac{\pi}{n}\right).$$

Thus,

$$\int_0^{\pi} \sin x\,dx = \lim_{n\to\infty} \frac{\pi}{n}\sum_{i=1}^{n} \sin\left(\left(i - \frac{1}{2}\right)\frac{\pi}{n}\right).$$

Evaluating the definite integral directly,

$$\int_0^{\pi} \sin x\,dx = \big[-\cos x\big]_0^{\pi} = (-\cos\pi) - (-\cos 0) = 2.$$

$$\boxed{2}$$

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Question 1

Evaluate the definite integral

$$\int_0^1 \left(x^2 e^x - 2xe^x\right)\, dx.$$

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Question 1 - Solution

Combine the terms:

$$\int_0^1 e^x(x^2 - 2x)\,dx.$$

Use integration by parts. Let

$$u = x^2 - 2x, \qquad dv = e^x\,dx.$$

Then

$$du = (2x - 2)\,dx, \qquad v = e^x.$$

Apply integration by parts:

$$\int e^x(x^2 - 2x)\,dx = (x^2 - 2x)e^x - \int e^x(2x - 2)\,dx.$$

Apply integration by parts again to the remaining integral. Let

$$u = 2x - 2, \qquad dv = e^x\,dx.$$

Then

$$du = 2\,dx, \qquad v = e^x.$$

Thus,

$$\int e^x(2x - 2)\,dx = (2x - 2)e^x - \int 2e^x\,dx = (2x - 2)e^x - 2e^x.$$

Substitute back:

$$\int e^x(x^2 - 2x)\,dx = (x^2 - 2x)e^x - \big[(2x - 2)e^x - 2e^x\big].$$

Simplify:

$$(x^2 - 4x + 4)e^x.$$

Now evaluate from 0 to 1:

$$\int_0^1 e^x(x^2 - 2x)\,dx = \big[(x^2 - 4x + 4)e^x\big]_0^1.$$

Compute:

$$(1 - 4 + 4)e - (0 - 0 + 4) = e - 4.$$

$$\boxed{e - 4}$$

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Question 2

Evaluate the definite integral

$$\int_0^{\pi/2} \left(\sin x + \sin^2 x\right)\, dx.$$

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Question 2 - Solution

Split the integral:

$$\int_0^{\pi/2} \sin x\,dx + \int_0^{\pi/2} \sin^2 x\,dx.$$

Evaluate the first integral:

$$\int_0^{\pi/2} \sin x\,dx = \big[-\cos x\big]_0^{\pi/2} = 1.$$

For the second integral, use the identity

$$\sin^2 x = \frac{1-\cos(2x)}{2}.$$

Then

$$\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/2} (1-\cos 2x)\,dx.$$

Integrate:

$$\frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{4}.$$

Combine results:

$$1 + \frac{\pi}{4}.$$

$$\boxed{1 + \frac{\pi}{4}}$$

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Question 3

Evaluate the definite integral

$$\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx.$$

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Question 3 - Solution

First notice a symmetry property. The function

$$x^2\sqrt{1-x^2}$$

is an *even function* because replacing x by $-x$ does not change its value. Therefore,

$$\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx = 2\int_{0}^{1} x^2\sqrt{1-x^2}\,dx.$$

Now focus on the integral from 0 to 1.

Use a trigonometric substitution. Let

$$x = \sin\theta, \qquad dx = \cos\theta\,d\theta.$$

When $x = 0$, $\theta = 0$, and when $x = 1$, $\theta = \frac{\pi}{2}$.
Substitute into the integral:

$$\int_{0}^{1} x^2\sqrt{1-x^2}\,dx = \int_{0}^{\pi/2} (\sin^2\theta)(\cos\theta)(\cos\theta)\,d\theta = \int_{0}^{\pi/2} \sin^2\theta\,\cos^2\theta\,d\theta.$$

Use the identity

$$\sin^2\theta\,\cos^2\theta = \frac{1}{4}\sin^2(2\theta).$$

Then

$$\int_{0}^{\pi/2} \sin^2\theta\,\cos^2\theta\,d\theta = \frac{1}{4}\int_{0}^{\pi/2} \sin^2(2\theta)\,d\theta.$$

Apply the identity

$$\sin^2 u = \frac{1-\cos(2u)}{2}.$$

Thus,

$$\frac{1}{4}\int_{0}^{\pi/2} \frac{1-\cos(4\theta)}{2}\,d\theta = \frac{1}{8}\left[\theta - \frac{1}{4}\sin(4\theta)\right]_{0}^{\pi/2}.$$

Evaluate:

$$\frac{1}{8}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{16}.$$

Finally, multiply by 2 to account for the symmetry:

$$\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx = 2\cdot\frac{\pi}{16} = \boxed{\frac{\pi}{8}}.$$

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Question 4

Evaluate the definite integral

$$\int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx.$$

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Question 4 - Solution

This integral is well-suited for a direct substitution because the derivative of $1+x^2$ appears in the numerator.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x\,dx \quad \Rightarrow \quad x\,dx = \frac{1}{2}\,du.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

Substitute into the integral:

$$\int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx = \frac{1}{2}\int_1^2 u^{-1/2}\,du.$$

Integrate:

$$\frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\cdot 2u^{1/2} = \sqrt{u}.$$

Apply the limits:

$$\sqrt{u}\Big|_1^2 = \sqrt{2} - 1.$$

$$\boxed{\sqrt{2}-1}$$

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Question 5

Evaluate the definite integral

$$\int_0^{\pi} x \sin x \, dx.$$

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Question 5 - Solution

This integral suggests integration by parts because it is a product of x and $\sin x$.

Let

$$u = x, \qquad dv = \sin x\, dx.$$

Then

$$du = dx, \qquad v = -\cos x.$$

Apply integration by parts:

$$\int x \sin x\, dx = -x\cos x + \int \cos x\, dx = -x\cos x + \sin x.$$

Now evaluate from 0 to π:

$$\left[-x\cos x + \sin x\right]_0^{\pi}.$$

Compute each endpoint:

$$-\pi\cos\pi + \sin\pi = \pi,$$

$$-0\cdot\cos 0 + \sin 0 = 0.$$

Subtract:

$$\pi - 0 = \pi.$$

$$\boxed{\pi}$$

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Question 6

Evaluate the definite integral

$$\int_{-2}^{2} \frac{1}{1+x^2}\, dx.$$

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Question 6 - Solution

First observe that the function

$$\frac{1}{1+x^2}$$

is an even function, since

$$\frac{1}{1+(-x)^2} = \frac{1}{1+x^2}.$$

Therefore, the integral over the symmetric interval $[-2, 2]$ can be written as

$$\int_{-2}^{2} \frac{1}{1+x^2}\,dx = 2\int_{0}^{2} \frac{1}{1+x^2}\,dx.$$

Now evaluate the integral. Recall that

$$\int \frac{1}{1+x^2}\,dx = \arctan x.$$

Thus,

$$2\int_{0}^{2} \frac{1}{1+x^2}\,dx = 2\left[\arctan x\right]_0^2.$$

Evaluate the bounds:

$$\arctan(2) - \arctan(0) = \arctan(2).$$

Therefore,

$$\int_{-2}^{2} \frac{1}{1+x^2}\,dx = \boxed{2\arctan(2)}.$$

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Question 7

Evaluate the definite integral

$$\int_0^1 (3x^2 + 2)\sqrt{x}\, dx.$$

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Question 7 - Solution

Begin by rewriting the integrand using powers of x:

$$(3x^2+2)\sqrt{x} = 3x^{5/2} + 2x^{1/2}.$$

Now integrate term by term:

$$\int 3x^{5/2}\,dx = 3 \cdot \frac{x^{7/2}}{7/2} = \frac{6}{7}x^{7/2},$$

$$\int 2x^{1/2}\,dx = 2 \cdot \frac{x^{3/2}}{3/2} = \frac{4}{3}x^{3/2}.$$

Combine the results:

$$\int (3x^2+2)\sqrt{x}\,dx = \frac{6}{7}x^{7/2} + \frac{4}{3}x^{3/2}.$$

Evaluate from 0 to 1:

$$\left[\frac{6}{7}x^{7/2} + \frac{4}{3}x^{3/2}\right]_0^1 = \frac{6}{7} + \frac{4}{3}.$$

Find a common denominator:

$$\frac{6}{7} + \frac{4}{3} = \frac{18+28}{21} = \frac{46}{21}.$$

$$\boxed{\frac{46}{21}}$$

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Question 8

Evaluate the definite integral

$$\int_0^{\ln 2} e^x \left(1 + e^x\right) dx.$$

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Question 8 - Solution

Begin by expanding the integrand:

$$e^x(1+e^x) = e^x + e^{2x}.$$

Now integrate term by term:

$$\int e^x\,dx = e^x, \qquad \int e^{2x}\,dx = \frac{1}{2}e^{2x}.$$

Thus,

$$\int e^x(1+e^x)\,dx = e^x + \frac{1}{2}e^{2x}.$$

Evaluate from 0 to $\ln 2$:

$$\left[e^x + \frac{1}{2}e^{2x}\right]_0^{\ln 2}.$$

Compute the upper bound:

$$e^{\ln 2} + \frac{1}{2}e^{2\ln 2} = 2 + \frac{1}{2}\cdot 4 = 4.$$

Compute the lower bound:

$$e^0 + \frac{1}{2}e^0 = 1 + \frac{1}{2} = \frac{3}{2}.$$

Subtract:

$$4 - \frac{3}{2} = \frac{5}{2}.$$

$$\boxed{\frac{5}{2}}$$

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Question 9

Evaluate the definite integral

$$\int_1^4 \left(\frac{2}{\sqrt{x}} - \frac{1}{x} \right) dx.$$

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Question 9 - Solution

Rewrite the integrand using exponents:

$$\frac{2}{\sqrt{x}} - \frac{1}{x} = 2x^{-1/2} - x^{-1}.$$

Integrate term by term:

$$\int 2x^{-1/2}\,dx = 2 \cdot \frac{x^{1/2}}{1/2} = 4\sqrt{x},$$

$$\int x^{-1}\,dx = \ln x.$$

Thus,

$$\int \left(\frac{2}{\sqrt{x}} - \frac{1}{x}\right) dx = 4\sqrt{x} - \ln x.$$

Evaluate from 1 to 4:

$$\left[4\sqrt{x} - \ln x\right]_1^4.$$

Compute the upper bound:

$$4\sqrt{4} - \ln 4 = 8 - \ln 4.$$

Compute the lower bound:

$$4\sqrt{1} - \ln 1 = 4.$$

Subtract:

$$(8 - \ln 4) - 4 = 4 - \ln 4.$$

$$\boxed{4 - \ln 4}$$

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Question 10

Evaluate the definite integral

$$\int_0^1 \frac{2x}{1+x^2}\,dx.$$

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Question 10 - Solution

This integral suggests a substitution because the derivative of the denominator appears in the numerator.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x\,dx.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

Substitute into the integral:

$$\int_0^1 \frac{2x}{1+x^2}\,dx = \int_1^2 \frac{1}{u}\,du.$$

Integrate:

$$\int \frac{1}{u}\,du = \ln u.$$

Apply the limits:

$$\ln u\Big|_1^2 = \ln 2.$$

$$\boxed{\ln 2}$$

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Question 1

Evaluate the definite integral

$$\int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx.$$

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Question 1 - Solution

Use substitution to simplify the integrand.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x\,dx \quad \Rightarrow \quad x\,dx = \frac{1}{2}\,du.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

Substitute into the integral:

$$\int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx = \frac{1}{2}\int_1^2 u^{-1/2}\,du.$$

Integrate:

$$\frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\cdot 2u^{1/2} = \sqrt{u}.$$

Apply the new limits:

$$\sqrt{u}\Big|_1^2 = \sqrt{2} - 1.$$

$$\boxed{\sqrt{2}-1}$$

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Question 2

Evaluate the definite integral

$$\int_0^{\pi/2} \sin x \sqrt{1+\cos x}\, dx.$$

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Question 2 - Solution

This integral suggests a substitution involving $\cos x$ because its derivative appears in the integrand.

Let

$$u = 1 + \cos x.$$

Then

$$du = -\sin x\, dx \quad \Rightarrow \quad -du = \sin x\, dx.$$

Change the limits of integration. When $x = 0$, $\cos 0 = 1$, so $u = 2$. When $x = \frac{\pi}{2}$, $\cos\left(\frac{\pi}{2}\right) = 0$, so $u = 1$.

Substitute into the integral:

$$\int_0^{\pi/2} \sin x \sqrt{1 + \cos x}\, dx = -\int_2^1 \sqrt{u}\, du = \int_1^2 \sqrt{u}\, du.$$

Integrate:

$$\int \sqrt{u}\, du = \frac{2}{3} u^{3/2}.$$

Apply the limits:

$$\frac{2}{3} u^{3/2}\Big|_1^2 = \frac{2}{3}\left(2^{3/2} - 1\right).$$

$$\boxed{\frac{2}{3}\left(2^{3/2} - 1\right)}$$

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Question 3

Evaluate the definite integral

$$\int_1^2 \frac{\ln x}{x}\, dx.$$

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Question 3 - Solution

This integral is well suited for substitution because the derivative of $\ln x$ appears in the denominator.

Let

$$u = \ln x.$$

Then

$$du = \frac{1}{x}\,dx.$$

Change the limits of integration. When $x = 1$, $u = \ln 1 = 0$. When $x = 2$, $u = \ln 2$.

Substitute into the integral:

$$\int_1^2 \frac{\ln x}{x}\,dx = \int_0^{\ln 2} u\,du.$$

Integrate:

$$\int u\,du = \frac{u^2}{2}.$$

Apply the limits:

$$\frac{u^2}{2}\Big|_0^{\ln 2} = \frac{(\ln 2)^2}{2}.$$

$$\boxed{\frac{(\ln 2)^2}{2}}$$

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Question 4

Evaluate the definite integral

$$\int_0^1 \frac{x^3}{\sqrt{1+x^4}}\,dx.$$

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Question 4 - Solution

This integral suggests a substitution involving the expression inside the square root.

Let

$$u = 1 + x^4.$$

Then

$$du = 4x^3\,dx \quad \Rightarrow \quad x^3\,dx = \frac{1}{4}\,du.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

Substitute into the integral:

$$\int_0^1 \frac{x^3}{\sqrt{1+x^4}}\,dx = \frac{1}{4}\int_1^2 u^{-1/2}\,du.$$

Integrate:

$$\frac{1}{4}\int u^{-1/2}\,du = \frac{1}{4}\cdot 2u^{1/2} = \frac{1}{2}\sqrt{u}.$$

Apply the limits:

$$\frac{1}{2}\sqrt{u}\Big|_1^2 = \frac{1}{2}(\sqrt{2}-1).$$

$$\boxed{\frac{\sqrt{2}-1}{2}}$$

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Question 5

Evaluate the definite integral

$$\int_0^{\pi/2} \frac{\cos x}{1+\sin x}\, dx.$$

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Question 5 - Solution

This integral is well suited for substitution because the derivative of $\sin x$ appears in the numerator.

Let

$$u = 1 + \sin x.$$

Then

$$du = \cos x \, dx.$$

Change the limits of integration. When $x = 0$, $\sin 0 = 0$, so $u = 1$. When $x = \frac{\pi}{2}$, $\sin\left(\frac{\pi}{2}\right) = 1$, so $u = 2$.

Substitute into the integral:

$$\int_0^{\pi/2} \frac{\cos x}{1 + \sin x} \, dx = \int_1^2 \frac{1}{u} \, du.$$

Integrate:

$$\int \frac{1}{u} \, du = \ln u.$$

Apply the limits:

$$\ln u \Big|_1^2 = \ln 2.$$

$$\boxed{\ln 2}$$

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Question 6

Evaluate the definite integral

$$\int_0^{\sqrt{3}} \frac{x}{1+x^2}\,dx.$$

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Question 6 - Solution

The integrand suggests a substitution involving $1 + x^2$.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x\,dx \quad \Rightarrow \quad x\,dx = \frac{1}{2}\,du.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = \sqrt{3}$, $u = 4$.

Substitute into the integral:

$$\int_0^{\sqrt{3}} \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_1^4 \frac{1}{u}\,du.$$

Integrate:

$$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln u.$$

Apply the limits:

$$\frac{1}{2}\ln u\Big|_1^4 = \frac{1}{2}\ln 4 = \ln 2.$$

$$\boxed{\ln 2}$$

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Question 7

Evaluate the definite integral

$$\int_1^e \frac{1}{x}\sqrt{1+\ln x}\,dx.$$

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Question 7 - Solution

The integrand suggests a substitution involving $\ln x$.

Let

$$u = 1 + \ln x.$$

Then

$$du = \frac{1}{x}\,dx.$$

Change the limits of integration. When $x = 1$, $u = 1$. When $x = e$, $u = 2$.

Substitute into the integral:

$$\int_1^e \frac{1}{x}\sqrt{1 + \ln x}\,dx = \int_1^2 \sqrt{u}\,du.$$

Integrate:

$$\int \sqrt{u}\,du = \frac{2}{3}u^{3/2}.$$

Apply the limits:

$$\frac{2}{3}u^{3/2}\Big|_1^2 = \frac{2}{3}\left(2^{3/2} - 1\right).$$

$$\boxed{\frac{2}{3}\left(2^{3/2} - 1\right)}$$

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Question 8

Evaluate the definite integral

$$\int_0^1 \frac{2x}{(1+x^2)^2}\,dx.$$

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Question 8 - Solution

The integrand suggests a substitution involving $1 + x^2$.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x\,dx.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

Substitute into the integral:

$$\int_0^1 \frac{2x}{(1+x^2)^2}\,dx = \int_1^2 u^{-2}\,du.$$

Integrate:

$$\int u^{-2}\,du = -u^{-1}.$$

Apply the limits:

$$-u^{-1}\Big|_1^2 = -\frac{1}{2} - (-1) = \frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$

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Question 9

Evaluate the definite integral

$$\int_0^1 \frac{\sqrt{1-x}}{1+x}\,dx.$$

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Question 9 - Solution

This integral becomes simpler if we eliminate the square root by substitution.

Let

$$u = \sqrt{1-x}.$$

Then

$$x = 1 - u^2, \qquad dx = -2u\,du.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 0$.

Substitute into the integral:

$$\int_0^1 \frac{\sqrt{1-x}}{1+x}\,dx = \int_1^0 \frac{u}{1+(1-u^2)}(-2u)\,du.$$

Simplify:

$$\int_1^0 \frac{-2u^2}{2-u^2}\,du = \int_0^1 \frac{2u^2}{2-u^2}\,du.$$

Rewrite the integrand:

$$\frac{2u^2}{2-u^2} = -2 + \frac{4}{2-u^2}.$$

Thus,

$$\int_0^1 \left(-2 + \frac{4}{2-u^2}\right)\,du.$$

Integrate term by term:

$$\int -2\,du = -2u,$$

$$\int \frac{4}{2-u^2}\,du = 2\int \frac{1}{1-(u/\sqrt{2})^2}\,du.$$

Use the standard result

$$\int \frac{1}{1-a^2}\,da = \operatorname{artanh}(a).$$

Thus,

$$2\int \frac{1}{1-(u/\sqrt{2})^2}\,du = 2\sqrt{2}\,\operatorname{artanh}\left(\frac{u}{\sqrt{2}}\right).$$

Evaluate from 0 to 1:

$$-2u + 2\sqrt{2}\,\operatorname{artanh}\left(\frac{u}{\sqrt{2}}\right)\Big|_0^1.$$

Final answer:

$$\boxed{-2 + 2\sqrt{2}\,\operatorname{artanh}\left(\frac{1}{\sqrt{2}}\right)}$$

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Question 10

Evaluate the definite integral

$$\int_0^1 \sqrt{1-x^2}\,\arcsin x\,dx.$$

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Question 10 - Solution

This integral is interesting because it mixes an inverse trigonometric function with an algebraic expression, suggesting a trigonometric substitution.

Let

$$x = \sin\theta.$$

Then

$$dx = \cos\theta\, d\theta, \qquad \sqrt{1-x^2} = \cos\theta.$$

Change the limits of integration. When $x = 0$, $\theta = 0$. When $x = 1$, $\theta = \frac{\pi}{2}$.

Substitute into the integral:

$$\int_0^1 \sqrt{1-x^2}\, \arcsin x\, dx = \int_0^{\pi/2} \theta\, \cos^2\theta\, d\theta.$$

Use the identity

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}.$$

Then

$$\int_0^{\pi/2} \theta\, \cos^2\theta\, d\theta = \frac{1}{2}\int_0^{\pi/2} \theta\, d\theta + \frac{1}{2}\int_0^{\pi/2} \theta\cos(2\theta)\, d\theta.$$

The first integral is

$$\frac{1}{2}\int_0^{\pi/2} \theta\, d\theta = \frac{1}{2}\cdot\frac{(\pi/2)^2}{2} = \frac{\pi^2}{16}.$$

For the second integral, use integration by parts. Let

$$u = \theta, \qquad dv = \cos(2\theta)\, d\theta.$$

Then

$$du = d\theta, \qquad v = \frac{1}{2}\sin(2\theta).$$

Thus,

$$\frac{1}{2}\int_0^{\pi/2} \theta\cos(2\theta)\, d\theta = \frac{1}{2}\left[\frac{1}{2}\theta\sin(2\theta) - \frac{1}{2}\int \sin(2\theta)\, d\theta\right]_0^{\pi/2}.$$

Since $\sin(\pi) = 0$ and $\sin(0) = 0$, the boundary term vanishes. Also,

$$\int \sin(2\theta)\, d\theta = -\frac{1}{2}\cos(2\theta).$$

Therefore,

$$\frac{1}{2}\int_0^{\pi/2} \theta\cos(2\theta)\, d\theta = \frac{1}{8}\big[\cos(2\theta)\big]_0^{\pi/2} = \frac{1}{8}(-1-1) = -\frac{1}{4}.$$

Combine results:

$$\frac{\pi^2}{16} - \frac{1}{4}.$$

$$\boxed{\frac{\pi^2}{16} - \frac{1}{4}}$$

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Question 1

Find the average value of the function

$$f(x) = x\sqrt{1+x^2}$$

on the interval $[0, 2]$.

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Question 1 - Solution

The average value of a function f on an interval $[a, b]$ is defined by

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Here, $a = 0$ and $b = 2$, so

$$f_{\text{avg}} = \frac{1}{2}\int_0^2 x\sqrt{1+x^2}\,dx.$$

Evaluate the integral using substitution. Let

$$u = 1 + x^2.$$

Then

$$du = 2x\,dx \quad \Rightarrow \quad x\,dx = \frac{1}{2}\,du.$$

Change the limits. When $x = 0$, $u = 1$. When $x = 2$, $u = 5$.
Substitute:

$$\int_0^2 x\sqrt{1+x^2}\,dx = \frac{1}{2}\int_1^5 u^{1/2}\,du.$$

Integrate:

$$\frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} = \frac{1}{3}u^{3/2}.$$

Apply the limits:

$$\frac{1}{3}u^{3/2}\Big|_1^5 = \frac{1}{3}\left(5^{3/2}-1\right).$$

Now divide by the interval length:

$$f_{\text{avg}} = \frac{1}{2}\cdot\frac{1}{3}\left(5^{3/2}-1\right) = \frac{1}{6}\left(5^{3/2}-1\right).$$

$$\boxed{\frac{1}{6}\left(5^{3/2}-1\right)}$$

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Question 2

Find the average value of the function

$$f(x) = \ln(1 + x^2)$$

on the interval $[-1, 1]$.

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Question 2 - Solution

The average value of a function f on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Here $a = -1$ and $b = 1$, so

$$f_{\text{avg}} = \frac{1}{2}\int_{-1}^{1} \ln(1+x^2)\,dx.$$

Notice that $\ln(1+x^2)$ is an even function, since

$$\ln(1+(-x)^2) = \ln(1+x^2).$$

Therefore,

$$\int_{-1}^{1} \ln(1+x^2)\,dx = 2\int_{0}^{1} \ln(1+x^2)\,dx.$$

Thus,

$$f_{\text{avg}} = \int_{0}^{1} \ln(1+x^2)\,dx.$$

Evaluate the integral using integration by parts. Let

$$u = \ln(1+x^2), \qquad dv = dx.$$

Then

$$du = \frac{2x}{1+x^2}\,dx, \qquad v = x.$$

Apply integration by parts:

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx.$$

Rewrite the integrand:

$$\frac{2x^2}{1+x^2} = 2 - \frac{2}{1+x^2}.$$

Thus,

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - \int \left(2 - \frac{2}{1+x^2}\right)dx.$$

Integrate:

$$x\ln(1+x^2) - 2x + 2\arctan x.$$

Evaluate from 0 to 1:

$$\left[x\ln(1+x^2) - 2x + 2\arctan x\right]_0^1 = \ln 2 - 2 + \frac{\pi}{2}.$$

Therefore,

$$\boxed{\ln 2 - 2 + \frac{\pi}{2}}$$

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Question 3

Find the average value of the function

$$f(x) = x^2 \cos x$$

on the interval $[0, \pi]$.

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Question 3 - Solution

The average value of a function f on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{\pi}\int_0^{\pi} x^2\cos x\,dx.$$

Evaluate the integral using integration by parts.

Let

$$u = x^2, \qquad dv = \cos x\,dx.$$

Then

$$du = 2x\,dx, \qquad v = \sin x.$$

Apply integration by parts:

$$\int x^2\cos x\,dx = x^2\sin x - \int 2x\sin x\,dx.$$

Apply integration by parts again to the remaining integral. Let

$$u = 2x, \qquad dv = \sin x\,dx.$$

Then

$$du = 2\,dx, \qquad v = -\cos x.$$

Thus,

$$\int 2x\sin x\,dx = -2x\cos x + 2\int \cos x\,dx = -2x\cos x + 2\sin x.$$

Substitute back:

$$\int x^2\cos x\,dx = x^2\sin x + 2x\cos x - 2\sin x.$$

Evaluate from 0 to π:

$$\left[x^2\sin x + 2x\cos x - 2\sin x\right]_0^{\pi}.$$

At $x = \pi$:

$$\pi^2\sin\pi + 2\pi\cos\pi - 2\sin\pi = -2\pi.$$

At $x = 0$:

$$0 + 0 - 0 = 0.$$

Thus,

$$\int_0^{\pi} x^2\cos x\,dx = -2\pi.$$

Divide by the interval length:

$$f_{\text{avg}} = \frac{1}{\pi}(-2\pi) = -2.$$

$$\boxed{-2}$$

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Question 4

Find the average value of the function

$$f(x) = \sqrt{x}(1 + x)$$

on the interval $[0, 1]$.

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Question 4 - Solution

The average value of a function on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Here,

$$f_{\text{avg}} = \int_0^1 \sqrt{x}(1+x)\,dx.$$

First expand the integrand:

$$\sqrt{x}(1+x) = x^{1/2} + x^{3/2}.$$

Integrate term by term:

$$\int x^{1/2}\,dx = \frac{2}{3}x^{3/2}, \qquad \int x^{3/2}\,dx = \frac{2}{5}x^{5/2}.$$

Thus,

$$\int_0^1 \sqrt{x}(1+x)\,dx = \left[\frac{2}{3}x^{3/2} + \frac{2}{5}x^{5/2}\right]_0^1.$$

Evaluate:

$$\frac{2}{3} + \frac{2}{5} = \frac{10+6}{15} = \frac{16}{15}.$$

$$\boxed{\frac{16}{15}}$$

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Question 5

Find the average value of the function

$$f(x) = \frac{x}{1+x^2}$$

on the interval $[0, 1]$.

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Question 5 - Solution

The average value of a function f on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Here $a = 0$ and $b = 1$, so

$$f_{\text{avg}} = \int_0^1 \frac{x}{1+x^2}\,dx.$$

Evaluate the integral using substitution.
Let

$$u = 1 + x^2.$$

Then

$$du = 2x\,dx \quad \Rightarrow \quad x\,dx = \frac{1}{2}\,du.$$

Change the limits. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.
Substitute:

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_1^2 \frac{1}{u}\,du.$$

Integrate:

$$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln u.$$

Apply the limits:

$$\frac{1}{2}\ln u\Big|_1^2 = \frac{1}{2}\ln 2.$$

$$\boxed{\frac{1}{2}\ln 2}$$

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Question 6

Find the average value of the function

$$f(x) = \sin^2 x$$

on the interval $[0, \pi]$.

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Question 6 - Solution

The average value of a function on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{\pi}\int_0^{\pi} \sin^2 x\,dx.$$

Use the identity

$$\sin^2 x = \frac{1-\cos(2x)}{2}.$$

Then

$$\int_0^{\pi} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi}(1-\cos 2x)\,dx.$$

Integrate:

$$\frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^{\pi} = \frac{1}{2}(\pi - 0) = \frac{\pi}{2}.$$

Now divide by the interval length:

$$f_{\text{avg}} = \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$

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Question 7

Find the average value of the function

$$f(x) = \sqrt{1 - x^2}$$

on the interval $[-1, 1]$.

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Question 7 - Solution

The average value of a function on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{2}\int_{-1}^{1}\sqrt{1-x^2}\,dx.$$

Method 1: Geometry
The graph of

$$y = \sqrt{1-x^2}$$

is the upper half of a circle of radius 1 centered at the origin.
The area under this curve from $x = -1$ to $x = 1$ is the area of a semicircle:

$$\frac{1}{2}\pi(1)^2 = \frac{\pi}{2}.$$

Therefore,

$$f_{\text{avg}} = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.$$

Method 2: Trigonometric Substitution
Evaluate the integral directly.
Let

$$x = \sin\theta, \qquad dx = \cos\theta\,d\theta.$$

Change the limits. When $x = -1$, $\theta = -\frac{\pi}{2}$. When $x = 1$, $\theta = \frac{\pi}{2}$.
Substitute:

$$\int_{-1}^{1}\sqrt{1-x^2}\,dx = \int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta.$$

Use the identity

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}.$$

Then

$$\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta.$$

Integrate:

$$\frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{-\pi/2}^{\pi/2}.$$

Since $\sin(\pi) = \sin(-\pi) = 0$, this becomes

$$\frac{1}{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{2}.$$

Now compute the average value:

$$f_{\text{avg}} = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.$$

$$\boxed{\frac{\pi}{4}}$$

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Question 8

Find the average value of the function

$$f(x) = |x|$$

on the interval $[-2, 2]$.

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Question 8 - Solution

The average value of a function on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Here $a = -2$ and $b = 2$, so

$$f_{\text{avg}} = \frac{1}{4}\int_{-2}^{2} |x|\,dx.$$

Since $|x|$ is an even function,

$$\int_{-2}^{2} |x|\,dx = 2\int_0^2 x\,dx.$$

Evaluate:

$$2\left[\frac{x^2}{2}\right]_0^2 = 2 \cdot 2 = 4.$$

Compute the average value:

$$f_{\text{avg}} = \frac{1}{4} \cdot 4 = 1.$$

$$\boxed{1}$$

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Question 9

Find the average value of the function

$$f(x) = \sqrt{x^2 + 1}$$

on the interval $[-1, 1]$.

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Question 9 - Solution

The average value of a function on an interval $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{2}\int_{-1}^{1}\sqrt{x^2+1}\,dx.$$

Notice that $\sqrt{x^2+1}$ is an even function, so

$$\int_{-1}^{1}\sqrt{x^2+1}\,dx = 2\int_0^1\sqrt{x^2+1}\,dx.$$

Thus,

$$f_{\text{avg}} = \int_0^1\sqrt{x^2+1}\,dx.$$

Evaluate the integral using trigonometric substitution. Let

$$x = \tan\theta, \qquad dx = \sec^2\theta\,d\theta.$$

When $x = 0$, $\theta = 0$. When $x = 1$, $\theta = \frac{\pi}{4}$.
Substitute:

$$\int_0^1\sqrt{x^2+1}\,dx = \int_0^{\pi/4}\sec\theta\cdot\sec^2\theta\,d\theta = \int_0^{\pi/4}\sec^3\theta\,d\theta.$$

Use the standard result

$$\int\sec^3\theta\,d\theta = \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right).$$

Apply the limits:

$$\frac{1}{2}\left(\sec\theta\tan\theta + \ln(\sec\theta+\tan\theta)\right)\Big|_0^{\pi/4}.$$

Evaluate:

$$\sec\left(\frac{\pi}{4}\right) = \sqrt{2}, \quad \tan\left(\frac{\pi}{4}\right) = 1,$$
$$\sec(0) = 1, \quad \tan(0) = 0.$$

Thus,

$$f_{\text{avg}} = \frac{1}{2}\left(\sqrt{2}+\ln(1+\sqrt{2})\right).$$

$$\boxed{\frac{1}{2}\left(\sqrt{2}+\ln(1+\sqrt{2})\right)}$$

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Question 10

Find the average value of the function

$$f(x) = x\,e^{-x}$$

on the interval $[0, 2]$.

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Question 10 - Solution

The average value of a function on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{2}\int_0^2 xe^{-x}\,dx.$$

Evaluate the integral using integration by parts.
Let

$$u = x, \qquad dv = e^{-x}\,dx.$$

Then

$$du = dx, \qquad v = -e^{-x}.$$

Apply integration by parts:

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x}.$$

Thus,

$$\int xe^{-x}\,dx = -(x+1)e^{-x}.$$

Evaluate from 0 to 2:

$$\left[-(x+1)e^{-x}\right]_0^2 = -3e^{-2} - (-1) = 1 - 3e^{-2}.$$

Now compute the average value:

$$f_{\text{avg}} = \frac{1}{2}\left(1 - 3e^{-2}\right).$$

$$\boxed{\frac{1}{2}\left(1 - 3e^{-2}\right)}$$

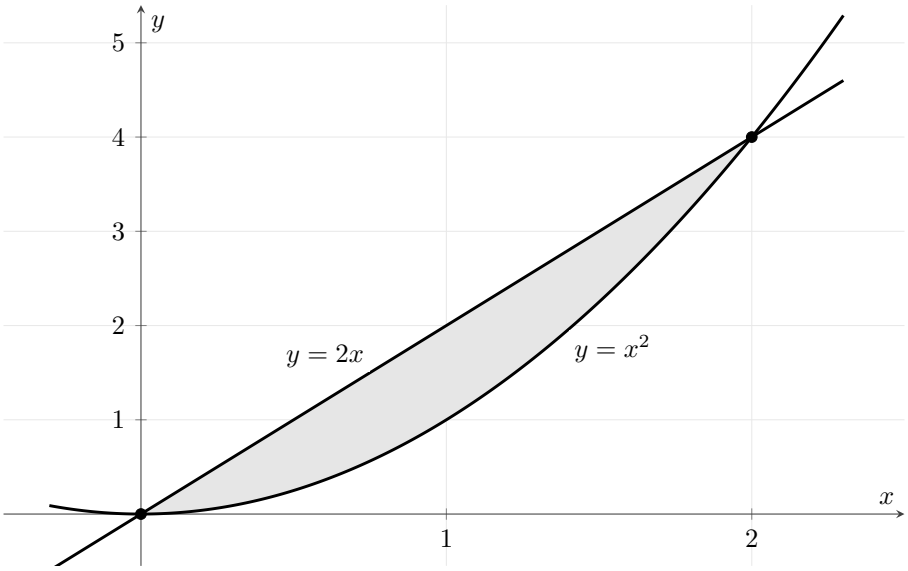
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Question 1

Find the exact area of the region enclosed by the curves

$$y = x^2 \quad \text{and} \quad y = 2x,$$

between their points of intersection.



The shaded region is the area to be found.

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Question 1 – Solution

1. Find the intersections.

$$x^2 = 2x \quad \Longrightarrow \quad x(x-2) = 0 \quad \Longrightarrow \quad x = 0,\ 2.$$

Factoring gives $x = 0$ or $x - 2 = 0$. Substituting into $y = 2x$ gives $y = 0$ or $y = 4$. The intersection points are $(0, 0)$ and $(2, 4)$.

2. Determine which curve is above the other.

On $0 \le x \le 2$, $2x - x^2 = x(2 - x) \ge 0$, so the line is above the parabola.

3. Set up upper minus lower.

$$A = \int_0^2 (\text{upper} - \text{lower})\, dx = \int_0^2 (2x - x^2)\, dx.$$

4. Find an antiderivative.

By the power rule, $\int 2x\, dx = x^2$ and $\int x^2\, dx = x^3/3$.

5. Apply the bounds and simplify.

$$\begin{aligned} A &= \left[x^2 - \frac{x^3}{3}\right]_0^2 \\ &= \left(4 - \frac{8}{3}\right) - 0 = \boxed{\frac{4}{3}}. \end{aligned}$$

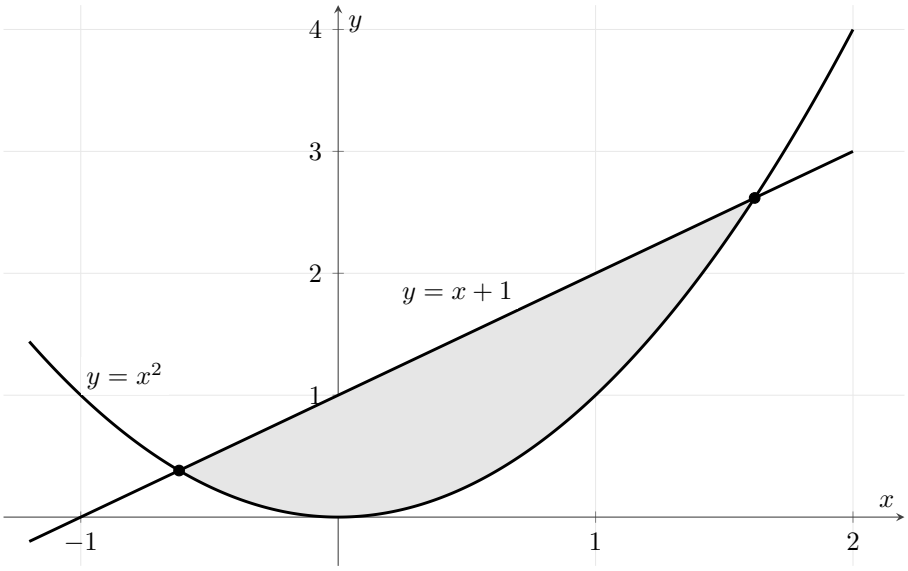
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Question 2

Find the exact area of the region enclosed by the curves

$$y = x + 1 \quad \text{and} \quad y = x^2,$$

between their points of intersection.



The shaded region is the area to be found.

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Question 2 – Solution

1. Find the intersections.

$$x^2 = x + 1 \quad \Longrightarrow \quad x^2 - x - 1 = 0.$$

The quadratic formula gives $x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$. Write the two roots as

$$a = \frac{1 - \sqrt{5}}{2}, \qquad b = \frac{1 + \sqrt{5}}{2}.$$

2. Determine which curve is above the other.

Test $x = 0$: the line has height 1 and the parabola has height 0. Between the roots, $(x + 1) - x^2 \geq 0$, so the line is above the parabola.

3. Set up upper minus lower.

$$A = \int_a^b (-x^2 + x + 1)\, dx.$$

4. Complete the square to simplify the bounds.

Complete the square:

$$-x^2 + x + 1 = \frac{5}{4} - \left(x - \frac{1}{2}\right)^2.$$

Let $u = x - \frac{1}{2}$. The new bounds are $-\sqrt{5}/2$ and $\sqrt{5}/2$. Here $du = dx$. The integrand is even, so the two halves have equal area.

5. Integrate and simplify.

$$\begin{aligned} A &= 2 \int_0^{\sqrt{5}/2} \left(\frac{5}{4} - u^2\right) du \\ &= 2 \left[\frac{5}{4}u - \frac{u^3}{3}\right]_0^{\sqrt{5}/2} \\ &= 2 \left(\frac{5\sqrt{5}}{8} - \frac{5\sqrt{5}}{24}\right) = \boxed{\frac{5\sqrt{5}}{6}}. \end{aligned}$$

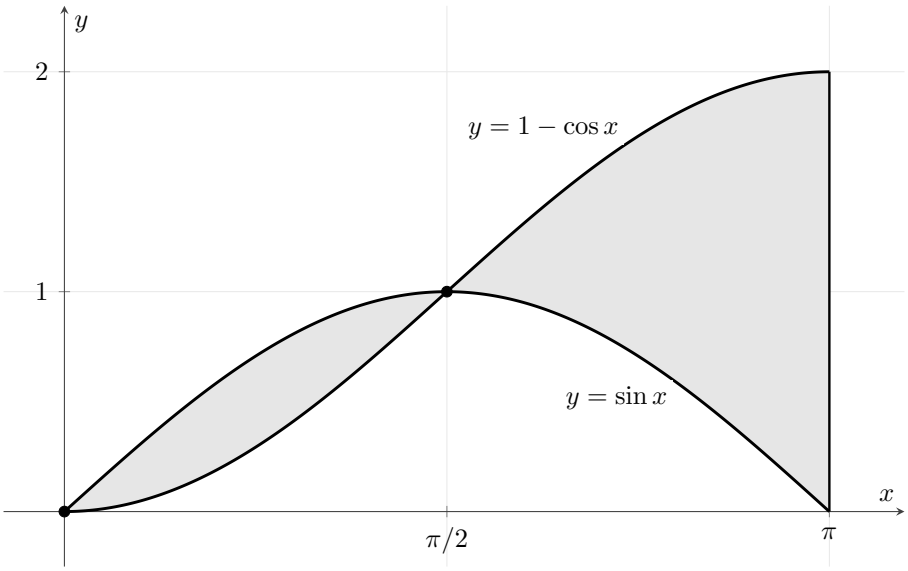
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Question 3

Find the exact area of the region between the curves

$$y = \sin x \quad \text{and} \quad y = 1 - \cos x,$$

between $x = 0$ and $x = \pi$.



The shaded region is the area to be found.

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Question 3 – Solution

1. Find where the curves cross on $[0, \pi]$.

Using half-angle identities,

$$\begin{aligned}\sin x = 1 - \cos x &\iff 2\sin(x/2)\big(\cos(x/2) - \sin(x/2)\big) = 0 \\ &\iff x = 0 \quad \text{or} \quad x = \frac{\pi}{2}.\end{aligned}$$

2. Determine the upper curve on each interval.

On $(0, \pi/2)$, $\sin x > 1 - \cos x$; on $(\pi/2, \pi)$, the order reverses. Test $x = \pi/4$: $\sin x = \sqrt{2}/2 > 1 - \sqrt{2}/2$. At $x = 3\pi/4$, $1 - \cos x = 1 + \sqrt{2}/2 > \sqrt{2}/2$. The line $x = \pi$ closes the region on the right.

3. Set up the sum of the two areas.

$$A = \int_0^{\pi/2} (\sin x - 1 + \cos x)\, dx + \int_{\pi/2}^{\pi} (1 - \cos x - \sin x)\, dx.$$

4. Integrate each piece and apply its bounds.

$$A_1 = [-\cos x - x + \sin x]_0^{\pi/2} = 2 - \frac{\pi}{2},$$

$$A_2 = [x - \sin x + \cos x]_{\pi/2}^{\pi} = (\pi - 1) - \left(\frac{\pi}{2} - 1\right) = \frac{\pi}{2}.$$

5. Add the positive areas.

$$A = A_1 + A_2 = \left(2 - \frac{\pi}{2}\right) + \frac{\pi}{2} = \boxed{2}.$$

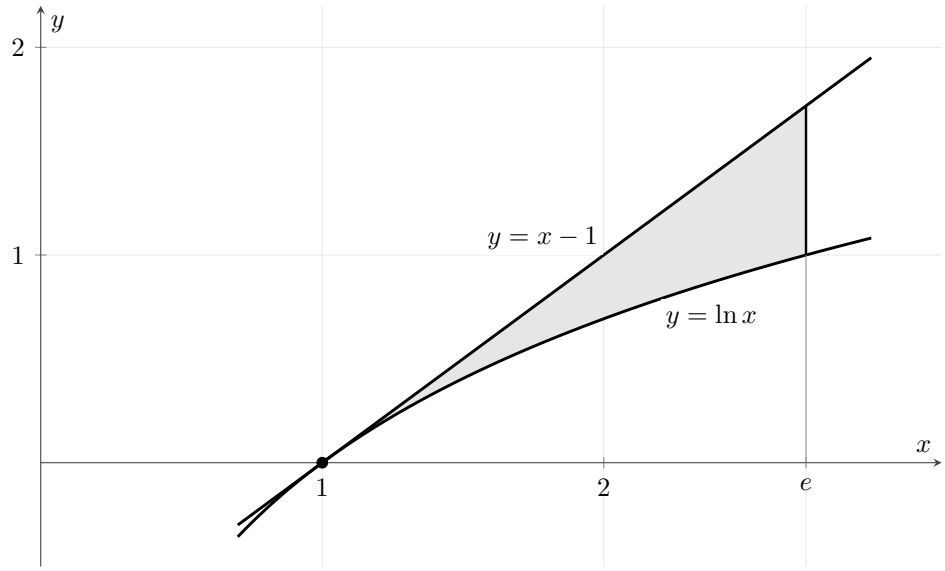
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Question 4

Find the exact area of the region between the curves

$$y = \ln x \quad \text{and} \quad y = x - 1,$$

between $x = 1$ and $x = e$.



The shaded region is the area to be found.

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Question 4 – Solution

1. Identify the upper curve on $[1, e]$.

Let $h(x) = x - 1 - \ln x$. Then $h(1) = 0$ and

$$h'(x) = 1 - \frac{1}{x} \geq 0 \qquad (1 \leq x \leq e).$$

Thus $x - 1 \geq \ln x$ throughout the interval.

2. Set up the area.

$$A = \int_1^e (x - 1 - \ln x)\, dx.$$

3. Find the logarithm antiderivative.

Integration by parts, with $u = \ln x$ and $dv = dx$, gives

$$\int \ln x\, dx = x \ln x - \int 1\, dx = x \ln x - x.$$

Here $du = dx/x$ and $v = x$. Combine the terms: $x^2/2 - x - (x \ln x - x) = x^2/2 - x \ln x$.

4. Apply the upper and lower bounds.

$$\begin{aligned} A &= \left[\frac{x^2}{2} - x - (x \ln x - x)\right]_1^e \\ &= \left[\frac{x^2}{2} - x \ln x\right]_1^e \\ &= \left(\frac{e^2}{2} - e\right) - \frac{1}{2} = \boxed{\frac{e^2}{2} - e - \frac{1}{2}}. \end{aligned}$$

5. Check the result.

Since $\ln e = 1$ and $\ln 1 = 0$, the endpoint terms above give an area of approximately 0.4762 square units, which is positive.

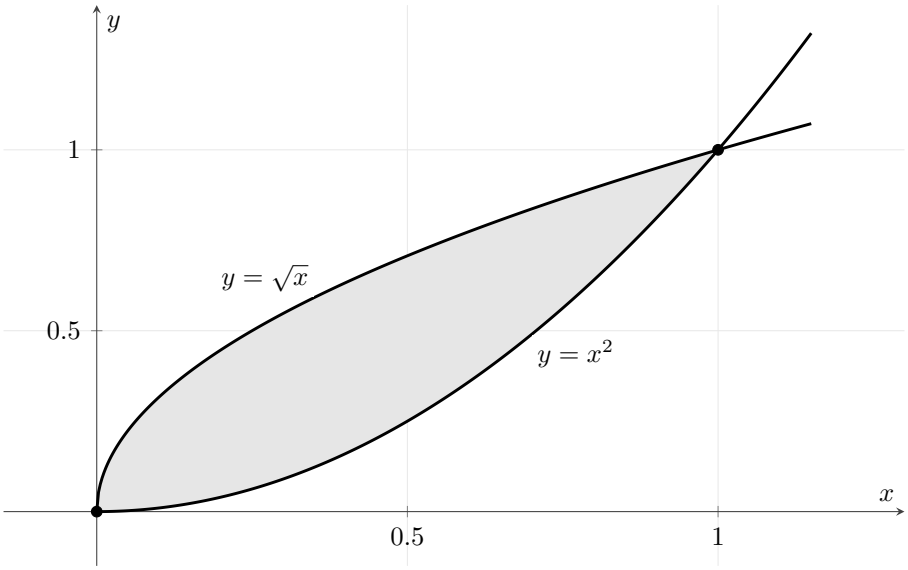
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Question 5

Find the exact area of the region enclosed by the curves

$$y = \sqrt{x} \quad \text{and} \quad y = x^2,$$

between their points of intersection.



The shaded region is the area to be found.

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Question 5 – Solution

1. Find the intersections.

The square root requires $x \geq 0$. Both sides are nonnegative, so squaring gives

$$\sqrt{x} = x^2 \quad \Longrightarrow \quad x = x^4 \quad \Longrightarrow \quad x(x^3 - 1) = 0.$$

The factor $x^3 - 1 = (x - 1)(x^2 + x + 1)$ has only one real zero, $x = 1$. The real solutions $x = 0$ and $x = 1$ both satisfy the original equation. The intersection points are $(0, 0)$ and $(1, 1)$.

2. Determine which curve is above the other.

For $0 < x < 1$, $x^{1/2} > x^2$. For example, at $x = 1/4$, $\sqrt{x} = 1/2$ while $x^2 = 1/16$.

3. Set up upper minus lower.

$$A = \int_0^1 \left(\sqrt{x} - x^2\right)\, dx.$$

4. Integrate each power.

Use $\int x^n\, dx = x^{n+1}/(n+1)$ for $n \neq -1$:

$$\int x^{1/2}\, dx = \frac{x^{3/2}}{3/2} = \frac{2}{3}x^{3/2}, \qquad \int x^2\, dx = \frac{x^3}{3}.$$

5. Apply the bounds and subtract.

$$\begin{aligned} A &= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1 \\ &= \left(\frac{2}{3} - \frac{1}{3}\right) - 0 = \boxed{\frac{1}{3}}. \end{aligned}$$

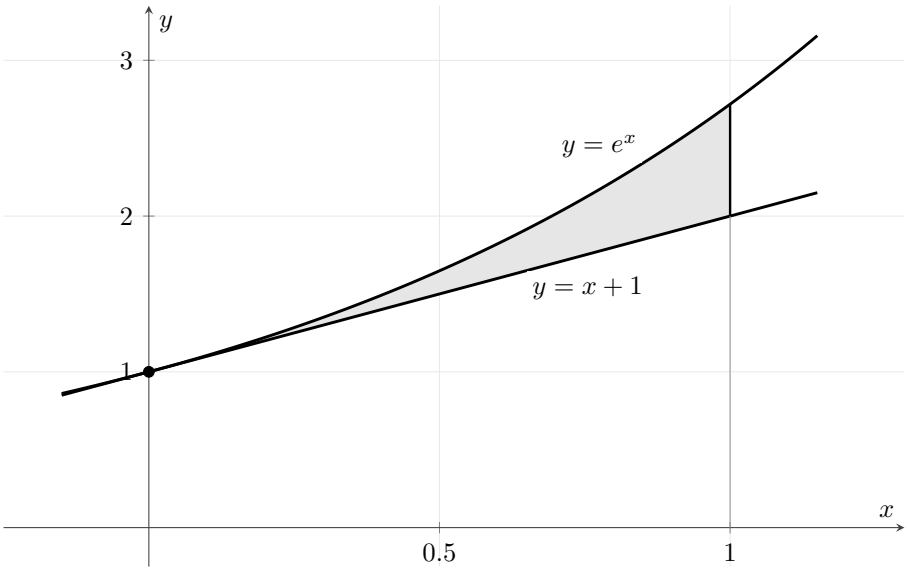
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Question 6

Find the exact area of the region bounded by the curves

$$y = e^x \qquad \text{and} \qquad y = x + 1,$$

and the vertical lines $x = 0$ and $x = 1$.



The shaded region is the area to be found.

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Question 6 – Solution

1. Identify the upper curve on $[0, 1]$.

Let $h(x) = e^x - x - 1$. Then $h(0) = 0$ and

$$h'(x) = e^x - 1 \geq 0 \qquad (0 \leq x \leq 1).$$

Thus $e^x \geq x + 1$, with equality only at $x = 0$.

2. Use the given vertical boundaries.

$$A = \int_0^1 \big(e^x - (x + 1)\big)\, dx.$$

3. Integrate term by term.

$$\int e^x\, dx = e^x, \qquad \int x\, dx = \frac{x^2}{2}, \qquad \int 1\, dx = x.$$

4. Evaluate at $x = 1$ and subtract the value at $x = 0$.

$$\begin{aligned} A &= \left[e^x - \frac{x^2}{2} - x\right]_0^1 \\ &= \left(e - \frac{1}{2} - 1\right) - (1 - 0 - 0) \\ &= \boxed{e - \frac{5}{2}}. \end{aligned}$$

5. Check the sign and units.

The area is approximately $2.71828 - 2.5 = 0.21828$ square units. It is positive because e^x lies above the line on $(0, 1]$.

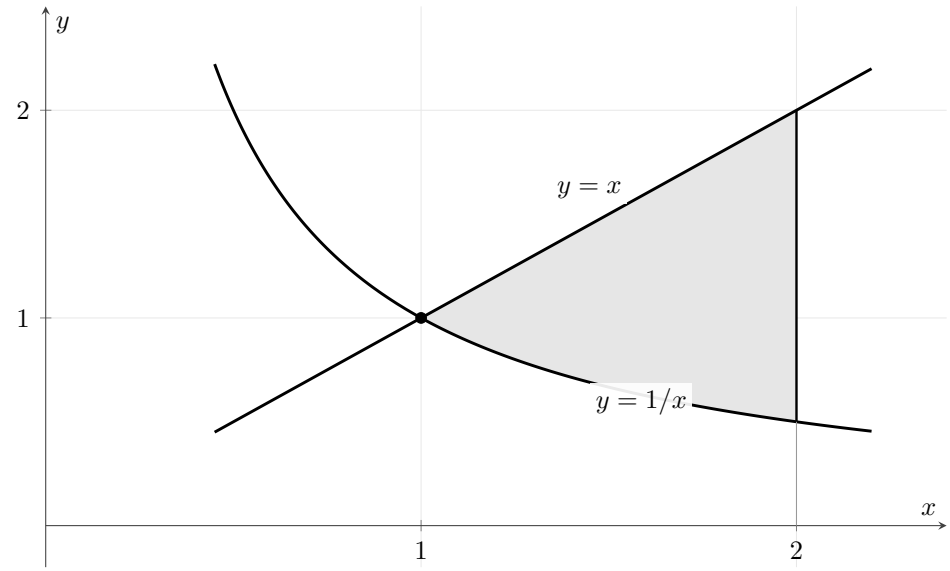
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Question 7

Find the exact area of the region between the curves

$$y = \frac{1}{x} \quad \text{and} \quad y = x,$$

between $x = 1$ and $x = 2$.



The shaded region is the area to be found.

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Question 7 – Solution

1. Identify the upper curve on $[1, 2]$.

For $x \geq 1$,

$$x - \frac{1}{x} = \frac{x^2 - 1}{x} \geq 0.$$

Thus the line $y = x$ lies above $y = 1/x$. They meet at $(1, 1)$, and the boundary $x = 2$ closes the region.

2. Set up the area.

$$A = \int_1^2 \left(x - \frac{1}{x} \right) dx.$$

3. Integrate each term.

Since $x > 0$ on this interval, an antiderivative of $1/x$ is $\ln x$.

$$\int x \, dx = \frac{x^2}{2}, \qquad \int \frac{1}{x} \, dx = \ln x.$$

4. Substitute the bounds and simplify.

$$\begin{aligned} A &= \left[\frac{x^2}{2} - \ln x \right]_1^2 \\ &= (2 - \ln 2) - \left(\frac{1}{2} - \ln 1 \right) \\ &= \boxed{\frac{3}{2} - \ln 2}. \end{aligned}$$

5. Check the result.

Using $\ln 2 \approx 0.69315$, the area is $1.5 - 0.69315 \approx 0.80685$ square units. The result is positive, as required.

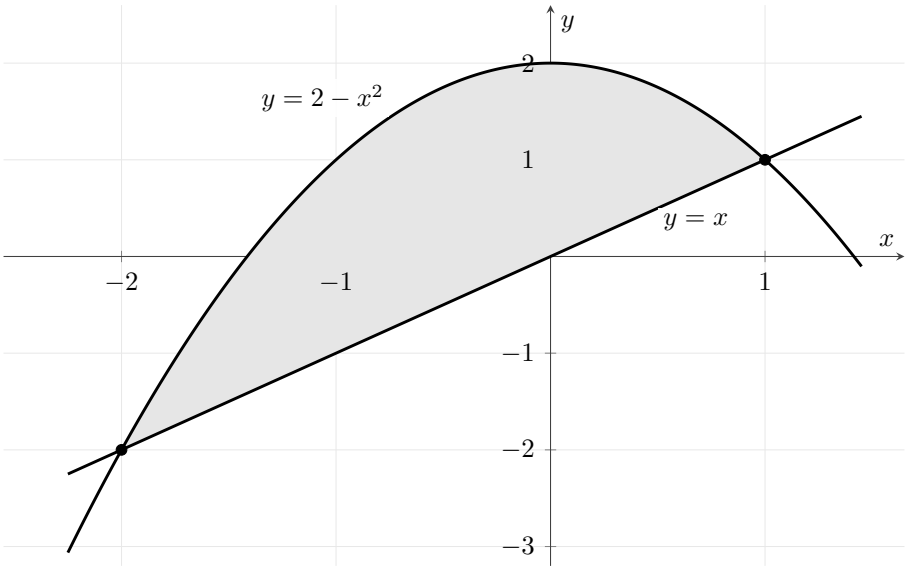
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Question 8

Find the exact area of the region enclosed by the curves

$$y = 2 - x^2 \quad \text{and} \quad y = x,$$

between their points of intersection.



The shaded region is the area to be found.

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Question 8 – Solution

1. Find the intersections.

$$2 - x^2 = x \quad \Longrightarrow \quad x^2 + x - 2 = 0 \quad \Longrightarrow \quad (x+2)(x-1) = 0.$$

The zero-product property gives $x+2=0$ or $x-1=0$. Thus $x=-2$ and $x=1$, giving points $(-2,-2)$ and $(1,1)$.

2. Determine which curve is above the other.

On $[-2,1]$, $(2-x^2)-x=(1-x)(x+2)\geq 0$. The parabola is above the line, even where both curves are below the x-axis.

3. Set up upper minus lower.

$$A = \int_{-2}^{1} (2 - x - x^2)\,dx.$$

4. Find an antiderivative.

$$\int 2\,dx = 2x, \qquad \int x\,dx = \frac{x^2}{2}, \qquad \int x^2\,dx = \frac{x^3}{3}.$$

5. Apply the bounds, taking care with negative signs.

$$\begin{aligned} A &= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1} \\ &= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) \\ &= \frac{7}{6} - \left(-\frac{10}{3}\right) = \boxed{\frac{9}{2}}. \end{aligned}$$

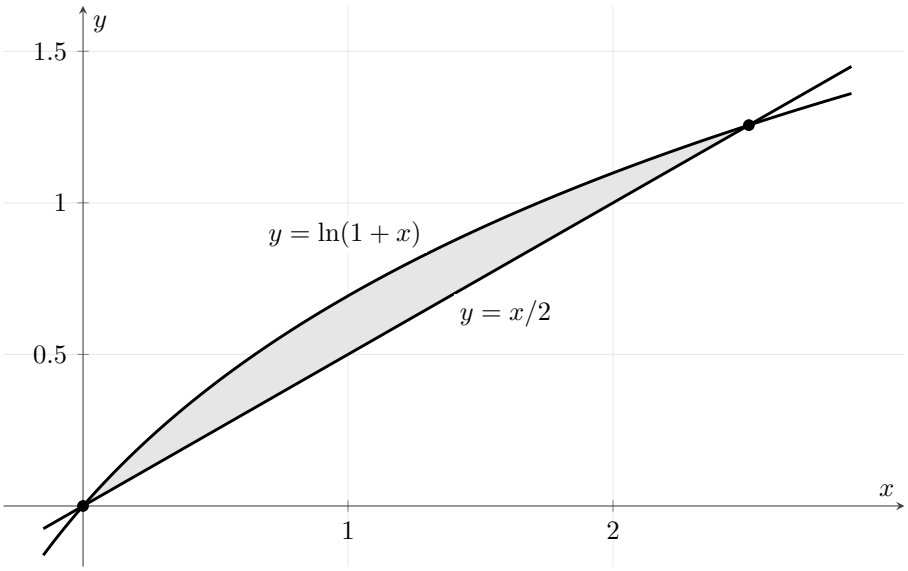
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Question 9

Find the exact area of the region enclosed by the curves

$$y = \ln(1 + x) \quad \text{and} \quad y = \frac{x}{2},$$

between their points of intersection.



The shaded region is the area to be found.

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Question 9 – Solution

1. Describe the intersections.

Let $h(x) = \ln(1+x) - x/2$, with domain $x > -1$. Then

$$h'(x) = \frac{1-x}{2(1+x)}.$$

So h increases up to $x = 1$ and decreases thereafter. Since $h(0) = 0$, $h(1) > 0$, and $h(x) \to -\infty$ as $x \to \infty$, the only roots are 0 and a unique number $a > 1$ satisfying

$$\ln(1+a) = \frac{a}{2}, \qquad a \approx 2.512862417.$$

The nonzero root has no elementary closed form; this equation defines a exactly.

2. Set up the area.

On $(0, a)$, $h(x) > 0$, so the logarithm is above the line.

$$A = \int_0^a \left(\ln(1+x) - \frac{x}{2}\right) dx.$$

3. Find an antiderivative.

With $u = 1 + x$, integration by parts gives

$$\int \ln(1+x)\, dx = (1+x)\ln(1+x) - x + C.$$

Also, $\int (x/2)\, dx = x^2/4$. At $x = 0$, all terms below are zero.

4. Evaluate, then substitute $\ln(1+a) = a/2$**.**

$$\begin{aligned} A &= \left[(1+x)\ln(1+x) - x - \frac{x^2}{4}\right]_0^a \\ &= (1+a)\frac{a}{2} - a - \frac{a^2}{4} \\ &= \boxed{\frac{a(a-2)}{4}} \approx 0.322188173. \end{aligned}$$

5. Interpret the exact answer.

The boxed expression is exact with a defined above; using its numerical approximation gives 0.322188173 square units.

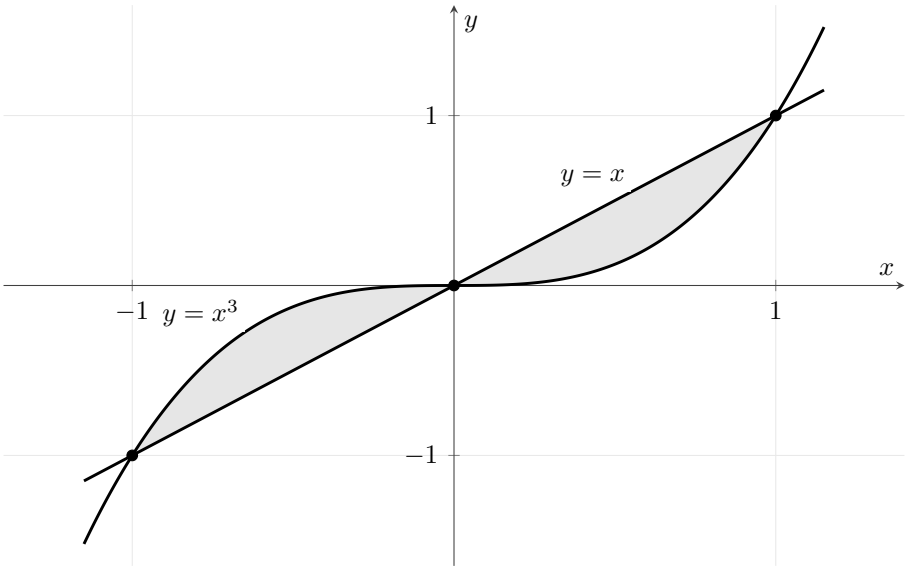
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Question 10

Find the exact total area of the regions enclosed by the curves

$$y = x^3 \quad \text{and} \quad y = x,$$

between their points of intersection.



The shaded region is the area to be found.

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Question 10 – Solution

1. Find all intersections.

$$x^3 = x \quad \Longrightarrow \quad x(x-1)(x+1) = 0 \quad \Longrightarrow \quad x = -1,\ 0,\ 1.$$

2. Split the area where the upper curve changes.

On $(-1, 0)$, $x^3 > x$; on $(0, 1)$, $x > x^3$. For example, at $x = -1/2$, $x^3 = -1/8 > -1/2$; at $x = 1/2$, $x = 1/2 > 1/8 = x^3$. Both enclosed regions contribute positive area:

3. Set up upper minus lower on each interval.

$$A = \int_{-1}^{0} (x^3 - x)\, dx + \int_{0}^{1} (x - x^3)\, dx.$$

4. Integrate and evaluate each region.

The power rule gives $\int x^3\, dx = x^4/4$ and $\int x\, dx = x^2/2$.

$$A_1 = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} = 0 - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{4},$$

$$A_2 = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

5. Add both areas.

$$A = A_1 + A_2 = \frac{1}{4} + \frac{1}{4} = \boxed{\frac{1}{2}}.$$

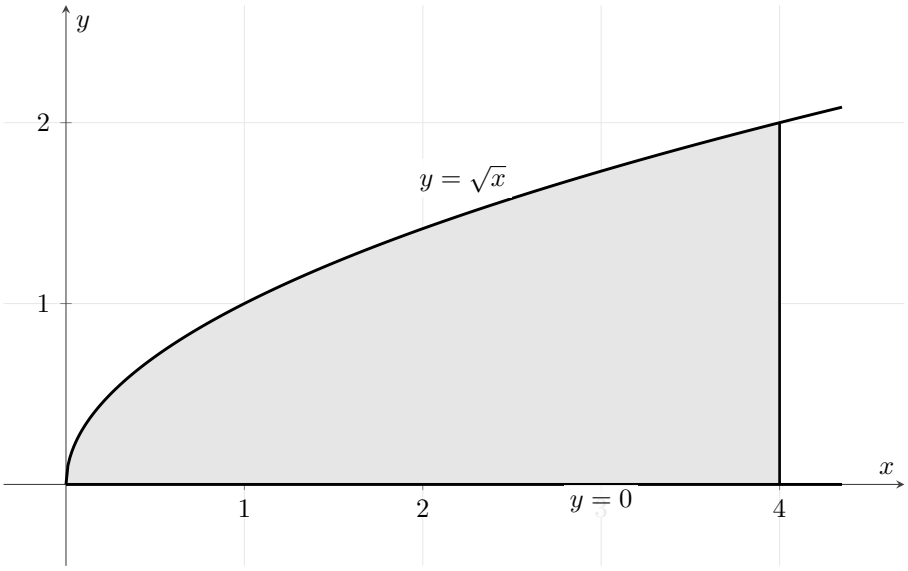
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Question 1

Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x} \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = 4$, about the x-axis.



Rotate the shaded region about the x-axis.

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Question 1 – Solution

1. Choose slices and bounds.

The rotation axis is horizontal, so use vertical slices of thickness dx. The stated interval is $0 \le x \le 4$. Each slice reaches the axis, producing a disk.

2. Measure the radii from the axis.

The distance from $y = 0$ to $y = \sqrt{x}$ is the outer radius. There is no central hole.

$$R(x) = \sqrt{x} - 0 = \sqrt{x}, \qquad r(x) = 0.$$

3. Write the cross-sectional area and volume.

$$A(x) = \pi R(x)^2 - \pi r(x)^2 = \pi\big((\sqrt{x})^2 - 0^2\big) = \pi x.$$
$$V = \int_0^4 A(x)\,dx = \pi \int_0^4 x\,dx.$$

4. Integrate using the power rule.

$$\int x\,dx = \frac{x^{1+1}}{1+1} = \frac{x^2}{2}, \qquad V = \pi \left[\frac{x^2}{2}\right]_0^4.$$

5. Apply the bounds and simplify.

$$V = \pi\left(\frac{4^2}{2} - \frac{0^2}{2}\right) = \pi\left(\frac{16}{2}\right) = \boxed{8\pi}.$$

All volumes are in cubic units.

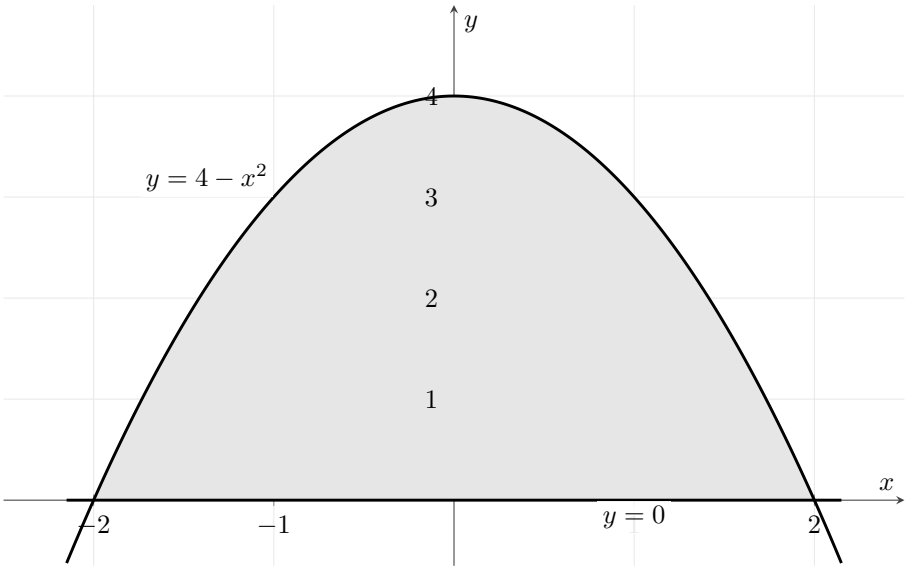
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Question 2

Find the volume of the solid obtained by rotating the region bounded by

$$y = 4 - x^2 \quad \text{and} \quad y = 0,$$

from $x = -2$ to $x = 2$, about the x-axis.



Rotate the shaded region about the x-axis.

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Question 2 – Solution

1. Choose slices and confirm the bounds.

Use vertical slices, perpendicular to the x-axis. The curve meets $y = 0$ where

$$4 - x^2 = 0 \quad \Longrightarrow \quad x = \pm 2.$$

2. Identify the radii.

On $[-2, 2]$, $4 - x^2 \geq 0$. The region touches the axis, so each cross-section is a disk:

$$R(x) = 4 - x^2, \qquad r(x) = 0.$$

3. Set up the volume and expand the square.

$$V = \pi \int_{-2}^{2} \left(R^2 - r^2\right) dx = \pi \int_{-2}^{2} (4 - x^2)^2 \, dx.$$
$$(4 - x^2)^2 = 16 - 8x^2 + x^4.$$

4. Use symmetry and integrate.

The integrand is even, so double the integral over $[0, 2]$:

$$V = 2\pi \int_{0}^{2} (16 - 8x^2 + x^4) \, dx = 2\pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_0^2.$$

5. Substitute and combine the fractions.

$$\begin{aligned} V &= 2\pi \left(32 - \frac{64}{3} + \frac{32}{5}\right) \\ &= 2\pi \left(\frac{480 - 320 + 96}{15}\right) = \boxed{\frac{512\pi}{15}}. \end{aligned}$$

All volumes are in cubic units.

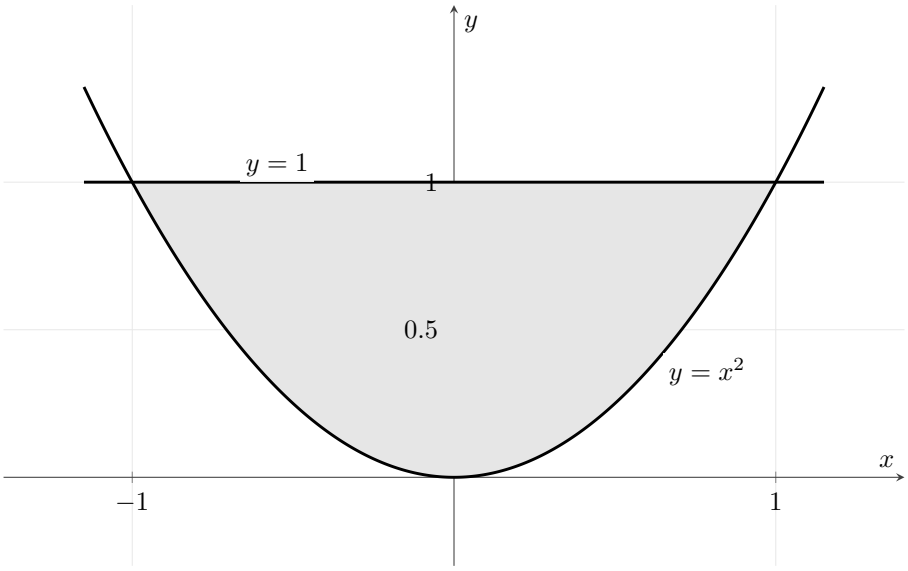
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Question 3

Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2 \quad \text{and} \quad y = 1,$$

from $x = -1$ to $x = 1$, about the x-axis.



Rotate the shaded region about the x-axis.

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Question 3 – Solution

1. Find the bounds and choose slices.

The intersections satisfy $x^2 = 1$, so $x = -1$ and $x = 1$. Use vertical slices of thickness dx because the rotation axis is horizontal.

2. Identify outer and inner radii.

On $[-1, 1]$, $0 \le x^2 \le 1$. Measured from the x-axis,

$$R(x) = 1, \qquad r(x) = x^2.$$

The missing inner disk must be subtracted from the outer disk.

3. Set up the washer integral.

$$A(x) = \pi\big(R(x)^2 - r(x)^2\big) = \pi\big(1^2 - (x^2)^2\big) = \pi(1 - x^4).$$

$$V = \pi \int_{-1}^{1} (1 - x^4)\, dx.$$

4. Integrate term by term.

$$\int 1\, dx = x, \qquad \int x^4\, dx = \frac{x^5}{5}.$$

$$V = \pi \left[x - \frac{x^5}{5} \right]_{-1}^{1}.$$

5. Evaluate both endpoints.

$$\begin{aligned} V &= \pi \left[\left(1 - \frac{1}{5}\right) - \left(-1 + \frac{1}{5}\right) \right] \\ &= \pi \left(\frac{4}{5} + \frac{4}{5} \right) = \boxed{\frac{8\pi}{5}}. \end{aligned}$$

All volumes are in cubic units.

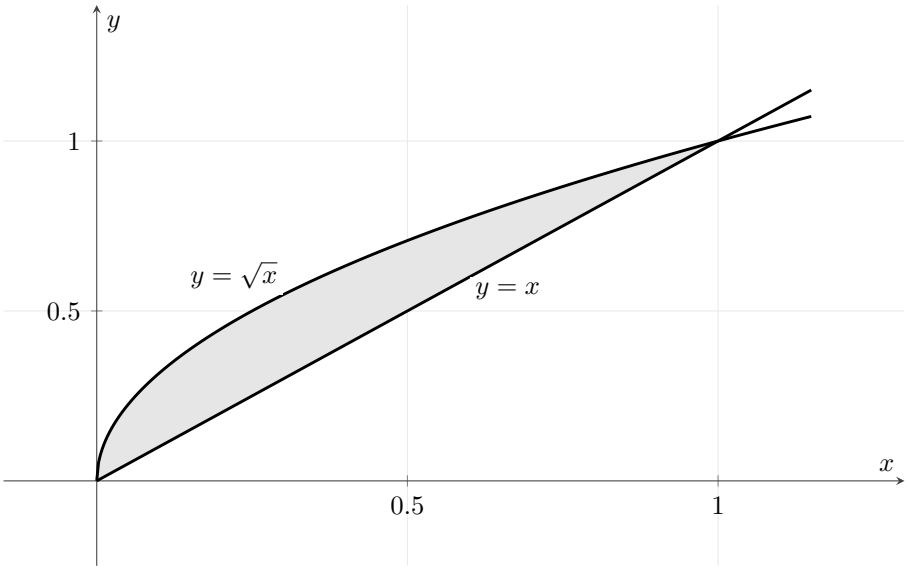
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Question 4

Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x} \quad \text{and} \quad y = x,$$

from $x = 0$ to $x = 1$, about the x-axis.



Rotate the shaded region about the x-axis.

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Question 4 – Solution

1. Find the bounds and choose slices.

The curves meet where $\sqrt{x} = x$. With $x \geq 0$, squaring gives $x = x^2$, so $x = 0$ or $x = 1$. Use vertical slices for rotation about the x-axis.

2. Measure the radii.

For $0 \leq x \leq 1$, $\sqrt{x} \geq x \geq 0$. Thus the square-root curve is farther from the axis:

$$R(x) = \sqrt{x}, \qquad r(x) = x.$$

3. Set up the washer integral.

Subtract the squared radii, then multiply by π:

$$V = \pi \int_0^1 \left(R^2 - r^2\right) dx = \pi \int_0^1 \left((\sqrt{x})^2 - x^2\right) dx = \pi \int_0^1 (x - x^2)\, dx.$$

4. Find the antiderivative.

$$\int x\, dx = \frac{x^2}{2}, \qquad \int x^2\, dx = \frac{x^3}{3}.$$

$$V = \pi \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1.$$

5. Apply the bounds and simplify.

$$V = \pi \left[\left(\frac{1}{2} - \frac{1}{3}\right) - 0\right] = \pi \left(\frac{3-2}{6}\right) = \boxed{\frac{\pi}{6}}.$$

All volumes are in cubic units.

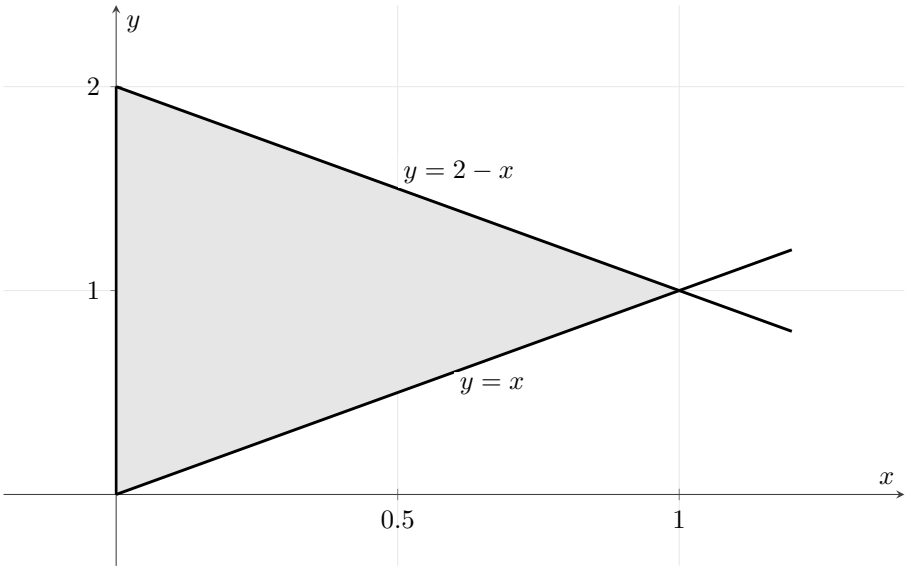
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Question 5

Find the volume of the solid obtained by rotating the region bounded by

$$y = 2 - x \quad \text{and} \quad y = x,$$

from $x = 0$ to $x = 1$, about the x-axis.



Rotate the shaded region about the x-axis.

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Question 5 – Solution

1. Identify the bounds and slice direction.

The given left boundary is $x = 0$. The lines meet where $2 - x = x$, giving $x = 1$. Use vertical slices, perpendicular to the x-axis.

2. Measure the radii from the x-axis.

On $[0, 1]$, $2 - x \geq x \geq 0$, so

$$R(x) = 2 - x, \qquad r(x) = x.$$

The rotated slice is a washer with area $\pi(R^2 - r^2)$.

3. Set up and simplify the volume integral.

$$V = \pi \int_0^1 \left[(2 - x)^2 - x^2\right] dx.$$

$$(2 - x)^2 - x^2 = (4 - 4x + x^2) - x^2 = 4 - 4x.$$

4. Integrate term by term.

$$\int 4\, dx = 4x, \qquad \int 4x\, dx = 4\frac{x^2}{2} = 2x^2.$$

$$V = \pi \left[4x - 2x^2\right]_0^1.$$

5. Apply the bounds.

$$V = \pi\big[(4(1) - 2(1)^2) - (4(0) - 2(0)^2)\big] = \pi(4 - 2) = \boxed{2\pi}.$$

All volumes are in cubic units.

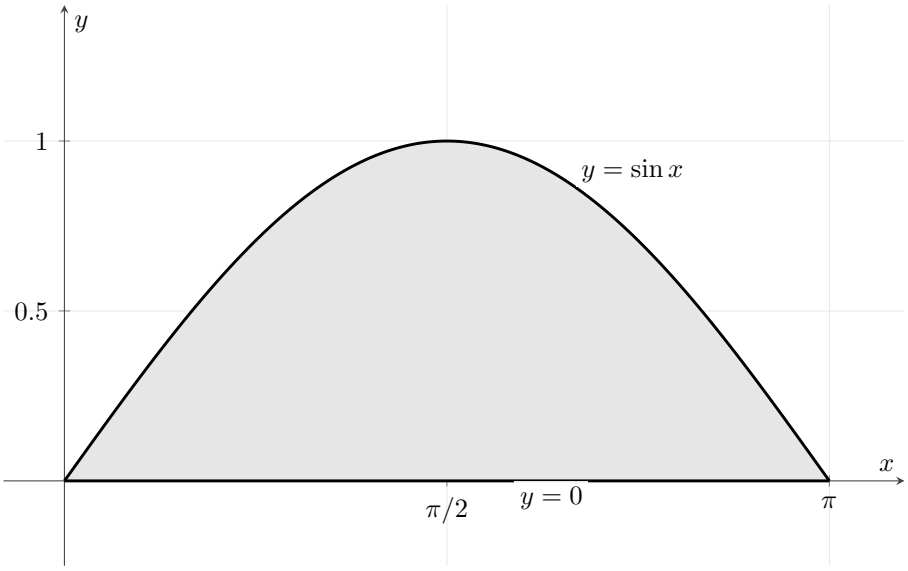
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Question 6

Find the volume of the solid obtained by rotating the region bounded by

$$y = \sin x \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = \pi$, about the x-axis.



Rotate the shaded region about the x-axis.

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Question 6 – Solution

1. Choose slices and identify the bounds.

Use vertical slices over $0 \le x \le \pi$. Since $\sin x \ge 0$ on this interval and the region touches the x-axis, rotation gives disks.

2. Identify radii and set up the volume.

$$R(x) = \sin x, \qquad r(x) = 0.$$
$$V = \pi \int_0^{\pi} \big((\sin x)^2 - 0^2\big) dx = \pi \int_0^{\pi} \sin^2 x \, dx.$$

3. Use the power-reduction identity.

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \quad \Longrightarrow \quad V = \frac{\pi}{2} \int_0^{\pi} (1 - \cos(2x)) \, dx.$$

4. Integrate, including the chain-rule factor.

Because $\dfrac{d}{dx} \sin(2x) = 2\cos(2x)$,

$$\int \cos(2x) \, dx = \frac{\sin(2x)}{2}, \qquad V = \frac{\pi}{2} \left[x - \frac{\sin(2x)}{2} \right]_0^{\pi}.$$

5. Evaluate at π and 0.

$$\begin{aligned} V &= \frac{\pi}{2} \left[\left(\pi - \frac{\sin(2\pi)}{2} \right) - \left(0 - \frac{\sin 0}{2} \right) \right] \\ &= \frac{\pi}{2}(\pi - 0) = \boxed{\frac{\pi^2}{2}}. \end{aligned}$$

All volumes are in cubic units.

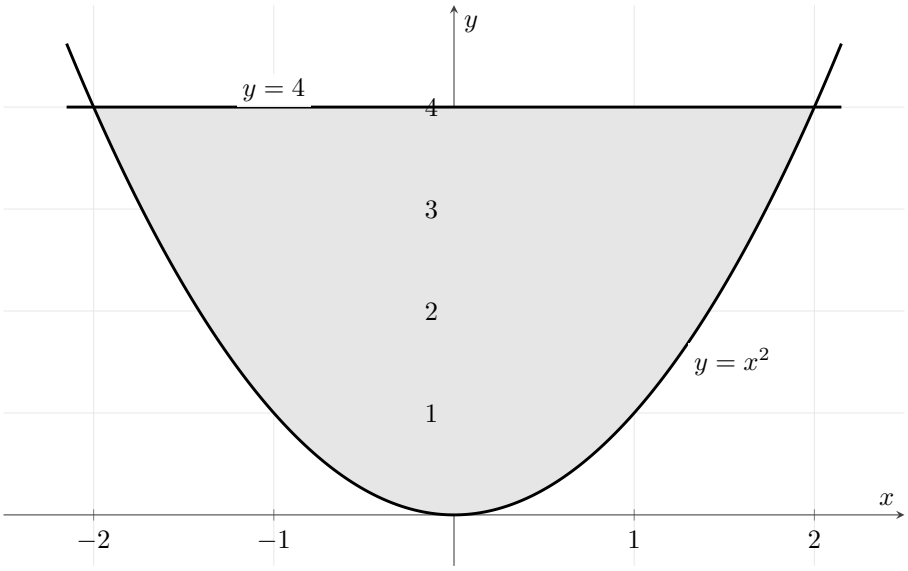
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Question 7

Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2 \quad \text{and} \quad y = 4,$$

about the y-axis.



Rotate the shaded region about the y-axis.

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Question 7 – Solution

1. Use horizontal slices and find the y-bounds.

Rotation about the y-axis requires slices of thickness dy. The region lies above $y = x^2$ and below $y = 4$, so $0 \le y \le 4$.

2. Express the horizontal boundaries in terms of y.

$$y = x^2 \quad \Longrightarrow \quad x = \pm\sqrt{y}.$$

At height y, the region extends from $-\sqrt{y}$ to $\sqrt{y}$ and crosses the rotation axis.

3. Identify the disk radius.

Rotating the entire horizontal segment gives one disk, not two separate disks. Its radius is the distance from the axis to either edge:

$$R(y) = \sqrt{y}, \qquad r(y) = 0.$$

4. Set up the volume and integrate.

$$V = \pi \int_0^4 \big(R^2 - r^2\big)dy = \pi \int_0^4 \big((\sqrt{y})^2 - 0^2\big)dy = \pi \int_0^4 y\,dy.$$

$$\int y\,dy = \frac{y^2}{2}.$$

5. Apply the bounds.

$$V = \pi \left[\frac{y^2}{2}\right]_0^4 = \pi \left(\frac{16}{2} - 0\right) = \boxed{8\pi}.$$

All volumes are in cubic units.

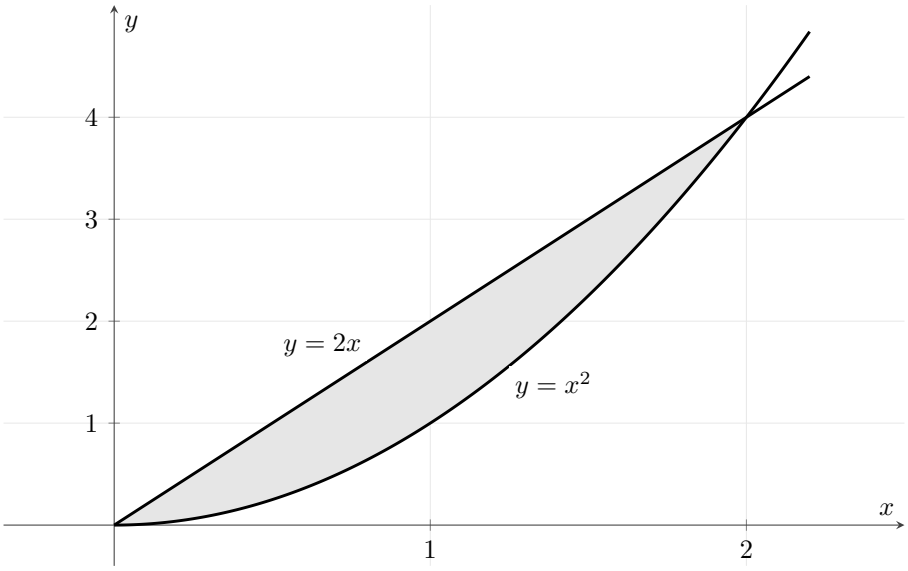
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Question 8

Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2 \quad \text{and} \quad y = 2x,$$

about the y-axis.



Rotate the shaded region about the y-axis.

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Question 8 – Solution

1. Find the intersections and y-bounds.

$$x^2 = 2x \quad \Longrightarrow \quad x(x-2) = 0 \quad \Longrightarrow \quad x = 0, 2.$$

The intersection points are $(0,0)$ and $(2,4)$, so integrate from $y = 0$ to $y = 4$.

2. Use horizontal slices and identify radii.

Rewrite the curves as $x = \sqrt{y}$ and $x = y/2$. For $0 \le y \le 4$, $\sqrt{y} \ge y/2$. Distances from the y-axis give

$$R(y) = \sqrt{y}, \qquad r(y) = \frac{y}{2}.$$

3. Set up the washer integral.

$$V = \pi \int_0^4 \left[(\sqrt{y})^2 - \left(\frac{y}{2}\right)^2 \right] dy = \pi \int_0^4 \left(y - \frac{y^2}{4} \right) dy.$$

4. Integrate term by term.

$$\int y\, dy = \frac{y^2}{2}, \qquad \int \frac{y^2}{4}\, dy = \frac{1}{4}\frac{y^3}{3} = \frac{y^3}{12}.$$

$$V = \pi \left[\frac{y^2}{2} - \frac{y^3}{12} \right]_0^4.$$

5. Apply the bounds and simplify.

$$V = \pi \left(\frac{16}{2} - \frac{64}{12} - 0 \right) = \pi \left(8 - \frac{16}{3} \right) = \boxed{\frac{8\pi}{3}}.$$

All volumes are in cubic units.

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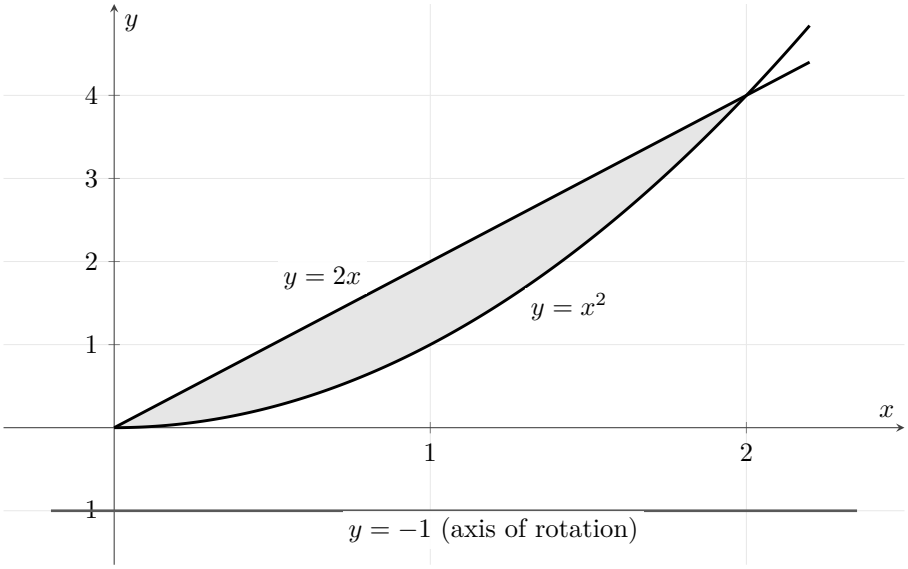
Question 9

Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2 \quad \text{and} \quad y = 2x,$$

about the horizontal line

$$y = -1.$$



Rotate the shaded region about the line $y = -1$.

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Question 9 – Solution

1. Find the bounds and choose vertical slices.

The curves meet where $x^2 = 2x$, or $x(x-2) = 0$. Thus $0 \le x \le 2$. The horizontal rotation axis $y = -1$ requires vertical slices of thickness dx.

2. Measure both radii from $y = -1$.

On $[0, 2]$, $2x \ge x^2$. The outer radius reaches the line, and the inner radius reaches the parabola:

$$R(x) = 2x - (-1) = 2x + 1, \qquad r(x) = x^2 - (-1) = x^2 + 1.$$

3. Set up the washer integral and expand.

$$V = \pi \int_0^2 \left[(2x+1)^2 - (x^2+1)^2\right] dx.$$

$$\begin{aligned}(2x+1)^2 - (x^2+1)^2 &= (4x^2 + 4x + 1) - (x^4 + 2x^2 + 1) \\ &= -x^4 + 2x^2 + 4x.\end{aligned}$$

4. Integrate each power.

$$V = \pi \left[-\frac{x^5}{5} + \frac{2x^3}{3} + 2x^2\right]_0^2.$$

The coefficients follow from $\int x^n\,dx = x^{n+1}/(n+1)$.

5. Evaluate and use a common denominator.

$$\begin{aligned}V &= \pi\left(-\frac{32}{5} + \frac{16}{3} + 8 - 0\right) \\ &= \pi\left(\frac{-96 + 80 + 120}{15}\right) = \boxed{\frac{104\pi}{15}}.\end{aligned}$$

All volumes are in cubic units.

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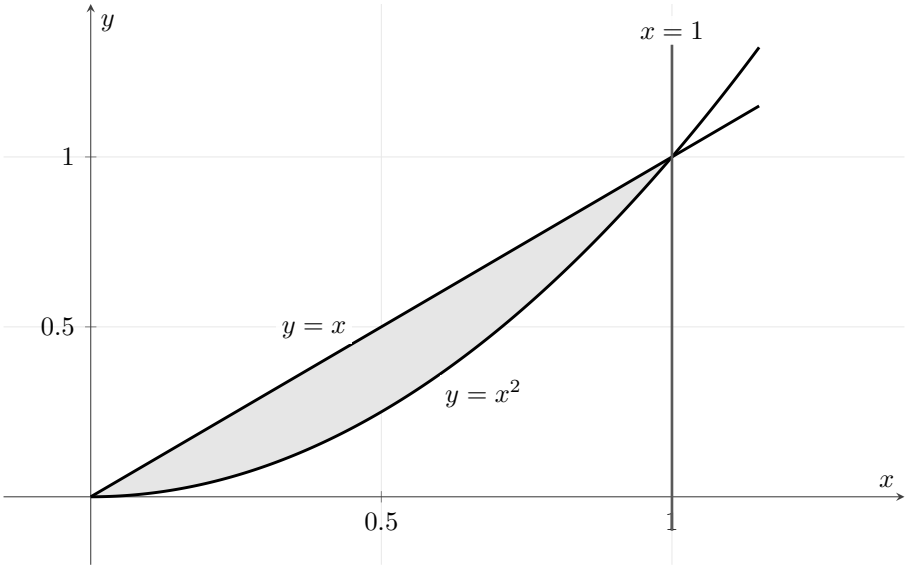
Question 10

Find the volume of the solid obtained by rotating the region bounded by

$$y = x \quad \text{and} \quad y = x^2,$$

about the vertical line

$$x = 1.$$



Rotate the shaded region about the line $x = 1$.

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Question 10 – Solution

1. Find the bounds and choose horizontal slices.

The intersections satisfy $x = x^2$, so $x = 0, 1$ and $y = 0, 1$. Rotation about the vertical line $x = 1$ requires horizontal slices of thickness dy.

2. Rewrite the curves and measure radii.

The boundaries are $x = y$ on the left and $x = \sqrt{y}$ on the right. The axis is to their right, so the farther, left boundary gives the outer radius:

$$R(y) = 1 - y, \qquad r(y) = 1 - \sqrt{y}.$$

3. Set up the washer integral and expand.

$$V = \pi \int_0^1 \left[(1-y)^2 - (1-\sqrt{y})^2\right] dy.$$

$$\begin{aligned}(1-y)^2 - (1-\sqrt{y})^2 &= (1 - 2y + y^2) - (1 - 2\sqrt{y} + y) \\ &= y^2 - 3y + 2y^{1/2}.\end{aligned}$$

4. Integrate using the power rule.

In particular, $\int 2y^{1/2}\, dy = 2 \cdot \frac{2}{3} y^{3/2} = \frac{4}{3} y^{3/2}$. Therefore,

$$V = \pi \left[\frac{y^3}{3} - \frac{3y^2}{2} + \frac{4}{3} y^{3/2}\right]_0^1.$$

5. Apply the bounds and simplify.

$$V = \pi \left(\frac{1}{3} - \frac{3}{2} + \frac{4}{3} - 0\right) = \pi \left(\frac{2 - 9 + 8}{6}\right) = \boxed{\frac{\pi}{6}}.$$

All volumes are in cubic units.

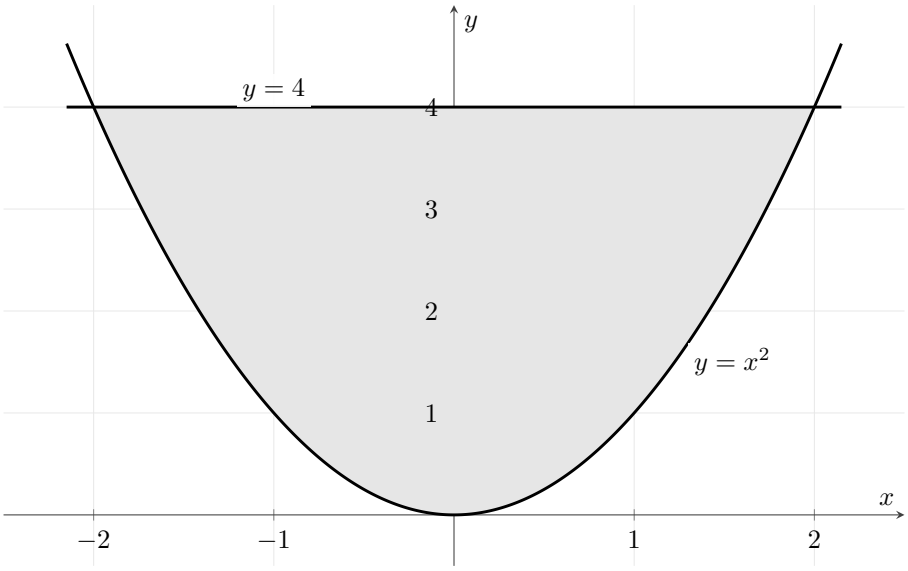
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Question 1

Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2 \qquad \text{and} \qquad y = 4$$

about the y-axis.



Rotate the shaded region about the y-axis.

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Question 1 – Solution

1. Find the intersections and choose slices.

For shells about the y-axis, use vertical slices parallel to the axis. The curves meet where $x^2 = 4$, so $x = \pm 2$.

2. Choose an interval that counts each shell once.

The half-region $0 \le x \le 2$ generates the entire solid. The negative half generates the same shells, so do not double the volume or integrate over both halves.

3. Identify radius, height, and the shell integral.

The radius is the distance to the y-axis; the height is top minus bottom:

$$r(x) = x, \qquad h(x) = 4 - x^2.$$

$$V = 2\pi \int_0^2 r(x)h(x)\,dx = 2\pi \int_0^2 x(4 - x^2)\,dx.$$

4. Expand and integrate.

$$x(4 - x^2) = 4x - x^3, \qquad \int (4x - x^3)\,dx = 2x^2 - \frac{x^4}{4}.$$

5. Apply the bounds and simplify.

$$\begin{aligned} V &= 2\pi \left[2x^2 - \frac{x^4}{4}\right]_0^2 \\ &= 2\pi \left[\left(8 - \frac{16}{4}\right) - 0\right] = 2\pi(4) = \boxed{8\pi}. \end{aligned}$$

All volumes are in cubic units.

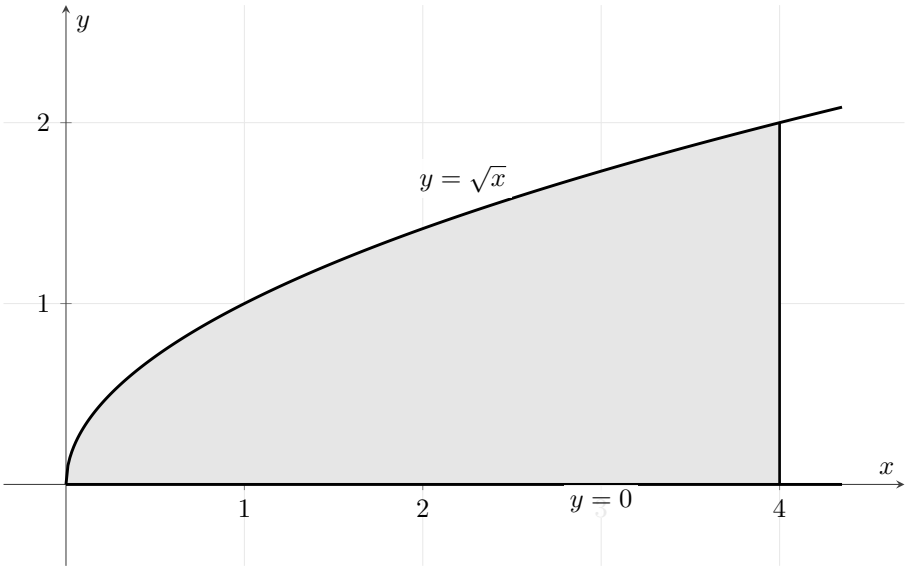
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Question 2

Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x} \qquad \text{and} \qquad y = 0,$$

from $x = 0$ to $x = 4$, about the x-axis.



Rotate the shaded region about the x-axis.

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Question 2 – Solution

1. Choose horizontal slices and y-bounds.

Shell slices run parallel to the x-axis. Rewrite $y = \sqrt{x}$ as $x = y^2$. The region extends from $y = 0$ to $y = \sqrt{4} = 2$.

2. Identify the shell radius and height.

The radius is the distance to $y = 0$. For a horizontal slice, shell height is its horizontal length, right minus left:

$$r(y) = y, \qquad h(y) = 4 - y^2.$$

3. Set up the shell integral.

A thin shell has volume approximately $(2\pi r)(h)\,dy$. Thus

$$V = 2\pi \int_0^2 y(4 - y^2)\,dy = 2\pi \int_0^2 (4y - y^3)\,dy.$$

4. Integrate with respect to y.

$$\int 4y\,dy = 2y^2, \qquad \int y^3\,dy = \frac{y^4}{4}.$$

$$V = 2\pi \left[2y^2 - \frac{y^4}{4}\right]_0^2.$$

5. Evaluate both endpoints.

$$V = 2\pi \left[\left(2(2)^2 - \frac{2^4}{4}\right) - 0\right] = 2\pi(8 - 4) = \boxed{8\pi}.$$

All volumes are in cubic units.

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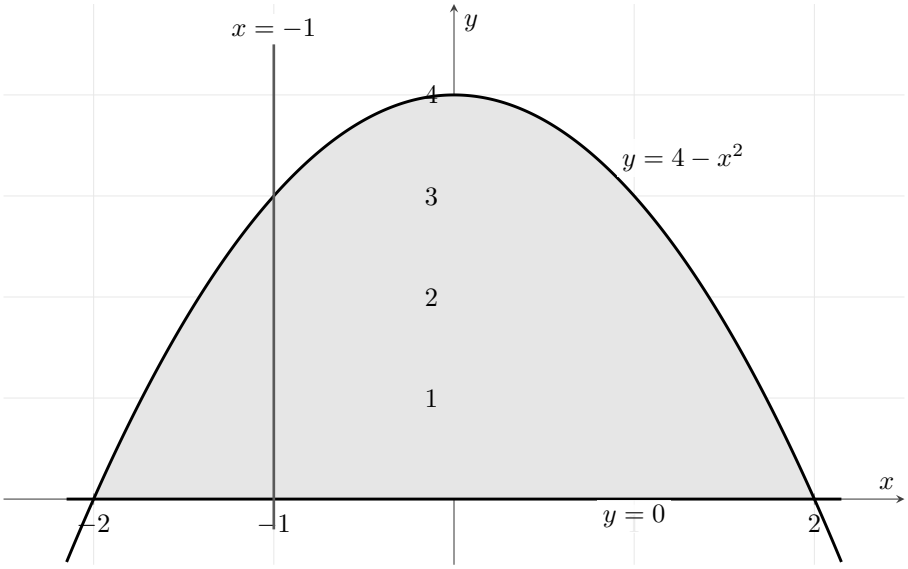
Question 3

Find the volume of the solid obtained by rotating the region bounded by

$$y = 4 - x^2 \qquad \text{and} \qquad y = 0,$$

about the vertical line

$$x = -1.$$



Rotate the shaded region about the line $x = -1$.

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Question 3 – Solution

1. Locate the region and rotation axis.

The parabola meets $y = 0$ at $x = \pm 2$. The axis $x = -1$ passes through the region. Vertical strips on opposite sides can rotate into the same shell.

2. Count each radius once.

Let r be distance from $x = -1$. The right strip is at $x = -1+r$, with $0 \le r \le 3$. Its height is $4-(r-1)^2$. For $0 \le r \le 1$, a left strip also exists at $x = -1 - r$, with height $4 - (r+1)^2$.

$$\left[4 - (r-1)^2\right] - \left[4 - (r+1)^2\right] = 4r \ge 0.$$

Both strips start at $y = 0$, so the taller right strip includes the entire left strip after rotation. Use only the right strip.

3. Set up the shell integral.

$$h(r) = 4 - (r-1)^2 = 3 + 2r - r^2.$$

$$V = 2\pi \int_0^3 r(3 + 2r - r^2)\, dr = 2\pi \int_0^3 (3r + 2r^2 - r^3)\, dr.$$

4. Integrate each power.

$$V = 2\pi \left[\frac{3}{2}r^2 + \frac{2}{3}r^3 - \frac{1}{4}r^4\right]_0^3.$$

5. Apply the bounds.

$$V = 2\pi \left(\frac{27}{2} + 18 - \frac{81}{4}\right) = 2\pi \left(\frac{54 + 72 - 81}{4}\right) = \boxed{\frac{45\pi}{2}}.$$

All volumes are in cubic units.

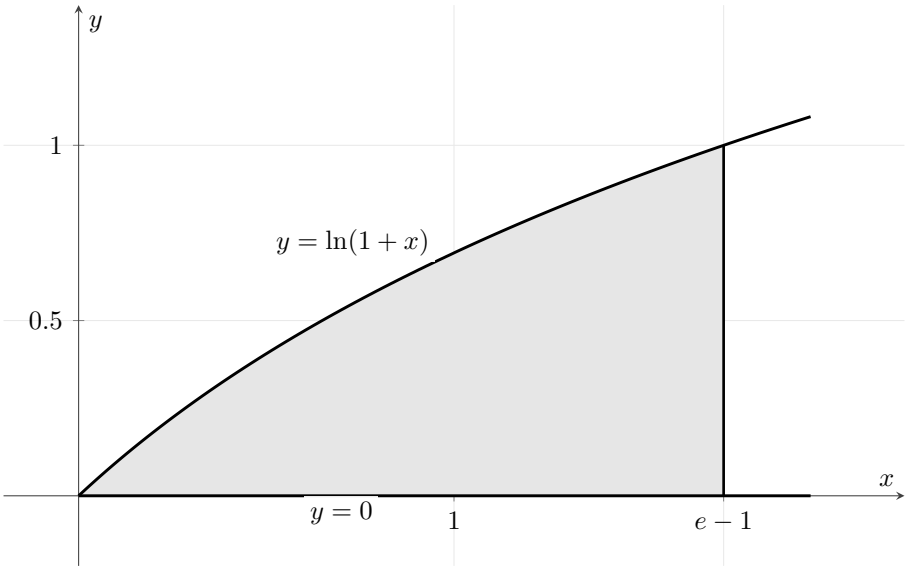
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Question 4

Find the volume of the solid obtained by rotating the region bounded by

$$y = \ln(1 + x) \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = e - 1$, about the y-axis.



Rotate the shaded region about the y-axis.

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Question 4 – Solution

1. Identify shell radius and height.

Use vertical slices about the y-axis, with $0 \le x \le e-1$:

$$r(x) = x, \qquad h(x) = \ln(1+x) - 0.$$

2. Set up the volume.

$$V = 2\pi \int_0^{e-1} x \ln(1+x)\, dx.$$

3. Apply integration by parts.

Choose $u = \ln(1+x)$ and $dv = x\,dx$. Then $du = dx/(1+x)$ and $v = x^2/2$:

$$\int x \ln(1+x)\, dx = \frac{x^2}{2}\ln(1+x) - \frac{1}{2}\int \frac{x^2}{1+x}\, dx.$$

4. Divide and integrate the remaining terms.

Since $x^2 = (x+1)(x-1) + 1$,

$$\frac{x^2}{1+x} = x - 1 + \frac{1}{1+x}.$$

An antiderivative is therefore

$$F(x) = \frac{x^2}{2}\ln(1+x) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(1+x).$$

5. Apply the bounds and multiply by 2π.

Using $\ln e = 1$ and $F(0) = 0$,

$$F(e-1) - F(0) = \frac{(e-1)^2}{4} + \frac{e-1}{2} - \frac{1}{2} = \frac{e^2-3}{4},$$

$$V = 2\pi \cdot \frac{e^2-3}{4} = \boxed{\frac{\pi}{2}(e^2-3)}.$$

All volumes are in cubic units.

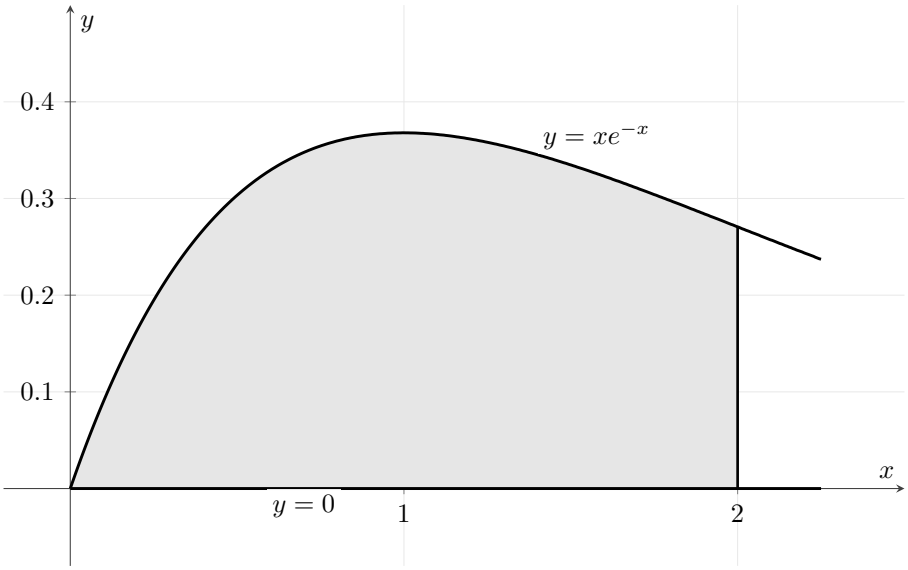
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Question 5

Find the volume of the solid obtained by rotating the region bounded by

$$y = xe^{-x} \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = 2$, about the y-axis.



Rotate the shaded region about the y-axis.

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Question 5 – Solution

1. Identify the shell dimensions.

Use vertical slices for rotation about the y-axis. On $0 \le x \le 2$,

$$r(x) = x, \qquad h(x) = xe^{-x}.$$

2. Set up the volume integral.

$$V = 2\pi \int_0^2 r(x)h(x)\,dx = 2\pi \int_0^2 x^2 e^{-x}\,dx.$$

3. Integrate by parts once.

Let $u = x^2$, $dv = e^{-x}dx$, so $du = 2x\,dx$ and $v = -e^{-x}$:

$$\int x^2 e^{-x}\,dx = -x^2e^{-x} + 2\int xe^{-x}\,dx.$$

4. Integrate by parts a second time.

For the remaining integral, choose $u = x$ and $dv = e^{-x}dx$:

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x}.$$

Substitution gives $F(x) = -e^{-x}(x^2 + 2x + 2)$.

5. Evaluate at 2 and 0.

$$\begin{aligned} V &= 2\pi\left[-e^{-x}(x^2+2x+2)\right]_0^2 \\ &= 2\pi\left[-10e^{-2} - (-2)\right] \\ &= 2\pi(2 - 10e^{-2}) = \boxed{4\pi(1-5e^{-2})}. \end{aligned}$$

All volumes are in cubic units.

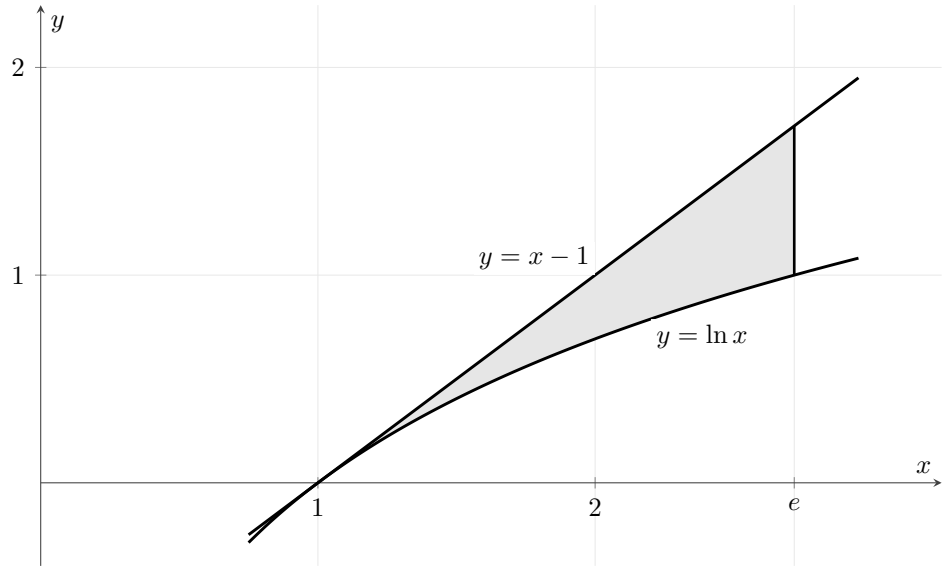
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Question 6

Use cylindrical shells to find the volume obtained by revolving the region bounded by

$$y = \ln x, \qquad y = x - 1, \qquad x = 1, \qquad x = e$$

about the y-axis.



Rotate the shaded region about the y-axis.

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Question 6 – Solution

1. Determine the upper curve.

Let $g(x) = x - 1 - \ln x$. On $[1, e]$, $g(1) = 0$ and $g'(x) = 1 - 1/x \geq 0$. Hence $x - 1 \geq \ln x$.

2. Identify shell dimensions and set up the volume.

Use vertical slices parallel to the y-axis:

$$r(x) = x, \qquad h(x) = x - 1 - \ln x.$$

$$V = 2\pi \int_1^e x(x - 1 - \ln x)\, dx = 2\pi \int_1^e (x^2 - x - x\ln x)\, dx.$$

3. Integrate the logarithmic term by parts.

Let $u = \ln x$, $dv = x\, dx$, so $du = dx/x$ and $v = x^2/2$:

$$\int x \ln x\, dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\, dx = \frac{x^2}{2}\ln x - \frac{x^2}{4}.$$

4. Combine the antiderivatives.

$$F(x) = \frac{x^3}{3} - \frac{x^2}{2} - \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right) = \frac{x^3}{3} - \frac{x^2}{4} - \frac{x^2}{2}\ln x.$$

5. Apply the bounds.

$$F(e) = \frac{e^3}{3} - \frac{3e^2}{4}, \qquad F(1) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

$$V = 2\pi\big(F(e) - F(1)\big) = \boxed{\pi\left(\frac{2e^3}{3} - \frac{3e^2}{2} - \frac{1}{6}\right)}.$$

All volumes are in cubic units.

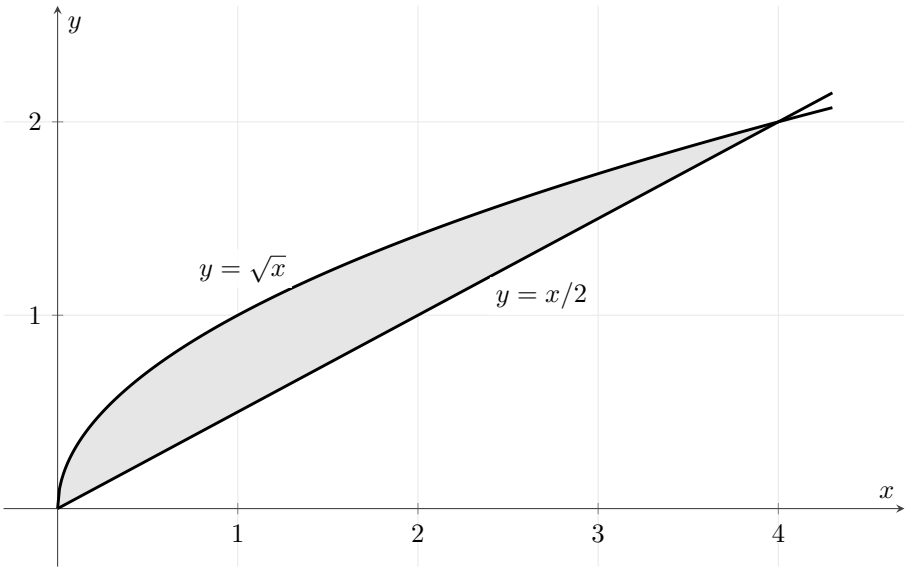
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Question 7

Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \qquad y = \frac{x}{2},$$

between their points of intersection, about the y-axis.



Rotate the shaded region about the y-axis.

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Question 7 – Solution

1. Find the intersections.

Both sides of $\sqrt{x} = x/2$ are nonnegative. Squaring and factoring gives

$$x = \frac{x^2}{4} \quad \Longrightarrow \quad x(x-4) = 0 \quad \Longrightarrow \quad x = 0, 4.$$

Both values satisfy the original equation.

2. Identify radius and height.

Use vertical shells about the y-axis. On $[0, 4]$, $\sqrt{x} \geq x/2$:

$$r(x) = x, \qquad h(x) = \sqrt{x} - \frac{x}{2}.$$

3. Set up and expand the shell integral.

$$V = 2\pi \int_0^4 x \left(\sqrt{x} - \frac{x}{2}\right) dx = 2\pi \int_0^4 \left(x^{3/2} - \frac{x^2}{2}\right) dx.$$

4. Integrate using the power rule.

$$\int x^{3/2}\, dx = \frac{2}{5}x^{5/2}, \qquad \int \frac{x^2}{2}\, dx = \frac{x^3}{6}.$$

$$V = 2\pi \left[\frac{2}{5}x^{5/2} - \frac{x^3}{6}\right]_0^4.$$

5. Apply the bounds and simplify.

Since $4^{5/2} = (\sqrt{4})^5 = 32$,

$$V = 2\pi \left(\frac{64}{5} - \frac{64}{6}\right) = 2\pi \left(\frac{192 - 160}{15}\right) = \boxed{\frac{64\pi}{15}}.$$

All volumes are in cubic units.

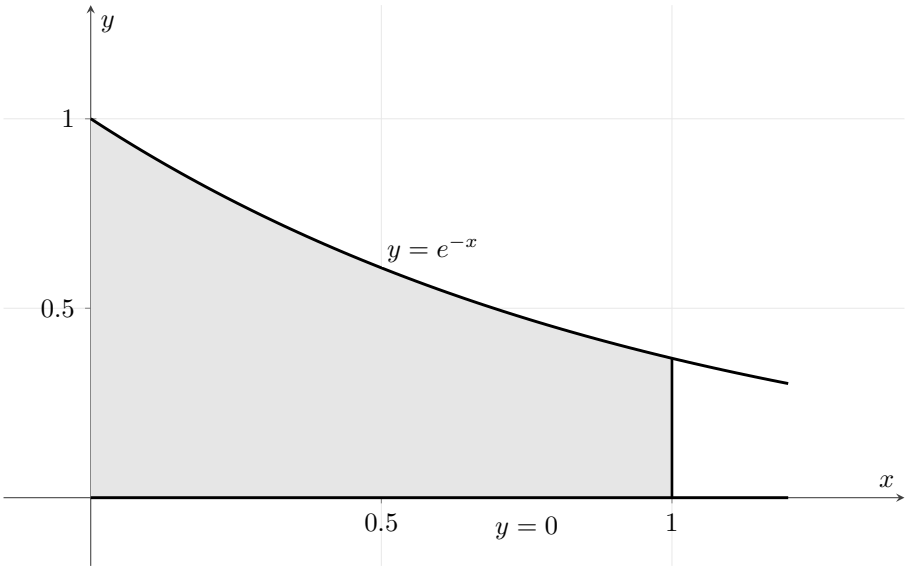
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Question 8

Find the volume of the solid obtained by rotating the region bounded by

$$y = e^{-x}, \qquad y = 0,$$

from $x = 0$ to $x = 1$, about the y-axis.



Rotate the shaded region about the y-axis.

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Question 8 – Solution

1. Identify the shell radius and height.

Use vertical slices over $0 \le x \le 1$, parallel to the y-axis:

$$r(x) = x, \qquad h(x) = e^{-x} - 0 = e^{-x}.$$

2. Set up the volume integral.

Circumference times height times thickness gives

$$V = 2\pi \int_0^1 r(x)h(x)\,dx = 2\pi \int_0^1 xe^{-x}\,dx.$$

3. Apply integration by parts.

Let $u = x$ and $dv = e^{-x}dx$, so $du = dx$ and $v = -e^{-x}$:

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x}.$$

4. Evaluate the definite integral.

Factor the antiderivative and substitute the bounds:

$$\int_0^1 xe^{-x}\,dx = \left[-(x+1)e^{-x}\right]_0^1 = -\frac{2}{e} - (-1) = 1 - \frac{2}{e}.$$

5. Multiply by the circumference factor.

$$V = \boxed{2\pi\left(1 - \frac{2}{e}\right)}.$$

Since $e > 2$, the expression is positive, as a volume must be.

All volumes are in cubic units.

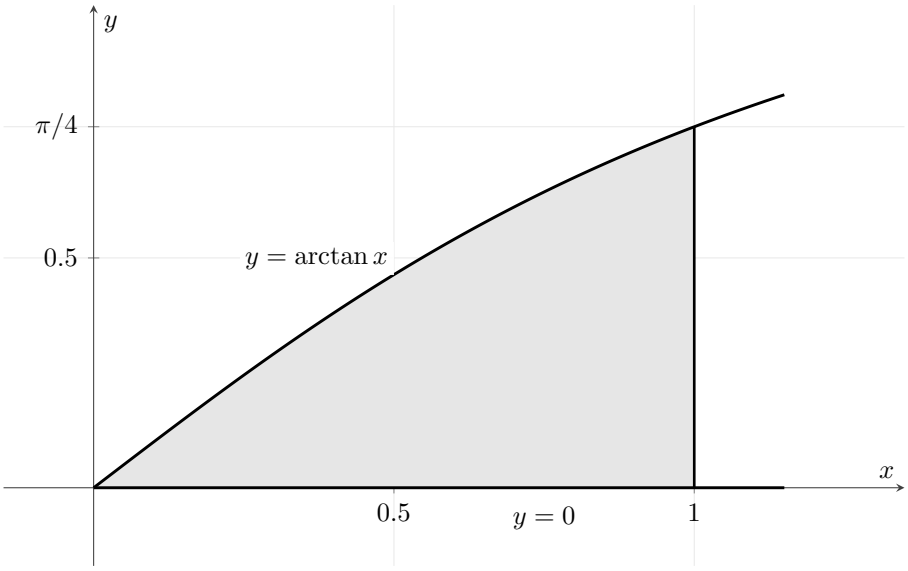
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Question 9

Find the volume of the solid obtained by rotating the region bounded by

$$y = \arctan x \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = 1$, about the y-axis.



Rotate the shaded region about the y-axis.

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Question 9 – Solution

1. Identify shell radius and height.

Use vertical slices about the y-axis on $[0, 1]$:

$$r(x) = x, \qquad h(x) = \arctan x.$$

Angles are measured in radians.

2. Set up the volume integral.

$$V = 2\pi \int_0^1 x \arctan x \, dx.$$

3. Apply integration by parts.

Let $u = \arctan x$, $dv = x\,dx$, so $du = dx/(1 + x^2)$ and $v = x^2/2$:

$$\int x \arctan x \, dx = \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2}{1 + x^2} \, dx.$$

4. Simplify the remaining rational function.

$$\frac{x^2}{1 + x^2} = 1 - \frac{1}{1 + x^2}.$$

Since $\int (1 + x^2)^{-1} dx = \arctan x$, an antiderivative is

$$F(x) = \frac{x^2}{2} \arctan x - \frac{x}{2} + \frac{1}{2} \arctan x.$$

5. Apply the bounds and simplify.

Using $\arctan 1 = \pi/4$ and $\arctan 0 = 0$,

$$F(1) - F(0) = \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}.$$

$$V = 2\pi \left(\frac{\pi}{4} - \frac{1}{2} \right) = \boxed{\pi \left(\frac{\pi}{2} - 1 \right)}.$$

All volumes are in cubic units.

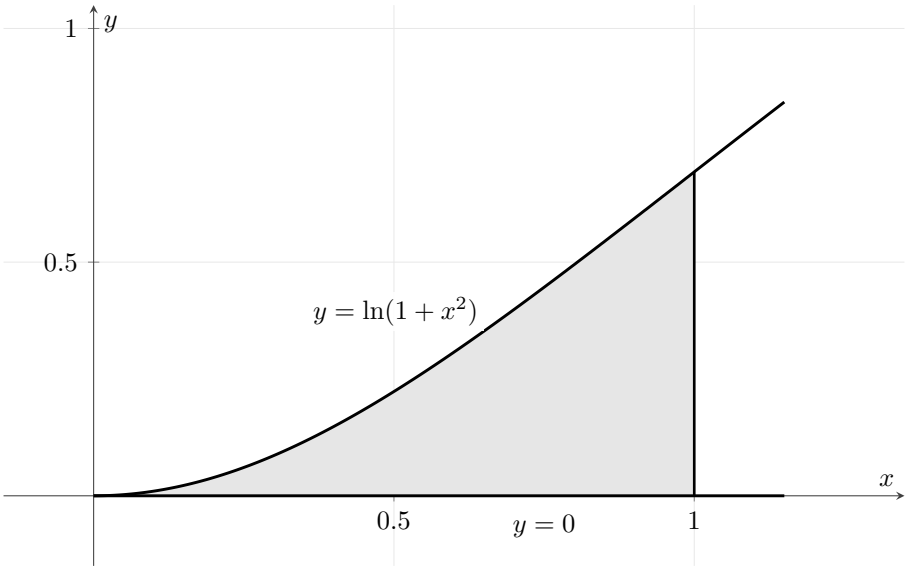
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Question 10

Find the volume of the solid obtained by rotating the region bounded by

$$y = \ln(1 + x^2) \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = 1$, about the y-axis.



Rotate the shaded region about the y-axis.

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Question 10 – Solution

1. Identify shell dimensions and set up the integral.

Use vertical shells about the y-axis, with $0 \le x \le 1$:

$$r(x) = x, \qquad h(x) = \ln(1 + x^2).$$

$$V = 2\pi \int_0^1 x \ln(1 + x^2)\, dx.$$

2. Substitute and change the bounds.

Let $u = 1 + x^2$. Then $du = 2x\, dx$, so $x\, dx = du/2$. The bounds change as follows:

$$x = 0 \Rightarrow u = 1, \qquad x = 1 \Rightarrow u = 2.$$

3. Rewrite the volume in terms of u.

$$V = 2\pi \cdot \frac{1}{2} \int_1^2 \ln u\, du = \pi \int_1^2 \ln u\, du.$$

4. Integrate the logarithm by parts.

Choose $a = \ln u$ and $db = du$, giving $da = du/u$ and $b = u$:

$$\int \ln u\, du = u \ln u - \int u \frac{1}{u}\, du = u \ln u - u.$$

5. Evaluate the endpoints.

$$\begin{aligned} V &= \pi \left[u \ln u - u \right]_1^2 \\ &= \pi \big[(2 \ln 2 - 2) - (\ln 1 - 1) \big] \\ &= \pi (2 \ln 2 - 1) = \boxed{2\pi \left(\ln 2 - \frac{1}{2} \right)}. \end{aligned}$$

All volumes are in cubic units.

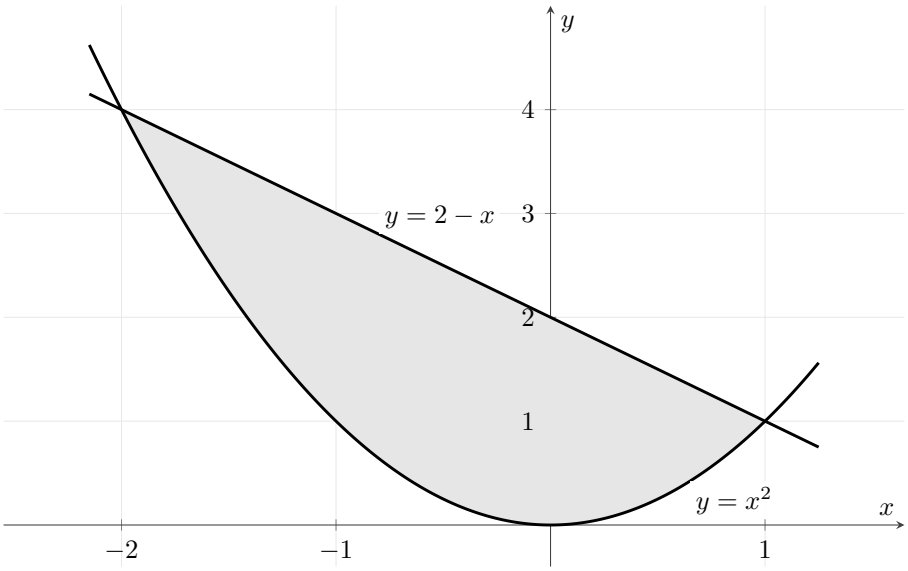
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Question 1

Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \qquad y = 2 - x,$$

about the x-axis.



Rotate the shaded region about the x-axis.

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Question 1 – Solution

1. Find the intersections.

$$x^2 = 2 - x \quad \Longrightarrow \quad x^2 + x - 2 = (x+2)(x-1) = 0.$$

Thus $x = -2$ and $x = 1$, giving intersection points $(-2, 4)$ and $(1, 1)$.

2. Choose washers and identify the radii.

Use vertical slices perpendicular to the x-axis. On $[-2, 1]$, $0 \le x^2 \le 2 - x$, so

$$R(x) = 2 - x, \qquad r(x) = x^2.$$

3. Set up the volume and expand.

$$V = \pi \int_{-2}^{1} \big(R^2 - r^2\big)dx = \pi \int_{-2}^{1} \big((2-x)^2 - x^4\big)dx.$$

$$(2-x)^2 - x^4 = 4 - 4x + x^2 - x^4.$$

4. Find an antiderivative.

Using the power rule for each term,

$$F(x) = 4x - 2x^2 + \frac{x^3}{3} - \frac{x^5}{5}, \qquad V = \pi[F(x)]_{-2}^{1}.$$

5. Evaluate the endpoints and subtract.

$$F(1) = 4 - 2 + \frac{1}{3} - \frac{1}{5} = \frac{32}{15}, \qquad F(-2) = -8 - 8 - \frac{8}{3} + \frac{32}{5} = -\frac{184}{15}.$$

$$V = \pi\left(\frac{32}{15} + \frac{184}{15}\right) = \boxed{\frac{72\pi}{5}}.$$

All volumes are in cubic units.

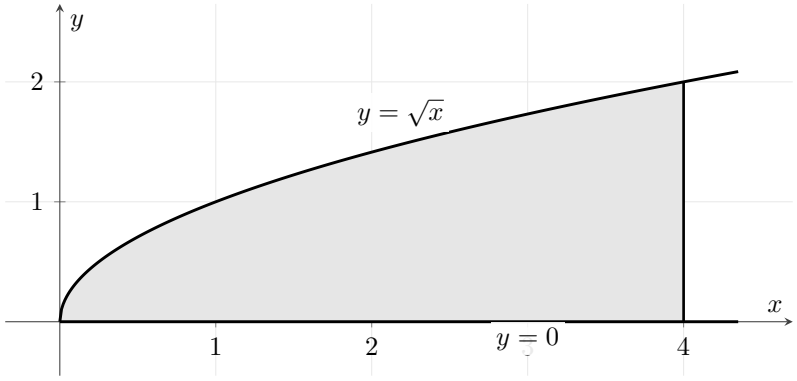
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Question 2

The base of a solid is the region bounded by

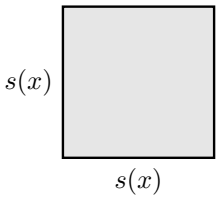
$$y = \sqrt{x} \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = 4$. Cross-sections perpendicular to the x-axis are squares. Find the volume of the solid.



Shaded base region in the xy-plane.

Cross-section perpendicular to the x-axis



Schematic cross-section; dimensions vary with x.

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Question 2 – Solution

1. Choose slices and bounds.

The cross-sections are perpendicular to the x-axis, so use slices of thickness dx on the given interval $0 \le x \le 4$.

2. Find the square side length.

The vertical segment in the base is the side of the square:

$$s(x) = y_{\text{top}} - y_{\text{bottom}} = \sqrt{x} - 0 = \sqrt{x}.$$

3. Write the cross-sectional area.

For a square, area is side squared:

$$A(x) = s(x)^2 = (\sqrt{x})^2 = x.$$

4. Set up and integrate the volume.

Add the areas of the thin cross-sections:

$$V = \int_0^4 A(x)\,dx = \int_0^4 x\,dx = \left[\frac{x^2}{2}\right]_0^4.$$

5. Evaluate both bounds.

$$V = \frac{4^2}{2} - \frac{0^2}{2} = \frac{16}{2} - 0 = \boxed{8}.$$

All volumes are in cubic units.

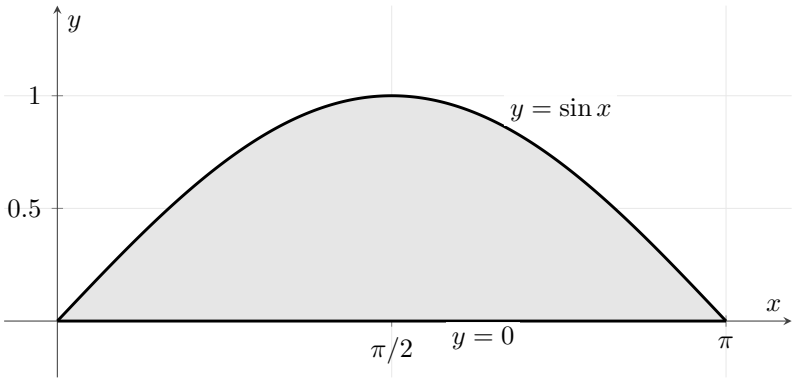
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Question 3

The base of a solid is the region enclosed by

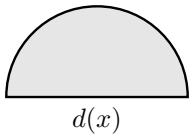
$$y = \sin x \quad \text{and} \quad y = 0$$

from $x = 0$ to $x = \pi$. Cross-sections perpendicular to the x-axis are semicircles whose diameters lie in the base region. Find the volume of the solid.



Shaded base region in the xy-plane.

Cross-section perpendicular to the x-axis



Schematic cross-section; dimensions vary with x.

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Question 3 – Solution

1. Identify the diameter and radius.

Use dx slices for $0 \le x \le \pi$. The base segment is a diameter, so

$$d(x) = \sin x - 0 = \sin x, \qquad r(x) = \frac{d(x)}{2} = \frac{\sin x}{2}.$$

2. Find the area of a semicircle.

$$A(x) = \frac{1}{2}\pi r(x)^2 = \frac{1}{2}\pi \left(\frac{\sin x}{2}\right)^2 = \frac{\pi}{8}\sin^2 x.$$

3. Set up the volume and use a trigonometric identity.

$$V = \frac{\pi}{8}\int_0^{\pi} \sin^2 x\, dx, \qquad \sin^2 x = \frac{1-\cos(2x)}{2}.$$

$$V = \frac{\pi}{16}\int_0^{\pi} (1-\cos(2x))\, dx.$$

4. Integrate with the chain-rule factor.

Since $\int \cos(2x)\, dx = \sin(2x)/2$,

$$V = \frac{\pi}{16}\left[x - \frac{\sin(2x)}{2}\right]_0^{\pi}.$$

5. Apply the bounds.

$$V = \frac{\pi}{16}\left[\left(\pi - \frac{\sin(2\pi)}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right] = \boxed{\frac{\pi^2}{16}}.$$

All volumes are in cubic units.

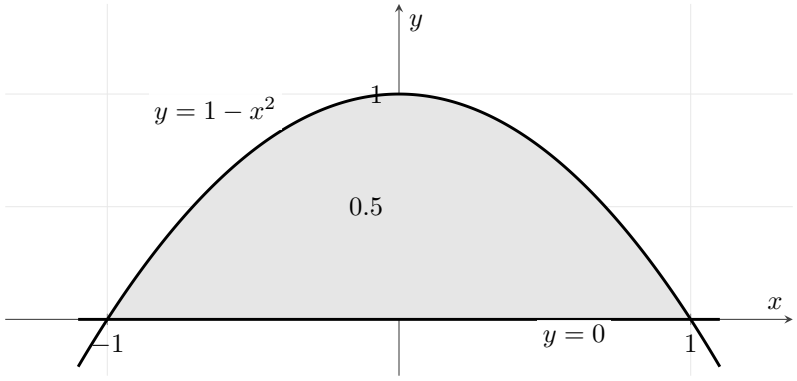
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Question 4

The base of a solid is the region enclosed by

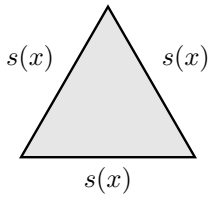
$$y = 1 - x^2 \quad \text{and} \quad y = 0$$

between their intersection points. Cross-sections perpendicular to the x-axis are equilateral triangles. Find the volume of the solid.



Shaded base region in the xy-plane.

Cross-section perpendicular to the x-axis



Schematic cross-section; dimensions vary with x.

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Question 4 – Solution

1. Find the bounds and side length.

The intersections satisfy $1 - x^2 = 0$, so $x = -1, 1$. A vertical base segment gives

$$s(x) = 1 - x^2 \qquad (-1 \le x \le 1).$$

2. Find the area of an equilateral triangle.

The altitude is $\sqrt{s^2 - (s/2)^2} = \sqrt{3}\, s/2$. Hence

$$A = \frac{1}{2} s \left(\frac{\sqrt{3}}{2} s \right) = \frac{\sqrt{3}}{4} s^2, \qquad A(x) = \frac{\sqrt{3}}{4}(1 - x^2)^2.$$

3. Set up the volume and expand.

$$V = \frac{\sqrt{3}}{4} \int_{-1}^{1} (1 - x^2)^2 dx = \frac{\sqrt{3}}{4} \int_{-1}^{1} (1 - 2x^2 + x^4) dx.$$

4. Use symmetry and integrate.

The integrand is even, so

$$V = \frac{\sqrt{3}}{2} \int_{0}^{1} (1 - 2x^2 + x^4) dx = \frac{\sqrt{3}}{2} \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_0^1.$$

5. Evaluate and simplify.

$$V = \frac{\sqrt{3}}{2} \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{\sqrt{3}}{2} \left(\frac{15 - 10 + 3}{15} \right) = \boxed{\frac{4\sqrt{3}}{15}}.$$

All volumes are in cubic units.

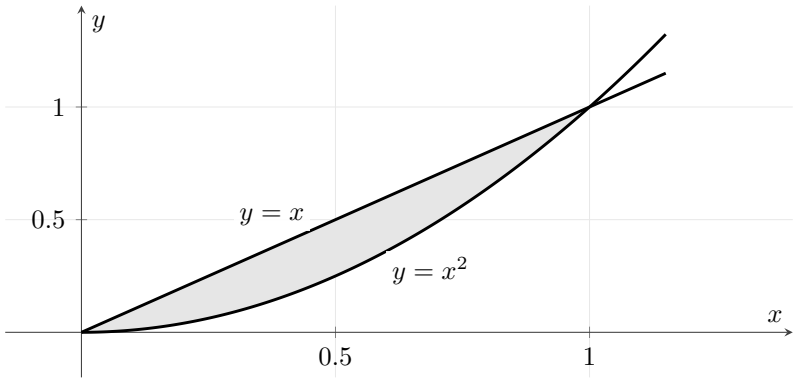
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Question 5

The base of a solid is the region bounded by

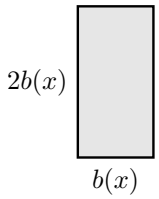
$$y = x \quad \text{and} \quad y = x^2$$

between their intersection points. Cross-sections perpendicular to the x-axis are rectangles whose height is *twice* the base in the region. Find the volume of the solid.



Shaded base region in the xy-plane.

Cross-section perpendicular to the x-axis



Schematic cross-section; dimensions vary with x.

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Question 5 – Solution

1. Find the interval and upper curve.

$$x = x^2 \quad \Longrightarrow \quad x(x-1) = 0 \quad \Longrightarrow \quad x = 0, 1.$$

On $[0, 1]$, $x \geq x^2$, so the line lies above the parabola.

2. Identify the rectangle dimensions.

The base is the vertical distance between curves; its height is twice that distance:

$$b(x) = x - x^2, \qquad h(x) = 2b(x) = 2(x - x^2).$$

3. Find the area and expand.

$$A(x) = b(x)h(x) = 2(x - x^2)^2 = 2x^2 - 4x^3 + 2x^4.$$

4. Set up and integrate the volume.

$$V = \int_0^1 A(x)\,dx = \left[\frac{2}{3}x^3 - x^4 + \frac{2}{5}x^5\right]_0^1.$$

Each term follows from $\int x^n\,dx = x^{n+1}/(n+1)$.

5. Evaluate and combine the fractions.

$$V = \left(\frac{2}{3} - 1 + \frac{2}{5}\right) - 0 = \frac{10 - 15 + 6}{15} = \boxed{\frac{1}{15}}.$$

All volumes are in cubic units.

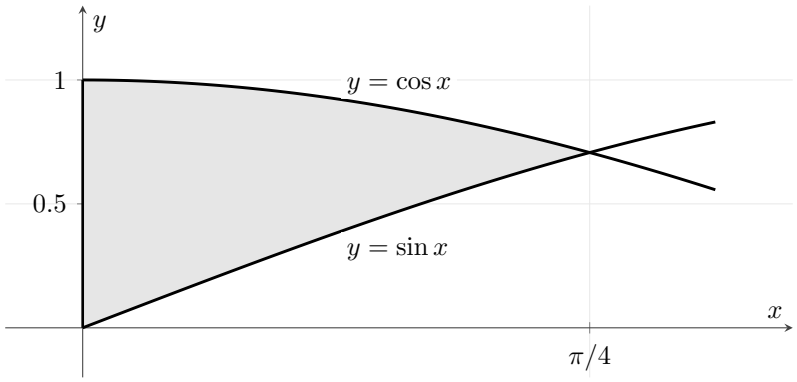
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Question 6

The base of a solid is the region bounded by

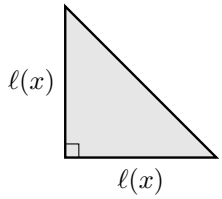
$$y = \cos x \quad \text{and} \quad y = \sin x$$

on the interval where $\cos x \geq \sin x$ in the first quadrant. Cross-sections perpendicular to the x-axis are isosceles right triangles with one leg lying in the base region. Find the volume of the solid.



Shaded base region in the xy-plane.

Cross-section perpendicular to the x-axis



Schematic cross-section; dimensions vary with x.

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Question 6 – Solution

1. Determine the bounds.

In the first quadrant, $\cos x = \sin x$ at $x = \pi/4$. The specified order $\cos x \geq \sin x$ holds for $0 \leq x \leq \pi/4$.

2. Identify the triangle leg and area.

The segment in the base is a leg, not the hypotenuse. Both perpendicular legs have length

$$\ell(x) = \cos x - \sin x, \qquad A(x) = \frac{1}{2}\ell(x)^2.$$

3. Set up the volume and simplify.

$$V = \frac{1}{2}\int_0^{\pi/4} (\cos x - \sin x)^2 dx.$$

$$(\cos x - \sin x)^2 = \cos^2 x + \sin^2 x - 2\sin x \cos x = 1 - \sin(2x).$$

4. Integrate each term.

$$V = \frac{1}{2}\int_0^{\pi/4} (1 - \sin(2x))dx = \frac{1}{2}\left[x + \frac{1}{2}\cos(2x)\right]_0^{\pi/4}.$$

5. Apply the bounds.

$$V = \frac{1}{2}\left[\left(\frac{\pi}{4} + \frac{1}{2}\cos\frac{\pi}{2}\right) - \left(0 + \frac{1}{2}\cos 0\right)\right] = \boxed{\frac{\pi}{8} - \frac{1}{4}}.$$

All volumes are in cubic units.

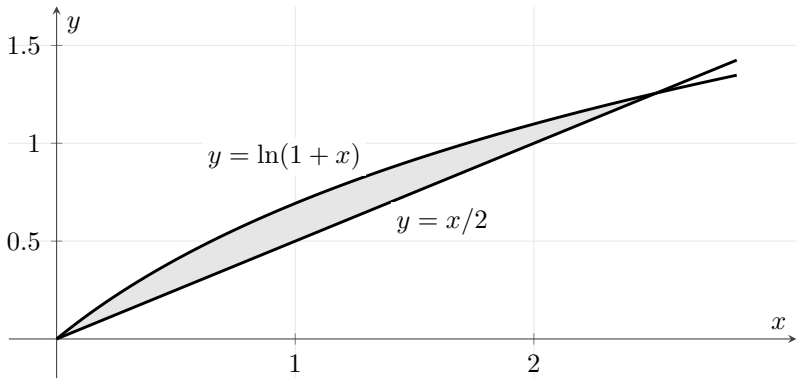
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Question 7

The base of a solid is the region bounded by

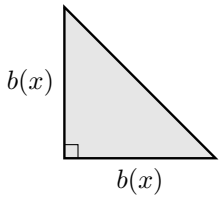
$$y = \ln(x+1) \quad \text{and} \quad y = \frac{x}{2},$$

between their points of intersection. Cross-sections perpendicular to the x-axis are right triangles whose height is equal to the base. Find the volume of the solid.



Shaded base region in the xy-plane.

Cross-section perpendicular to the x-axis



Schematic cross-section; dimensions vary with x.

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Question 7 – Solution

1. Define the intersection bounds.

Let $b(x) = \ln(1+x) - x/2$. Since $b'(x) = (1-x)/(2(1+x))$, b increases to $x = 1$ and then decreases. Its zeros are 0 and a unique $a > 1$ with

$$\ln(1+a) = a/2, \qquad a \approx 2.512862417.$$

This equation defines a exactly; $b(x) > 0$ on $(0, a)$.

2. Find the triangular area and set up the volume.

Both perpendicular legs have length $b(x)$, so

$$V = \frac{1}{2}\int_0^a \left(\ln(1+x) - \frac{x}{2}\right)^2 dx.$$

3. Expand and find the needed antiderivatives.

Write $L = \ln(1+x)$ and $u = 1+x$. The square is $L^2 - xL + x^2/4$. Integration by parts gives

$$\int L^2 dx = u(L^2 - 2L + 2), \qquad \int xL\,dx = \frac{x^2}{2}L - \frac{x^2}{4} + \frac{x}{2} - \frac{L}{2}.$$

For the first identity, use $\int \ln^2 u\,du = u\ln^2 u - 2\int \ln u\,du$.

4. Combine the terms and apply the endpoints.

An antiderivative of the squared expression is

$$H(x) = uL^2 - \left(u + \frac{u^2}{2}\right)L + u + \frac{u^2}{4} + \frac{x^3}{12}.$$

At $x = 0$, $H(0) = 5/4$. Substituting $u = 1+a$ and $L = a/2$ gives

$$H(a) - H(0) = \frac{a^3}{12} - \frac{a^2}{2} + \frac{3a}{4} = \frac{a(a-3)^2}{12}.$$

5. Include the triangular-area factor.

$$V = \frac{1}{2}[H(a) - H(0)] = \boxed{\frac{a(a-3)^2}{24}} \approx 0.024846244.$$

All volumes are in cubic units.

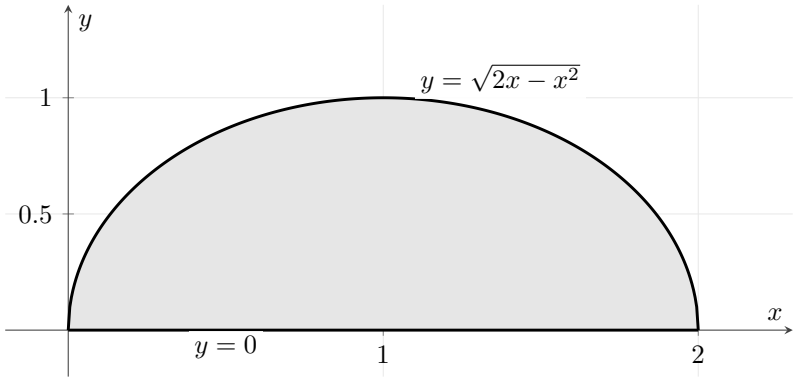
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Question 8

The base of a solid is the region bounded by

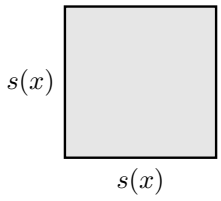
$$y = \sqrt{2x - x^2} \quad \text{and} \quad y = 0.$$

Cross-sections perpendicular to the x-axis are squares. Find the volume of the solid.



Shaded base region in the xy-plane.

Cross-section perpendicular to the x-axis



Schematic cross-section; dimensions vary with x.

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Question 8 – Solution

1. Determine the bounds.

The square root is real when $2x - x^2 = x(2 - x) \geq 0$, giving $0 \leq x \leq 2$. These endpoints are also the intersections with $y = 0$.

2. Find the square side length.

The cross-sections are perpendicular to the x-axis. Their side length is the vertical segment in the base:

$$s(x) = \sqrt{2x - x^2} - 0.$$

3. Find the cross-sectional area.

$$A(x) = s(x)^2 = \left(\sqrt{2x - x^2}\right)^2 = 2x - x^2.$$

4. Set up and integrate the volume.

$$V = \int_0^2 A(x)\,dx = \int_0^2 (2x - x^2)dx = \left[x^2 - \frac{x^3}{3}\right]_0^2.$$

5. Evaluate and simplify.

$$V = \left(4 - \frac{8}{3}\right) - 0 = \frac{12 - 8}{3} = \boxed{\frac{4}{3}}.$$

All volumes are in cubic units.

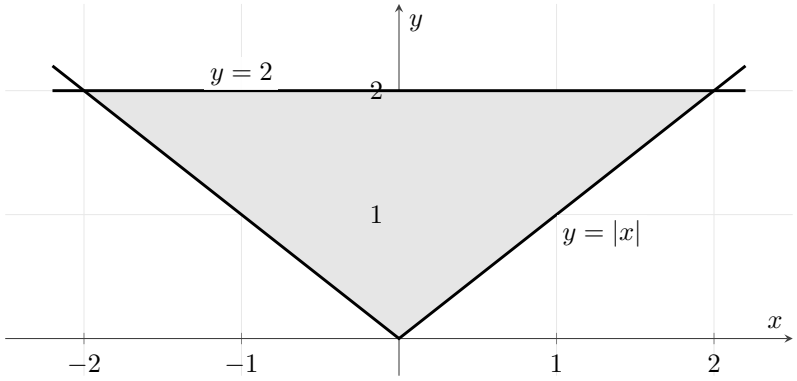
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Question 9

The base of a solid is the region bounded by

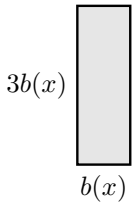
$$y = |x| \quad \text{and} \quad y = 2,$$

between their intersection points. Cross-sections perpendicular to the x-axis are rectangles whose height is three times their base. Find the volume of the solid.



Shaded base region in the xy-plane.

Cross-section perpendicular to the x-axis



Schematic cross-section; dimensions vary with x.

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Question 9 – Solution

1. Find the bounds.

The intersections satisfy $|x| = 2$, giving $x = -2$ and $x = 2$. Use dx slices because the cross-sections are perpendicular to the x-axis.

2. Identify the rectangle dimensions.

The base length is top minus bottom, and the height is three times the base:

$$b(x) = 2 - |x|, \qquad h(x) = 3(2 - |x|).$$

3. Set up the area and volume.

$$A(x) = b(x)h(x) = 3(2 - |x|)^2, \qquad V = 3\int_{-2}^{2}(2 - |x|)^2 dx.$$

4. Use symmetry and integrate.

The integrand is even; on $[0, 2]$, $|x| = x$. Thus

$$V = 6\int_{0}^{2}(2 - x)^2 dx = 6\int_{0}^{2}(4 - 4x + x^2)dx.$$

$$V = 6\left[4x - 2x^2 + \frac{x^3}{3}\right]_0^2.$$

5. Apply the bounds.

$$V = 6\left[(8 - 8 + \frac{8}{3}) - 0\right] = 6 \cdot \frac{8}{3} = \boxed{16}.$$

All volumes are in cubic units.

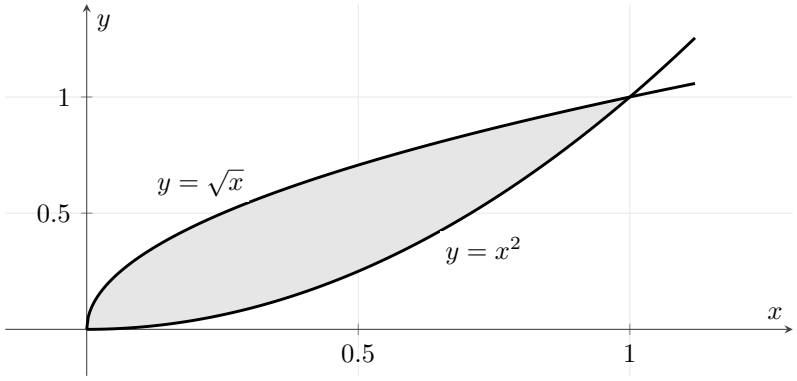
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Question 10

The base of a solid is the region bounded by

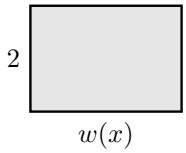
$$y = \sqrt{x} \quad \text{and} \quad y = x^2,$$

between their points of intersection. Cross-sections parallel to the y-axis are rectangles whose width is equal to the distance between the curves and whose height is constant and equal to 2. Find the volume of the solid.



Shaded base region in the xy-plane.

Cross-section perpendicular to the x-axis



Schematic cross-section; dimensions vary with x.

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Question 10 – Solution

1. Interpret the slice direction and find the bounds.

The base segments are parallel to the y-axis, so they are vertical; integrate with respect to x. Solving $\sqrt{x} = x^2$ gives $x = x^4$, or $x(x^3 - 1) = 0$. The real intersections occur at $x = 0, 1$.

2. Identify width and height.

For $0 \le x \le 1$, $\sqrt{x} \ge x^2$. Thus each rectangle has

$$w(x) = \sqrt{x} - x^2, \qquad h = 2.$$

The height of 2 extends out of the base plane.

3. Find the area and set up the integral.

$$A(x) = 2(\sqrt{x} - x^2), \qquad V = \int_0^1 A(x)\,dx = 2\int_0^1 (x^{1/2} - x^2)dx.$$

4. Integrate using the power rule.

$$\int x^{1/2}\,dx = \frac{2}{3}x^{3/2}, \qquad \int x^2\,dx = \frac{x^3}{3}.$$

$$V = 2\left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1.$$

5. Apply the bounds and simplify.

$$V = 2\left[\left(\frac{2}{3} - \frac{1}{3}\right) - 0\right] = \boxed{\frac{2}{3}}.$$

All volumes are in cubic units.

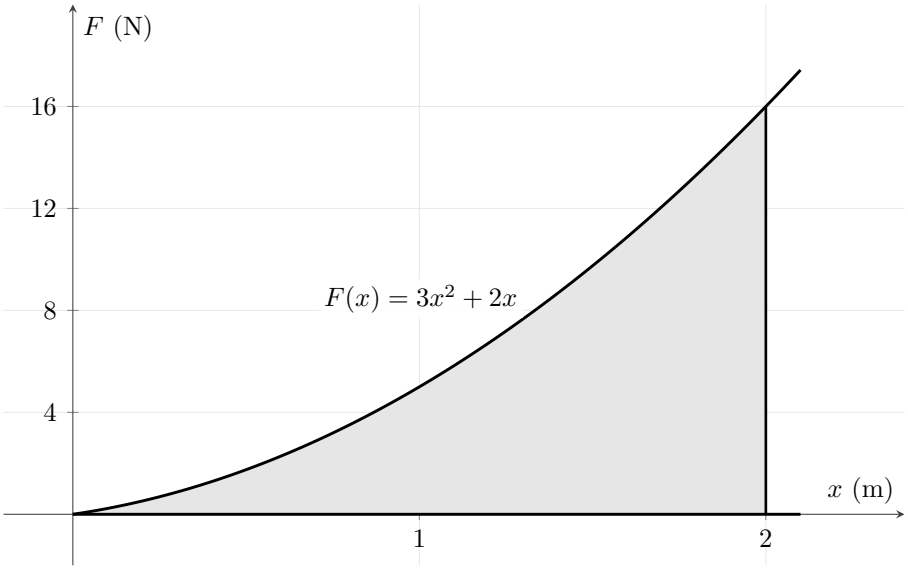
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Question 1

A force acting along a straight line is given by

$$F(x) = 3x^2 + 2x \quad \text{(newtons)},$$

where x is the displacement in meters from the origin. Find the work done by the force in moving an object from $x = 0$ to $x = 2$.



The shaded area represents work over the specified interval.

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Question 1 – Solution

1. Identify the force and interval.

The force acts along the displacement and is $F(x) = 3x^2 + 2x$ newtons. The object moves from $x = 0$ to $x = 2$ meters.

2. Write the work integral.

Over a small displacement dx, $dW = F(x)\,dx$. Therefore,

$$W = \int_0^2 F(x)\,dx = \int_0^2 (3x^2 + 2x)\,dx.$$

3. Integrate term by term.

By the power rule,

$$\int 3x^2\,dx = 3\frac{x^3}{3} = x^3, \qquad \int 2x\,dx = 2\frac{x^2}{2} = x^2.$$

4. Apply the bounds.

$$W = \left[x^3 + x^2\right]_0^2 = (2^3 + 2^2) - (0^3 + 0^2) = 8 + 4 = 12.$$

5. State the result with units.

$$\boxed{W = 12 \text{ J}}.$$

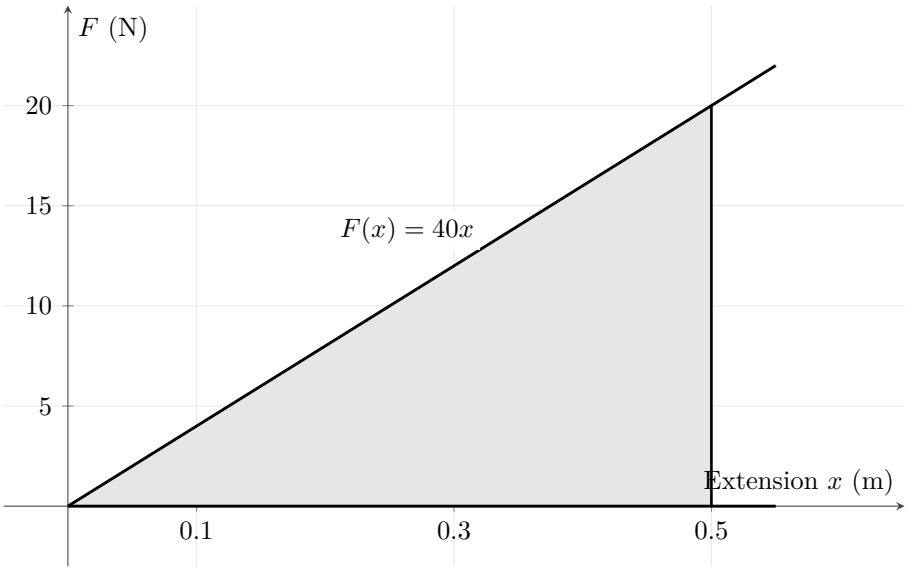
A newton times a meter is a joule. Since $F(x) \geq 0$ on $[0, 2]$, the work is the positive shaded area under the force curve.

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Question 2

A spring has natural length 0.5 m. A force of 12 N is required to stretch the spring to a length of 0.8 m.

Find the work required to stretch the spring from its natural length to a length of 1.0 m.



The shaded area represents work over the specified interval.

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Question 2 – Solution

1. Use extension beyond natural length.

Let x be extension, not total spring length. The calibration force produces

$$x = 0.8 - 0.5 = 0.3 \text{ m}.$$

2. Find the spring constant.

Hooke's law for the required stretching force is $F = kx$:

$$12 = k(0.3) \quad \Longrightarrow \quad k = \frac{12}{0.3} = 40 \text{ N/m}.$$

3. Translate the requested lengths into bounds.

The initial and final extensions are

$$x_0 = 0.5 - 0.5 = 0, \qquad x_1 = 1.0 - 0.5 = 0.5 \text{ m}.$$

Thus $F(x) = 40x$ and $W = \int_0^{0.5} 40x\,dx$.

4. Integrate and evaluate.

$$W = \left[20x^2\right]_0^{0.5} = 20(0.5)^2 - 0 = 20(0.25) = 5.$$

5. State and check the work.

$$\boxed{W = 5 \text{ J}}.$$

The force rises linearly from 0 to 20 N, so the triangular area is also $\frac{1}{2}(0.5)(20) = 5$ J.

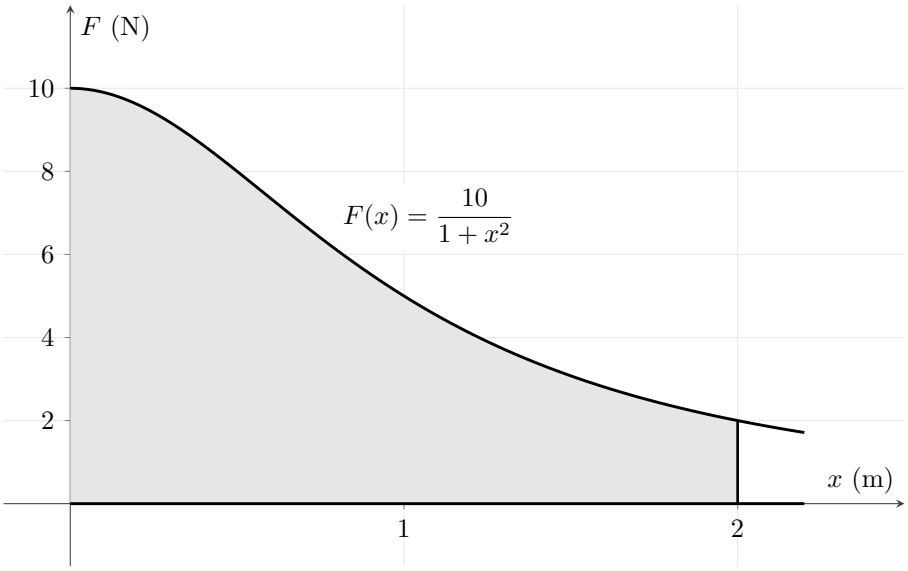
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Question 3

A force acting on an object is given by

$$F(x) = \frac{10}{1+x^2} \quad \text{(newtons)},$$

where x is the distance (in meters) the object has moved along a straight line. Find the work done in moving the object from $x = 0$ to $x = 2$.



The shaded area represents work over the specified interval.

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Question 3 – Solution

1. Identify the work model.

The force is positive over $0 \le x \le 2$, so the work equals the area under the force curve:

$$W = \int_0^2 F(x)\,dx.$$

2. Substitute the force.

$$W = \int_0^2 \frac{10}{1+x^2}\,dx = 10\int_0^2 \frac{1}{1+x^2}\,dx.$$

3. Use the inverse-tangent antiderivative.

Since $\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2}$,

$$W = 10\left[\arctan x\right]_0^2.$$

Inverse-trigonometric values are in radians.

4. Apply the endpoints.

$$W = 10(\arctan 2 - \arctan 0) = 10\arctan 2.$$

5. State the exact and approximate work.

$$\boxed{W = 10\arctan 2 \text{ J}} \approx 11.0715 \text{ J}.$$

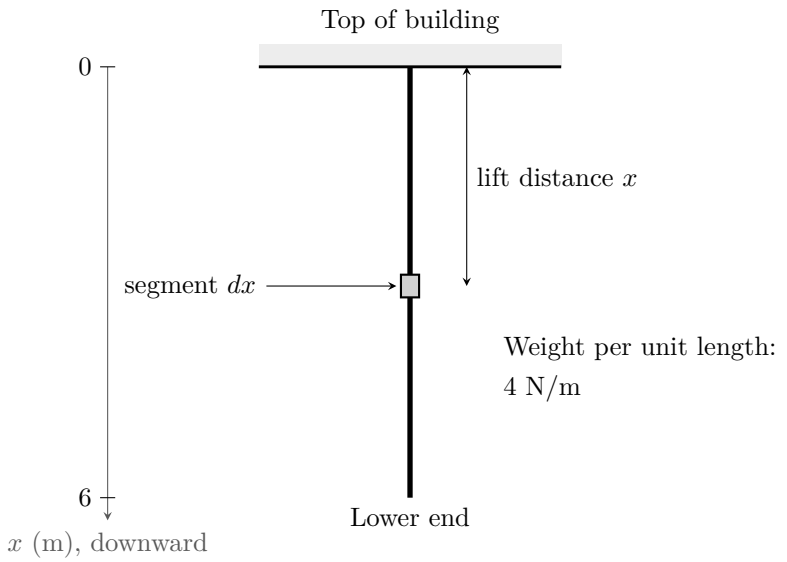
The force decreases from 10 N to 2 N; the resulting work lies between $2(2) = 4$ J and $10(2) = 20$ J.

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Question 4

A 6-meter cable hangs vertically from the top of a building. The cable has constant linear weight density 4 N/m.

Find the work required to lift the entire cable to the top of the building.



Initial configuration; x is measured down from the top.

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Question 4 – Solution

1. Choose a coordinate for each cable segment.

Let x measure the initial distance downward from the top of the building. The cable occupies $0 \le x \le 6$ meters.

2. Find the weight of a small segment.

The given quantity is weight per unit length. A segment of length dx weighs

$$dF = (4 \text{ N/m})\, dx.$$

No additional factor of gravity is needed.

3. Find its lifting distance and work.

The segment initially at depth x must rise x meters. Thus

$$dW = (\text{weight})(\text{distance}) = 4x\, dx.$$

4. Add the work for all segments.

$$W = \int_0^6 4x\, dx = \left[2x^2\right]_0^6 = 2(36) - 0 = 72.$$

5. State and check the result.

$$\boxed{W = 72 \text{ J}}.$$

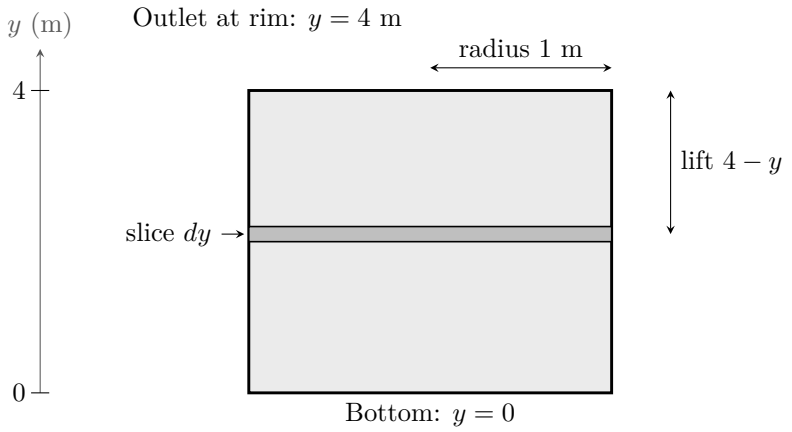
The cable weighs $4(6) = 24$ N, and its center of mass rises 3 m. This gives the same work: $24(3) = 72$ J.

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Question 5

A cylindrical tank of radius 1 m and height 4 m is full of water. Water weighs 9800 N/m^3.

Find the work required to pump all the water to the top of the tank.



Vertical section through the tank; the shaded band represents a horizontal slice.

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Question 5 – Solution

1. Choose horizontal slices.

Let y be height above the tank bottom, with $0 \le y \le 4$. A slice at height y has thickness dy.

2. Find the slice volume and weight.

The horizontal circular area is $A = \pi(1)^2 = \pi$ m^2. Hence

$$dV = \pi\, dy, \qquad dF = 9800\, dV = 9800\pi\, dy.$$

The density is already a weight density.

3. Find the distance to the outlet.

The outlet is at the top, $y = 4$, so the slice must rise $4 - y$ meters:

$$dW = 9800\pi(4 - y)\, dy.$$

4. Set up and evaluate the work integral.

$$W = 9800\pi \int_0^4 (4 - y)\, dy = 9800\pi \left[4y - \frac{y^2}{2}\right]_0^4.$$
$$W = 9800\pi(16 - 8) = 78400\pi.$$

5. State and check the work.

$$\boxed{W = 78400\pi \text{ J}}.$$

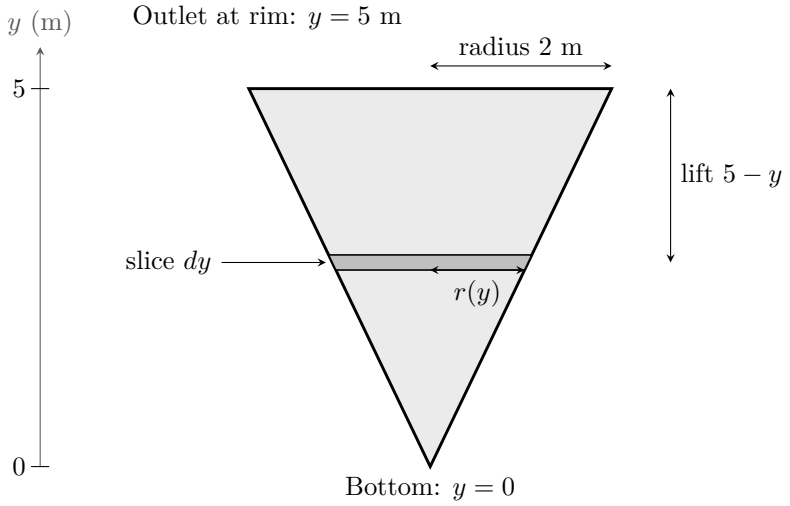
The total water weight is $9800(4\pi) = 39200\pi$ N. Its average lifting distance is 2 m, giving 78400π J.

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Question 6

A conical tank with its vertex at the bottom, height 5 m, and top radius 2 m is full of water. Water weighs 9800 N/m^3.

Find the work required to pump all the water to the top of the tank.



Vertical section through the tank; the shaded band represents a horizontal slice.

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Question 6 – Solution

1. Choose a height coordinate and determine the radius.

Measure y upward from the bottom vertex, so $0 \le y \le 5$. Similar triangles give

$$\frac{r(y)}{y} = \frac{2}{5} \quad \Longrightarrow \quad r(y) = \frac{2y}{5}.$$

2. Find the volume and weight of a slice.

$$dV = \pi r(y)^2 dy = \frac{4\pi}{25} y^2 dy, \qquad dF = 9800\, dV = \frac{39200\pi}{25} y^2 dy.$$

3. Include the lifting distance.

The outlet is at $y = 5$, so a slice at y rises $5 - y$ meters:

$$W = \frac{39200\pi}{25} \int_0^5 y^2 (5 - y)\, dy.$$

4. Expand and integrate.

$$y^2(5 - y) = 5y^2 - y^3, \qquad W = \frac{39200\pi}{25} \left[\frac{5}{3} y^3 - \frac{y^4}{4} \right]_0^5.$$

5. Apply the bounds and simplify.

$$\begin{aligned} W &= \frac{39200\pi}{25} \left(\frac{625}{3} - \frac{625}{4} \right) \\ &= \frac{39200\pi}{25} \cdot \frac{625}{12} = \boxed{\frac{245000\pi}{3}\ \text{J}}. \end{aligned}$$

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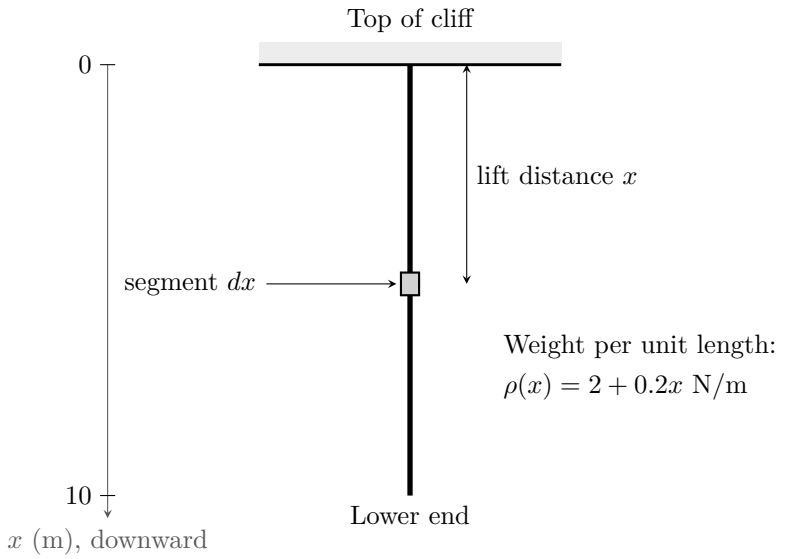
Question 7

A 10-meter rope hangs vertically from the edge of a cliff. The rope has variable linear weight density given by

$$\rho(x) = 2 + 0.2x \quad \text{N/m},$$

where x is the distance (in meters) measured downward from the top of the cliff.

Find the work required to pull the entire rope to the top of the cliff.



Initial configuration; x is measured down from the top.

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Question 7 – Solution

1. Label each segment by its initial depth.

Let x be distance downward from the cliff top, as in the question. The rope occupies $0 \le x \le 10$ meters; $\rho(x)$ describes the initial weight distribution.

2. Find the segment weight.

For a short piece of length dx,

$$dF = \rho(x)\,dx = (2 + 0.2x)\,dx.$$

The units are newtons because ρ is given in N/m.

3. Multiply by the distance lifted.

The segment at depth x is lifted x meters:

$$dW = x(2 + 0.2x)\,dx, \qquad W = \int_0^{10} x(2 + 0.2x)\,dx.$$

4. Expand and integrate.

$$W = \int_0^{10} \left(2x + \frac{x^2}{5}\right) dx = \left[x^2 + \frac{x^3}{15}\right]_0^{10}.$$

5. Evaluate and state the work.

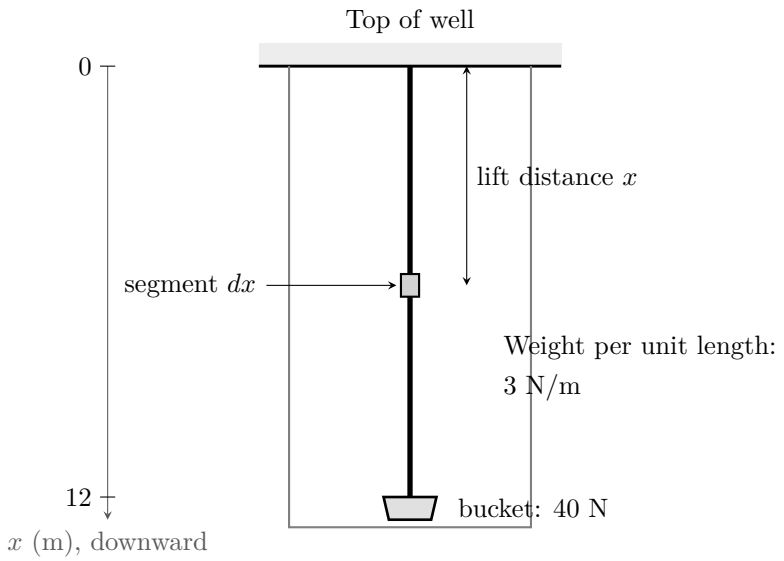
$$W = 100 + \frac{1000}{15} = 100 + \frac{200}{3} = \boxed{\frac{500}{3}\ \text{J}} \approx 166.667\ \text{J}.$$

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Question 8

A bucket is lifted from the bottom of a well that is 12 m deep. The bucket itself weighs 40 N. Attached to the bucket is a rope that weighs 3 N/m.

Find the work required to lift the bucket to the top of the well.



Initial configuration; x is measured down from the top.

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Question 8 – Solution

1. Separate the bucket and rope contributions.

The bucket has constant weight, while different pieces of rope rise different distances:

$$W_{\text{total}} = W_{\text{bucket}} + W_{\text{rope}}.$$

2. Calculate the bucket work.

The 40 N bucket rises the full 12 m:

$$W_{\text{bucket}} = 40(12) = 480 \text{ J}.$$

3. Define a consistent coordinate for the rope.

Let x be the initial depth below the top of the well, with $0 \le x \le 12$. A segment of length dx weighs $3\,dx$ newtons and rises x meters.

4. Integrate the rope work.

$$W_{\text{rope}} = \int_0^{12} 3x\,dx = \left[\frac{3}{2}x^2\right]_0^{12} = \frac{3}{2}(144) = 216 \text{ J}.$$

5. Add and check the total.

$$W_{\text{total}} = 480 + 216 = \boxed{696 \text{ J}}.$$

As a check, the rope weighs $3(12) = 36$ N and its center of mass rises 6 m, giving $36(6) = 216$ J.

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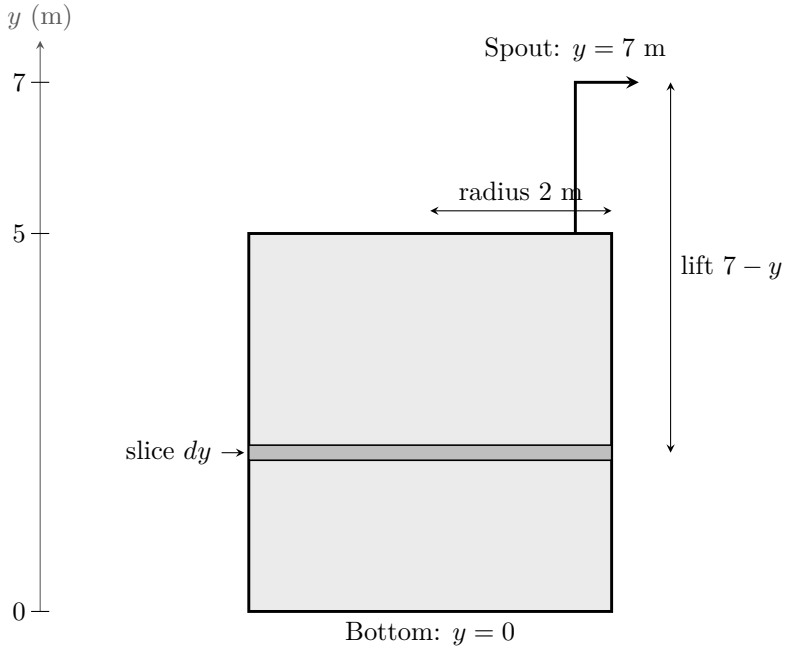
Question 9

A cylindrical tank of radius 2 m and height 5 m is full of oil. The oil has weight density

$$w(y) = 9000(1 + 0.1y) \text{ N/m}^3,$$

where y is the height (in meters) above the bottom of the tank.

Find the work required to pump all the oil to a spout located 2 m above the top of the tank.



Vertical section through the tank; the shaded band represents a horizontal slice.

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Question 9 – Solution

1. Locate the slices and outlet.

Let y be height above the bottom. Oil occupies $0 \le y \le 5$, and the spout is at height $5 + 2 = 7$ m. The lift distance is $7 - y$.

2. Find the slice volume and weight.

The tank radius is 2 m, so

$$dV = \pi(2)^2 dy = 4\pi\, dy.$$

$$dF = w(y)\, dV = 9000(1 + 0.1y)(4\pi)\, dy = 36000\pi(1 + 0.1y)\, dy.$$

3. Set up the work integral.

Multiply each slice weight by its lifting distance:

$$W = 36000\pi \int_0^5 (1 + 0.1y)(7 - y)\, dy.$$

4. Expand and integrate exactly.

$$(1 + 0.1y)(7 - y) = 7 - \frac{3}{10}y - \frac{1}{10}y^2.$$

$$W = 36000\pi \left[7y - \frac{3}{20}y^2 - \frac{1}{30}y^3\right]_0^5.$$

5. Apply the bounds and simplify.

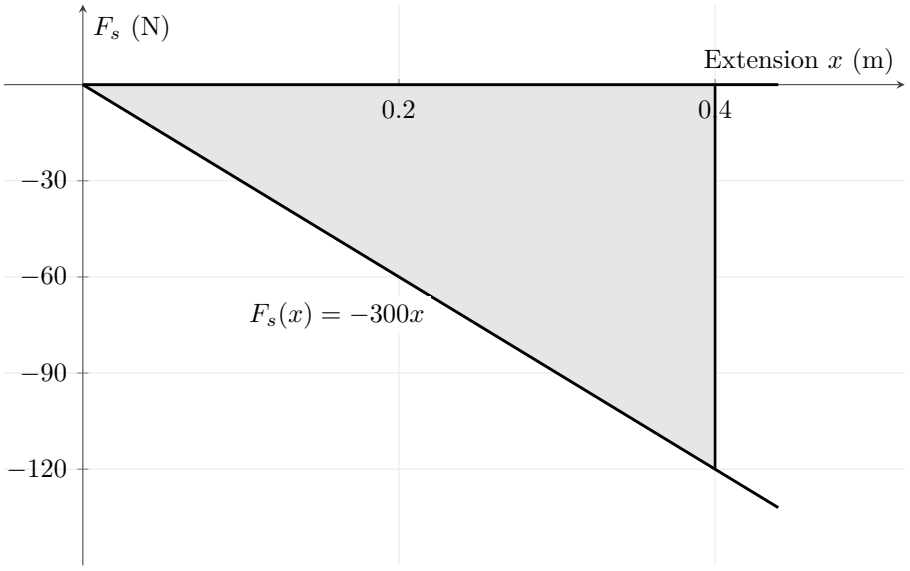
$$\begin{aligned} W &= 36000\pi \left(35 - \frac{15}{4} - \frac{25}{6}\right) \\ &= 36000\pi \left(\frac{420 - 45 - 50}{12}\right) = \boxed{975000\pi \text{ J}}. \end{aligned}$$

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Question 10

A vertical spring attached to a ceiling has natural length 1 m and spring constant $k = 300$ N/m. An external agent stretches it downward from length 1 m to 1.4 m.

Find the work done by the spring during this motion.



The shaded area represents negative work done by the spring.

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Question 10 – Solution

1. Define displacement from natural length.

Let x be extension measured positively downward. The spring lengths correspond to

$$x_0 = 1 - 1 = 0, \qquad x_1 = 1.4 - 1 = 0.4 \text{ m}.$$

2. Use the signed force exerted by the spring.

The spring pulls upward, opposite the positive direction. Therefore,

$$F_s(x) = -kx = -300x \text{ N}.$$

The question asks for work done by the spring itself.

3. Set up the signed work integral.

$$W_s = \int_0^{0.4} F_s(x)\,dx = \int_0^{0.4} -300x\,dx.$$

4. Integrate and apply the bounds.

$$W_s = \left[-150x^2\right]_0^{0.4} = -150(0.4)^2 = -150(0.16) = -24.$$

5. State the result and interpret its sign.

$$\boxed{W_s = -24 \text{ J}}.$$

The negative sign indicates that the spring force opposes the motion. Its elastic potential energy increases by $\frac{1}{2}k(0.4)^2 = 24$ J, consistent with $W_s = -\Delta U$.

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Question 1

Assume $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Prove that

$$\lim_{x\to a}\big(f(x)g(x)\big) = LM.$$

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Question 1 - Solution

Let $\varepsilon > 0$ be given.

We must show there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then

$$|f(x)g(x) - LM| < \varepsilon.$$

Start by adding and subtracting $Lg(x)$:

$$f(x)g(x) - LM = f(x)g(x) - Lg(x) + Lg(x) - LM = g(x)\big(f(x) - L\big) + L\big(g(x) - M\big).$$

Take absolute values and use the triangle inequality:

$$|f(x)g(x) - LM| \le |g(x)|\,|f(x) - L| + |L|\,|g(x) - M|.$$

To control $|g(x)|$, use the fact that $g(x) \to M$.
Since $\lim_{x\to a} g(x) = M$, there exists $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$, then

$$|g(x) - M| < 1.$$

Then, for $0 < |x - a| < \delta_1$,

$$|g(x)| = |g(x) - M + M| \le |g(x) - M| + |M| < 1 + |M|.$$

Now choose $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then

$$|f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}.$$

This is possible because $\lim_{x\to a} f(x) = L$.
Next choose $\delta_3 > 0$ such that if $0 < |x - a| < \delta_3$, then

$$|g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}.$$

This is possible because $\lim_{x\to a} g(x) = M$.
Let

$$\delta = \min(\delta_1, \delta_2, \delta_3).$$

Then whenever $0 < |x - a| < \delta$, we have simultaneously:

$$|g(x)| < 1 + |M|, \quad |f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}, \quad |g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}.$$

Substitute into the estimate:

$$|f(x)g(x) - LM| \le |g(x)|\,|f(x) - L| + |L|\,|g(x) - M|.$$

For the first term:

$$|g(x)|\,|f(x) - L| < (1 + |M|) \cdot \frac{\varepsilon}{2(1 + |M|)} = \frac{\varepsilon}{2}.$$

For the second term:

$$|L|\,|g(x) - M| \le \frac{\varepsilon|L|}{2(1 + |L|)} < \frac{\varepsilon}{2}.$$

Therefore,

$$|f(x)g(x) - LM| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\lim_{x\to a}\big(f(x)g(x)\big) = LM.$$

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Question 2

Assume that $\lim_{x\to a} f(x) = L$ and that c is a constant. Prove that

$$\lim_{x\to a} \big(cf(x)\big) = cL.$$

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Question 2 - Solution

Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that if

$$0 < |x - a| < \delta,$$

then

$$|cf(x) - cL| < \varepsilon.$$

Start by factoring out the constant:

$$|cf(x) - cL| = |c| \cdot |f(x) - L|.$$

Since $\lim_{x \to a} f(x) = L$, there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| < \frac{\varepsilon}{|c|},$$

provided $c \neq 0$.

Substitute this into the inequality:

$$|cf(x) - cL| = |c| \cdot |f(x) - L| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

Thus, whenever $0 < |x - a| < \delta$, we have

$$|cf(x) - cL| < \varepsilon.$$

Therefore,

$$\lim_{x \to a} \big(cf(x)\big) = cL.$$

If $c = 0$, then $cf(x) = 0$ for all x, and the result holds trivially.

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Question 3

Assume that $\lim_{x\to a} f(x) = L$. Prove that

$$\lim_{x\to a} |f(x)| = |L|.$$

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Question 3 - Solution

Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that if

$$0 < |x - a| < \delta,$$

then

$$\big||f(x)| - |L|\big| < \varepsilon.$$

Recall the inequality

$$\big||u| - |v|\big| \leq |u - v|$$

for all real numbers u and v.

Apply this inequality with $u = f(x)$ and $v = L$:

$$\big||f(x)| - |L|\big| \leq |f(x) - L|.$$

Since $\lim_{x \to a} f(x) = L$, there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| < \varepsilon.$$

Combining the inequalities gives

$$\big||f(x)| - |L|\big| \leq |f(x) - L| < \varepsilon.$$

Thus, whenever $0 < |x - a| < \delta$,

$$\big||f(x)| - |L|\big| < \varepsilon.$$

Therefore,

$$\lim_{x \to a} |f(x)| = |L|.$$

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Question 4

Assume that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, with $M \neq 0$. Prove that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

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Question 4 - Solution

Let $\varepsilon > 0$. Since $g(x) \to M \neq 0$, choose $\delta_1 > 0$ such that $|g(x) - M| < |M|/2$ when $0 < |x - a| < \delta_1$. Then $|g(x)| > |M|/2$.

By the two assumed limits, choose $\delta_2, \delta_3 > 0$ to ensure respectively

$$|f(x) - L| < \frac{\varepsilon |M|}{4}, \qquad |g(x) - M| < \frac{\varepsilon |M|^2}{4(1 + |L|)}.$$

Take $\delta = \min(\delta_1, \delta_2, \delta_3)$. For $0 < |x - a| < \delta$,

$$\begin{aligned} \left| \frac{f(x)}{g(x)} - \frac{L}{M} \right| &\leq \frac{|f(x) - L|}{|g(x)|} + \frac{|L|\,|g(x) - M|}{|M|\,|g(x)|} \\ &\leq \frac{2|f(x) - L|}{|M|} + \frac{2|L|\,|g(x) - M|}{|M|^2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon |L|}{2(1 + |L|)} < \varepsilon. \end{aligned}$$

(The first term makes the penultimate inequality strict even if $L = 0$.) This proves $\lim\limits_{x \to a} f(x)/g(x) = L/M$.

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Question 5

Assume that $\lim_{x\to a} f(x) = L$. Prove that

$$\lim_{x\to a}\big(f(x)+c\big) = L + c,$$

where c is a constant.

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Question 5 - Solution

Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|(f(x) + c) - (L + c)| < \varepsilon.$$

Simplify the expression inside the absolute value:

$$|(f(x) + c) - (L + c)| = |f(x) - L|.$$

Since $\lim_{x \to a} f(x) = L$, by definition of the limit there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| < \varepsilon.$$

Therefore, for the same δ,

$$|(f(x) + c) - (L + c)| < \varepsilon.$$

This proves that

$$\lim_{x \to a} \big(f(x) + c\big) = L + c.$$

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Question 6

Assume that $\lim_{x\to a} f(x) = L$. Prove that

$$\lim_{x\to a}\big(f(x) - c\big) = L - c,$$

where c is a constant.

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Question 6 - Solution

Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that if

$$0 < |x - a| < \delta,$$

then

$$|(f(x) - c) - (L - c)| < \varepsilon.$$

Simplify the expression:

$$|(f(x) - c) - (L - c)| = |f(x) - L|.$$

Since $\lim_{x \to a} f(x) = L$, by the definition of the limit there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| < \varepsilon.$$

Thus, for this same choice of δ,

$$|(f(x) - c) - (L - c)| < \varepsilon.$$

Therefore,

$$\lim_{x \to a} \big(f(x) - c\big) = L - c.$$

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Question 7

Assume that

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M,$$

and that

$$f(x) \leq g(x) \quad \text{for all } x \text{ near } a.$$

Prove that

$$L \leq M.$$

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Question 7 - Solution

We argue by contradiction.

Suppose, to the contrary, that

$$L > M.$$

Then the positive number

$$\varepsilon = \frac{L - M}{2}$$

satisfies $\varepsilon > 0$.

Since $\lim_{x \to a} f(x) = L$, there exists $\delta_1 > 0$ such that whenever

$$0 < |x - a| < \delta_1,$$

we have

$$|f(x) - L| < \varepsilon.$$

This implies

$$f(x) > L - \varepsilon = \frac{L + M}{2}.$$

Similarly, since $\lim_{x \to a} g(x) = M$, there exists $\delta_2 > 0$ such that whenever

$$0 < |x - a| < \delta_2,$$

we have

$$|g(x) - M| < \varepsilon.$$

This implies

$$g(x) < M + \varepsilon = \frac{L + M}{2}.$$

Let

$$\delta = \min(\delta_1, \delta_2).$$

Then for all x satisfying $0 < |x - a| < \delta$, we have

$$f(x) > \frac{L + M}{2} \quad \text{and} \quad g(x) < \frac{L + M}{2}.$$

Thus,

$$f(x) > g(x),$$

which contradicts the assumption that $f(x) \leq g(x)$ near a.

Therefore, our assumption that $L > M$ must be false, and we conclude that

$$L \leq M.$$

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Question 8

Assume that

$$\lim_{x \to a} f(x) = L \quad \text{and} \quad f(x) \geq 0 \quad \text{for all } x \text{ near } a.$$

Prove that

$$L \geq 0.$$

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Question 8 - Solution

We prove this by contradiction.

Assume, to the contrary, that

$$L < 0.$$

Then the positive number

$$\varepsilon = \frac{-L}{2}$$

satisfies $\varepsilon > 0$.

Since $\lim_{x \to a} f(x) = L$, there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| < \varepsilon.$$

This inequality implies

$$f(x) < L + \varepsilon = L + \frac{-L}{2} = \frac{L}{2}.$$

Because $L < 0$, we have

$$\frac{L}{2} < 0.$$

Thus, for all x with $0 < |x - a| < \delta$,

$$f(x) < 0.$$

This contradicts the assumption that $f(x) \geq 0$ for all x near a.

Therefore, our assumption that $L < 0$ must be false. We conclude that

$$L \geq 0.$$

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Question 9

Assume that

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = L.$$

Prove that

$$\lim_{x\to a} \big(f(x) - g(x)\big) = 0.$$

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Question 9 - Solution

Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - g(x)| < \varepsilon.$$

Begin by rewriting the difference:

$$f(x) - g(x) = \big(f(x) - L\big) - \big(g(x) - L\big).$$

Take absolute values and apply the triangle inequality:

$$|f(x) - g(x)| \le |f(x) - L| + |g(x) - L|.$$

Since $\lim_{x\to a} f(x) = L$, there exists $\delta_1 > 0$ such that

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{whenever } 0 < |x - a| < \delta_1.$$

Similarly, since $\lim_{x\to a} g(x) = L$, there exists $\delta_2 > 0$ such that

$$|g(x) - L| < \frac{\varepsilon}{2} \quad \text{whenever } 0 < |x - a| < \delta_2.$$

Let

$$\delta = \min(\delta_1, \delta_2).$$

Then for all x satisfying $0 < |x - a| < \delta$,

$$|f(x) - g(x)| \le |f(x) - L| + |g(x) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore,

$$\lim_{x\to a} \big(f(x) - g(x)\big) = 0.$$

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Question 10

Assume that

$$\lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} f(x) = M.$$

Prove that

$$L = M.$$

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Question 10 - Solution

We prove this by contradiction.

Assume, to the contrary, that

$$L \neq M.$$

Then the positive number

$$\varepsilon = \frac{|L - M|}{2}$$

satisfies $\varepsilon > 0$.

Since $\lim_{x \to a} f(x) = L$, there exists $\delta_1 > 0$ such that whenever

$$0 < |x - a| < \delta_1,$$

we have

$$|f(x) - L| < \varepsilon.$$

Similarly, since $\lim_{x \to a} f(x) = M$, there exists $\delta_2 > 0$ such that whenever

$$0 < |x - a| < \delta_2,$$

we have

$$|f(x) - M| < \varepsilon.$$

Let

$$\delta = \min(\delta_1, \delta_2).$$

Then for all x such that $0 < |x - a| < \delta$, both inequalities hold.

Using the triangle inequality,

$$|L - M| = |L - f(x) + f(x) - M| \leq |f(x) - L| + |f(x) - M|.$$

Substitute the bounds:

$$|L - M| < \varepsilon + \varepsilon = 2\varepsilon = |L - M|.$$

This is a contradiction.

Therefore, our assumption that $L \neq M$ is false. We conclude that

$$L = M.$$

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Question 1

Assume that f and g are differentiable at $x = a$. Prove that

$$(f + g)'(a) = f'(a) + g'(a).$$

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Question 1 - Solution

By definition of the derivative,

$$(f+g)'(a) = \lim_{h\to 0} \frac{(f+g)(a+h) - (f+g)(a)}{h}.$$

Expand the numerator:

$$(f+g)(a+h) - (f+g)(a) = \big(f(a+h) + g(a+h)\big) - \big(f(a) + g(a)\big).$$

Group terms:

$$= \big(f(a+h) - f(a)\big) + \big(g(a+h) - g(a)\big).$$

Substitute back into the limit:

$$(f+g)'(a) = \lim_{h\to 0} \left[\frac{f(a+h) - f(a)}{h} + \frac{g(a+h) - g(a)}{h}\right].$$

Use the limit law for sums:

$$(f+g)'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h\to 0} \frac{g(a+h) - g(a)}{h}.$$

Since f and g are differentiable at a, both limits exist and equal $f'(a)$ and $g'(a)$, respectively. Therefore,

$$(f+g)'(a) = f'(a) + g'(a).$$

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Question 2

Assume that f is differentiable at $x = a$ and that c is a constant. Prove that

$$(cf)'(a) = c\,f'(a).$$

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Question 2 - Solution

By definition of the derivative,

$$(cf)'(a) = \lim_{h\to 0} \frac{cf(a+h) - cf(a)}{h}.$$

Factor out the constant c from the numerator:

$$(cf)'(a) = \lim_{h\to 0} \frac{c\big(f(a+h) - f(a)\big)}{h}.$$

Rewrite the fraction:

$$(cf)'(a) = c \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Since f is differentiable at a, the limit exists and equals $f'(a)$. Therefore,

$$(cf)'(a) = c\, f'(a).$$

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Question 3

Assume that f and g are differentiable at $x = a$. Prove the product rule:

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

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Question 3 - Solution

By definition of the derivative,

$$(fg)'(a) = \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}.$$

Add and subtract $f(a+h)g(a)$ in the numerator:

$$= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a+h)g(a) + f(a+h)g(a) - f(a)g(a)}{h}.$$

Group the terms:

$$= \lim_{h\to 0} \left[\frac{f(a+h)\big(g(a+h) - g(a)\big)}{h} + \frac{g(a)\big(f(a+h) - f(a)\big)}{h} \right].$$

Separate the limit:

$$= \lim_{h\to 0} f(a+h) \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} + g(a) \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Since f is differentiable at a, it is continuous at a, so

$$\lim_{h\to 0} f(a+h) = f(a).$$

Also, by differentiability,

$$\lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = g'(a), \qquad \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

Substitute these limits:

$$(fg)'(a) = f(a)g'(a) + g(a)f'(a).$$

Thus,

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

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Question 4

Assume that f and g are differentiable at $x = a$ and that $g(a) \neq 0$. Prove the quotient rule:

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.$$

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Question 4 - Solution

By definition of the derivative,

$$\left(\frac{f}{g}\right)'(a) = \lim_{h\to 0} \frac{\dfrac{f(a+h)}{g(a+h)} - \dfrac{f(a)}{g(a)}}{h}.$$

Combine the fractions in the numerator:

$$= \lim_{h\to 0} \frac{f(a+h)g(a) - f(a)g(a+h)}{h\, g(a+h)g(a)}.$$

Rewrite the numerator by adding and subtracting $f(a)g(a)$:

$$f(a+h)g(a) - f(a)g(a+h) = g(a)\big(f(a+h) - f(a)\big) - f(a)\big(g(a+h) - g(a)\big).$$

Substitute this expression:

$$= \lim_{h\to 0} \frac{g(a)\big(f(a+h) - f(a)\big) - f(a)\big(g(a+h) - g(a)\big)}{h\, g(a+h)g(a)}.$$

Separate the fraction:

$$= \lim_{h\to 0} \left[\frac{g(a)}{g(a+h)g(a)} \cdot \frac{f(a+h) - f(a)}{h} - \frac{f(a)}{g(a+h)g(a)} \cdot \frac{g(a+h) - g(a)}{h}\right].$$

Simplify coefficients:

$$= \lim_{h\to 0} \left[\frac{1}{g(a+h)} \cdot \frac{f(a+h) - f(a)}{h} - \frac{f(a)}{g(a)g(a+h)} \cdot \frac{g(a+h) - g(a)}{h}\right].$$

Since g is differentiable at a, it is continuous at a, so

$$\lim_{h\to 0} g(a+h) = g(a).$$

Taking limits term by term,

$$\left(\frac{f}{g}\right)'(a) = \frac{1}{g(a)} f'(a) - \frac{f(a)}{[g(a)]^2} g'(a).$$

Combine the terms:

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.$$

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Question 4

Assume that f is differentiable at $x = a$. Prove that f is continuous at $x = a$.

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Question 4 - Solution

To show that f is continuous at $x = a$, we must prove that

$$\lim_{x \to a} f(x) = f(a).$$

Since f is differentiable at a, by definition,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Because this limit exists, the expression

$$\frac{f(a+h) - f(a)}{h}$$

remains bounded and approaches a finite value as $h \to 0$.

Rewrite $f(a+h) - f(a)$ as

$$f(a+h) - f(a) = \left(\frac{f(a+h) - f(a)}{h} \right) h.$$

Now take limits as $h \to 0$:

$$\lim_{h \to 0} \big(f(a+h) - f(a) \big) = \left(\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \right) \left(\lim_{h \to 0} h \right).$$

Since

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a) \quad \text{and} \quad \lim_{h \to 0} h = 0,$$

we obtain

$$\lim_{h \to 0} \big(f(a+h) - f(a) \big) = 0.$$

Therefore,

$$\lim_{h \to 0} f(a+h) = f(a).$$

This shows that

$$\lim_{x \to a} f(x) = f(a),$$

so f is continuous at $x = a$.

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Question 6

Assume that f is differentiable at $x = a$ and that $f'(a) = 0$. Prove that

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 0.$$

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Question 6 - Solution

By the definition of the derivative,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

We are given that

$$f'(a) = 0.$$

Thus,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 0.$$

To make this explicit using ε–δ language, let $\varepsilon > 0$ be given.
Since the limit defining $f'(a)$ exists and equals 0, there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$\left| \frac{f(x) - f(a)}{x - a} - 0 \right| < \varepsilon.$$

That is,

$$\left| \frac{f(x) - f(a)}{x - a} \right| < \varepsilon.$$

This proves that

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 0.$$

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Question 7

Prove that the derivative of a constant function is zero. That is, if

$$f(x) = c$$

for all x, where c is a constant, then

$$f'(a) = 0$$

for every real number a.

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Question 7 - Solution

By definition of the derivative,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Since $f(x) = c$ for all x, we have

$$f(a+h) = c \quad \text{and} \quad f(a) = c.$$

Substitute into the difference quotient:

$$f'(a) = \lim_{h \to 0} \frac{c - c}{h}.$$

Simplify the numerator:

$$f'(a) = \lim_{h \to 0} \frac{0}{h}.$$

Since $\frac{0}{h} = 0$ for all $h \neq 0$, the limit is

$$f'(a) = 0.$$

Therefore, the derivative of a constant function is zero:

$$\boxed{f'(a) = 0}.$$

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Question 8

Prove from first principles that if

$$f(x) = x^2,$$

then

$$f'(a) = 2a.$$

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Question 8 - Solution

By definition of the derivative,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Substitute $f(x) = x^2$:

$$f'(a) = \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h}.$$

Expand the square:

$$(a+h)^2 = a^2 + 2ah + h^2.$$

Substitute back:

$$f'(a) = \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - a^2}{h}.$$

Simplify the numerator:

$$f'(a) = \lim_{h \to 0} \frac{2ah + h^2}{h}.$$

Factor out h:

$$f'(a) = \lim_{h \to 0} \frac{h(2a+h)}{h}.$$

Cancel h (for $h \neq 0$):

$$f'(a) = \lim_{h \to 0} (2a + h).$$

Now take the limit:

$$f'(a) = 2a.$$

Thus, using first principles,

$$\boxed{f'(a) = 2a}.$$

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Question 9

Prove from first principles that if

$$f(x) = x^3,$$

then

$$f'(a) = 3a^2.$$

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Question 9 - Solution

By definition of the derivative,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Substitute $f(x) = x^3$:

$$f'(a) = \lim_{h \to 0} \frac{(a+h)^3 - a^3}{h}.$$

Expand the cube:

$$(a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3.$$

Substitute into the difference quotient:

$$f'(a) = \lim_{h \to 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h}.$$

Simplify the numerator:

$$f'(a) = \lim_{h \to 0} \frac{3a^2h + 3ah^2 + h^3}{h}.$$

Factor out h:

$$f'(a) = \lim_{h \to 0} \frac{h(3a^2 + 3ah + h^2)}{h}.$$

Cancel h (for $h \neq 0$):

$$f'(a) = \lim_{h \to 0} (3a^2 + 3ah + h^2).$$

Now take the limit:

$$f'(a) = 3a^2.$$

Thus, using first principles,

$$\boxed{f'(a) = 3a^2}.$$

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Question 10

Assume that f is differentiable at $x = a$ and that g is differentiable at $x = f(a)$. Prove the Chain Rule:

$$(g \circ f)'(a) = g'(f(a))\, f'(a).$$

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Question 10 - Solution

Put $b = f(a)$. Differentiability of g at b means

$$g(b+u) - g(b) = g'(b)u + u\eta(u),$$

where $\eta(u) \to 0$ as $u \to 0$; define $\eta(0) = 0$ so this identity also holds at $u = 0$.
For $h \neq 0$, set $u = f(a+h) - f(a)$. Then

$$\frac{g(f(a+h)) - g(f(a))}{h} = \big(g'(b) + \eta(f(a+h) - f(a))\big)\frac{f(a+h) - f(a)}{h}.$$

Differentiability implies continuity of f at a, so the first factor tends to $g'(b)$ and the second to $f'(a)$.

$$\boxed{(g \circ f)'(a) = g'(f(a))f'(a).}$$

This argument remains valid when $f(a+h) = f(a)$ for some or all nearby h.

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Question 1

Prove that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

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Question 1 - Solution

We prove this limit using geometric inequalities.

Consider a unit circle centered at the origin.
Let $0 < x < \pi/2$ be measured in radians, and consider the angle x.

The area of the sector with angle x is

$$\text{Area of sector} = \frac{1}{2}x.$$

The area of the triangle formed by the radius and the angle x is

$$\text{Area of triangle} = \frac{1}{2}\sin x.$$

The area of the triangle formed by the tangent line is

$$\text{Area of outer triangle} = \frac{1}{2}\tan x.$$

Thus, the areas satisfy

$$\frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}\tan x.$$

Multiply through by 2:

$$\sin x < x < \tan x.$$

Divide all parts by $\sin x$ (which is positive for $0 < x < \pi/2$):

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

Taking reciprocals reverses the inequalities:

$$\cos x < \frac{\sin x}{x} < 1.$$

Now take the limit as $x \to 0^+$.
Since

$$\lim_{x\to 0} \cos x = 1,$$

the Squeeze Theorem gives

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1.$$

A similar argument applies for $x \to 0^-$, so the two-sided limit exists.

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

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Question 2

Prove that

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$$

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Question 2 - Solution

We begin by rewriting the expression using a trigonometric identity.

Recall the identity

$$1-\cos x = 2\sin^2\left(\frac{x}{2}\right).$$

Substitute this into the limit:

$$\frac{1-\cos x}{x} = \frac{2\sin^2\left(\frac{x}{2}\right)}{x}.$$

Rewrite the fraction:

$$\frac{2\sin^2\left(\frac{x}{2}\right)}{x} = \sin\left(\frac{x}{2}\right)\cdot\frac{2\sin\left(\frac{x}{2}\right)}{x}.$$

Now rewrite the second factor:

$$\frac{2\sin\left(\frac{x}{2}\right)}{x} = \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}.$$

Thus,

$$\frac{1-\cos x}{x} = \sin\left(\frac{x}{2}\right)\cdot\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}.$$

Now take limits of each factor as $x \to 0$.

Since $\lim_{u\to 0}\sin u = 0$, we have

$$\lim_{x\to 0}\sin\left(\frac{x}{2}\right) = 0.$$

Also, using the fundamental trig limit

$$\lim_{u\to 0}\frac{\sin u}{u} = 1,$$

we obtain

$$\lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} = 1.$$

Therefore, by the product law for limits,

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0\cdot 1 = 0.$$

$$\boxed{0}$$

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Question 3

Prove that

$$\lim_{x \to 0} \frac{\sin(3x)}{x} = 3.$$

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Question 3 - Solution

We begin by rewriting the expression so that the fundamental trigonometric limit appears.

Write

$$\frac{\sin(3x)}{x} = \frac{\sin(3x)}{3x} \cdot 3.$$

Now take limits as $x \to 0$.

Since $3x \to 0$ as $x \to 0$, the fundamental trigonometric limit gives

$$\lim_{x\to 0} \frac{\sin(3x)}{3x} = 1.$$

Therefore,

$$\lim_{x\to 0} \frac{\sin(3x)}{x} = \left(\lim_{x\to 0} \frac{\sin(3x)}{3x}\right) \cdot 3 = 1 \cdot 3 = 3.$$

$$\boxed{3}$$

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Question 4

Prove that

$$\lim_{x \to 0} \frac{\tan x}{x} = 1.$$

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Question 4 - Solution

We rewrite the expression in terms of sine and cosine.

Recall that

$$\tan x = \frac{\sin x}{\cos x}.$$

Substitute this into the limit:

$$\frac{\tan x}{x} = \frac{\sin x}{x \cos x} = \left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right).$$

Now take limits of each factor as $x \to 0$.

From the fundamental trigonometric limit,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Also, since cosine is continuous at 0,

$$\lim_{x \to 0} \cos x = \cos 0 = 1,$$

and therefore

$$\lim_{x \to 0} \frac{1}{\cos x} = 1.$$

By the product law for limits,

$$\lim_{x \to 0} \frac{\tan x}{x} = 1 \cdot 1 = 1.$$

$$\boxed{1}$$

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Question 5

Prove that

$$\lim_{x\to 0} \frac{1-\cos(2x)}{x^2} = 2.$$

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Question 5 - Solution

We begin by using a trigonometric identity.

Recall that

$$1 - \cos(2x) = 2\sin^2 x.$$

Substitute this into the expression:

$$\frac{1 - \cos(2x)}{x^2} = \frac{2\sin^2 x}{x^2}.$$

Rewrite the fraction:

$$\frac{2\sin^2 x}{x^2} = 2\left(\frac{\sin x}{x}\right)^2.$$

Now take limits as $x \to 0$.

Using the fundamental trigonometric limit,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Therefore,

$$\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 = 1^2 = 1.$$

Multiply by 2:

$$\lim_{x\to 0} \frac{1 - \cos(2x)}{x^2} = 2 \cdot 1 = 2.$$

$$\boxed{2}$$

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Question 6

Prove that

$$\lim_{x \to 0} \frac{\sin x - x}{x} = 0.$$

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Question 6 - Solution

We begin by factoring the expression.

Rewrite the fraction as

$$\frac{\sin x - x}{x} = \frac{\sin x}{x} - 1.$$

Now take the limit as $x \to 0$.

Using the fundamental trigonometric limit,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Therefore,

$$\lim_{x \to 0} \left(\frac{\sin x}{x} - 1 \right) = 1 - 1 = 0.$$

Hence,

$$\boxed{\lim_{x \to 0} \frac{\sin x - x}{x} = 0}.$$

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Question 7

Prove that

$$\lim_{x \to 0} \frac{\sin x}{\tan x} = 1.$$

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Question 7 - Solution

Rewrite the expression using the identity

$$\tan x = \frac{\sin x}{\cos x}.$$

Substitute into the fraction:

$$\frac{\sin x}{\tan x} = \frac{\sin x}{\frac{\sin x}{\cos x}} = \cos x,$$

for all x where $\tan x$ is defined and $\sin x \neq 0$.

Now take the limit as $x \to 0$.

Since cosine is continuous at 0,

$$\lim_{x \to 0} \cos x = \cos 0 = 1.$$

Therefore,

$$\lim_{x \to 0} \frac{\sin x}{\tan x} = 1.$$

$$\boxed{1}$$

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Question 8

Prove that

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0.$$

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Question 8 - Solution

We begin by rewriting the expression using a trigonometric identity.

Recall that

$$\cos x - 1 = -\big(1 - \cos x\big).$$

Thus,

$$\frac{\cos x - 1}{x} = -\frac{1 - \cos x}{x}.$$

From a previously established trigonometric limit,

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$$

Therefore,

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = -\lim_{x \to 0} \frac{1 - \cos x}{x} = -0 = 0.$$

$$\boxed{0}$$

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Question 9

Prove that

$$\lim_{x\to 0} \frac{\sin(2x)}{\tan x} = 2.$$

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Question 9 - Solution

We begin by rewriting the expression using trigonometric identities.

Recall that

$$\tan x = \frac{\sin x}{\cos x}.$$

Substitute this into the fraction:

$$\frac{\sin(2x)}{\tan x} = \sin(2x) \cdot \frac{\cos x}{\sin x}.$$

Next, use the identity

$$\sin(2x) = 2\sin x \cos x.$$

Substitute:

$$\frac{\sin(2x)}{\tan x} = (2\sin x \cos x) \cdot \frac{\cos x}{\sin x}.$$

Cancel $\sin x$ (for $x \neq 0$):

$$\frac{\sin(2x)}{\tan x} = 2\cos^2 x.$$

Now take the limit as $x \to 0$.

Since cosine is continuous,

$$\lim_{x\to 0} \cos x = \cos 0 = 1.$$

Therefore,

$$\lim_{x\to 0} 2\cos^2 x = 2(1)^2 = 2.$$

$$\boxed{2}$$

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Question 10

Prove that

$$\lim_{x \to 0} \frac{\tan x - x}{x} = 0.$$

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Question 10 - Solution

We begin by rewriting the expression to isolate a known limit.

Factor out x from the numerator:

$$\frac{\tan x - x}{x} = \frac{\tan x}{x} - 1.$$

Now rewrite $\tan x$ in terms of sine and cosine:

$$\frac{\tan x}{x} = \frac{\sin x}{x \cos x} = \left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right).$$

Take limits of each factor as $x \to 0$.

Using the fundamental trigonometric limit,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Since cosine is continuous at 0,

$$\lim_{x \to 0} \cos x = \cos 0 = 1, \quad \text{so} \quad \lim_{x \to 0} \frac{1}{\cos x} = 1.$$

Thus,

$$\lim_{x \to 0} \frac{\tan x}{x} = 1.$$

Finally,

$$\lim_{x \to 0} \left(\frac{\tan x}{x} - 1\right) = 1 - 1 = 0.$$

$$\boxed{0}$$

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Question 1

Assume that f has a local maximum at $x = a$ and that f is differentiable at $x = a$. Prove that

$$f'(a) = 0.$$

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Question 1 - Solution

Since f has a local maximum at $x = a$, there exists $\delta > 0$ such that

$$f(x) \leq f(a) \quad \text{for all } x \text{ with } |x - a| < \delta.$$

Consider the difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

for small values of h.

Case 1: $h > 0$

If $0 < h < \delta$, then $a + h$ lies within the interval where $f(a)$ is a local maximum. Hence,

$$f(a+h) - f(a) \leq 0.$$

Since $h > 0$, dividing by h preserves the inequality:

$$\frac{f(a+h) - f(a)}{h} \leq 0.$$

Case 2: $h < 0$

If $-\delta < h < 0$, then again $a + h$ lies within the interval of the local maximum, so

$$f(a+h) - f(a) \leq 0.$$

Since $h < 0$, dividing by h reverses the inequality:

$$\frac{f(a+h) - f(a)}{h} \geq 0.$$

Step 3: Take the limit

For $h > 0$, the difference quotient is less than or equal to 0. For $h < 0$, the difference quotient is greater than or equal to 0.

Since f is differentiable at a, the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists and must be the same from both sides.

The only number that is both ≤ 0 and ≥ 0 is 0.

Therefore,

$$f'(a) = 0.$$

$$\boxed{f'(a) = 0}$$

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Question 2

Assume that f is continuous on $[a, b]$, differentiable on (a, b), and

$$f(a) = f(b).$$

Prove that there exists a number $c \in (a, b)$ such that

$$f'(c) = 0.$$

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Question 2 - Solution

Since f is continuous on the closed interval $[a, b]$, the Extreme Value Theorem guarantees that f attains both a maximum and a minimum on $[a, b]$.

Let

$$M = \max_{x\in[a,b]} f(x), \qquad m = \min_{x\in[a,b]} f(x).$$

If $M = m$, then f is constant on $[a, b]$, and hence

$$f'(x) = 0 \quad \text{for all } x \in (a, b).$$

In particular, there exists $c \in (a, b)$ such that $f'(c) = 0$.

Now assume that $M \neq m$. Since $f(a) = f(b)$, the maximum or minimum cannot occur at both endpoints.

Thus, at least one of the extreme values M or m must occur at some point $c \in (a, b)$.

At such a point c, the function f has either a local maximum or a local minimum. Because f is differentiable at c, the derivative must vanish there.

Therefore,

$$f'(c) = 0.$$

$$\boxed{\text{There exists } c \in (a, b) \text{ such that } f'(c) = 0}$$

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Question 3

Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Prove that there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

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Question 3 - Solution

Define a new function

$$g(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a).$$

The function g is continuous on $[a, b]$ and differentiable on (a, b) because it is formed from functions with these properties.

Evaluate g at the endpoints:

$$g(a) = f(a) - \left(\frac{f(b) - f(a)}{b - a}\right)(a - a) = f(a),$$

$$g(b) = f(b) - \left(\frac{f(b) - f(a)}{b - a}\right)(b - a) = f(b) - (f(b) - f(a)) = f(a).$$

Thus,

$$g(a) = g(b).$$

By Rolle's Theorem, there exists a number $c \in (a, b)$ such that

$$g'(c) = 0.$$

Differentiate $g(x)$:

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Substitute $x = c$:

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Solve for $f'(c)$:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

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Question 4

Assume that f is differentiable on an interval (a, b) and that

$$f'(x) > 0 \quad \text{for all } x \in (a, b).$$

Prove that f is increasing on (a, b).

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Question 4 - Solution

To show that f is increasing on (a, b), we must prove that for any $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, we have

$$f(x_1) < f(x_2).$$

Let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. Since f is differentiable on (a, b), it is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2).

By the Mean Value Theorem, there exists a number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By assumption, $f'(c) > 0$, and since $x_2 - x_1 > 0$, it follows that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0.$$

Multiply both sides by $x_2 - x_1$:

$$f(x_2) - f(x_1) > 0.$$

Thus,

$$f(x_2) > f(x_1).$$

Since this holds for all $x_1 < x_2$ in (a, b), the function f is increasing on (a, b).

$$\boxed{f \text{ is increasing on } (a, b)}$$

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Question 5

Assume that f is differentiable on an interval (a, b) and that

$$f'(x) < 0 \quad \text{for all } x \in (a, b).$$

Prove that f is decreasing on (a, b).

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Question 5 - Solution

To prove that f is decreasing on (a, b), we must show that for any $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, the inequality

$$f(x_1) > f(x_2)$$

holds.

Let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. Because f is differentiable on (a, b), it is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2).

By the Mean Value Theorem, there exists a number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By assumption, $f'(c) < 0$, and since $x_2 - x_1 > 0$, it follows that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0.$$

Multiply both sides by $x_2 - x_1$:

$$f(x_2) - f(x_1) < 0.$$

Thus,

$$f(x_2) < f(x_1).$$

Since this holds for all $x_1 < x_2$ in (a, b), the function f is decreasing on (a, b).

$$\boxed{f \text{ is decreasing on } (a, b)}$$

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Question 6

Assume that f is twice differentiable on an interval (a, b) and that

$$f''(x) > 0 \quad \text{for all } x \in (a, b).$$

Prove that f is concave up on (a, b).

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Question 6 - Solution

To show that f is concave up on (a, b), we must prove that the derivative $f'(x)$ is increasing on (a, b).

Let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. Since f is twice differentiable on (a, b), the function f' is differentiable on (a, b) and therefore continuous on $[x_1, x_2]$.

Apply the Mean Value Theorem to f' on the interval $[x_1, x_2]$. There exists a number $c \in (x_1, x_2)$ such that

$$f''(c) = \frac{f'(x_2) - f'(x_1)}{x_2 - x_1}.$$

By assumption, $f''(c) > 0$ and since $x_2 - x_1 > 0$, it follows that

$$\frac{f'(x_2) - f'(x_1)}{x_2 - x_1} > 0.$$

Multiply both sides by $x_2 - x_1$:

$$f'(x_2) - f'(x_1) > 0.$$

Thus,

$$f'(x_2) > f'(x_1).$$

Since this holds for all $x_1 < x_2$ in (a, b), the derivative $f'(x)$ is increasing on (a, b).

Therefore, the function f is concave up on (a, b).

$$\boxed{f \text{ is concave up on } (a, b)}$$

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Question 7

Assume that f is twice differentiable on an interval (a, b) and that

$$f''(x) < 0 \quad \text{for all } x \in (a, b).$$

Prove that f is concave down on (a, b).

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Question 7 - Solution

To show that f is concave down on (a, b), we must show that the derivative $f'(x)$ is decreasing on (a, b).

Let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. Since f is twice differentiable on (a, b), the function f' is differentiable on (a, b) and therefore continuous on $[x_1, x_2]$.

Apply the Mean Value Theorem to f' on the interval $[x_1, x_2]$. There exists a number $c \in (x_1, x_2)$ such that

$$f''(c) = \frac{f'(x_2) - f'(x_1)}{x_2 - x_1}.$$

By assumption, $f''(c) < 0$, and since $x_2 - x_1 > 0$, it follows that

$$\frac{f'(x_2) - f'(x_1)}{x_2 - x_1} < 0.$$

Multiply both sides by $x_2 - x_1$:

$$f'(x_2) - f'(x_1) < 0.$$

Thus,

$$f'(x_2) < f'(x_1).$$

Since this holds for all $x_1 < x_2$ in (a, b), the derivative $f'(x)$ is decreasing on (a, b).

Therefore, the function f is concave down on (a, b).

$$\boxed{f \text{ is concave down on } (a, b)}$$

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Question 8

Assume that f is differentiable on an open interval I containing a, and that

$$f'(a) = 0 \qquad \text{and} \qquad f''(a) > 0.$$

Prove that f has a strict local minimum at $x = a$.

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Question 8 - Solution

By the definition of the second derivative and $f'(a) = 0$,

$$\lim_{h \to 0} \frac{f'(a+h)}{h} = f''(a).$$

Because this limit is nonzero, the quotient has the same sign as $f''(a)$ for all sufficiently small nonzero h.

Consequently, $f'(a+h) < 0$ for negative h and $f'(a+h) > 0$ for positive h. Thus f is strictly decreasing just left of a and strictly increasing just right of a, by the Mean Value Theorem.

Therefore f has a strict local minimum at a. No continuity assumption on f'' near a is needed.

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Question 9

Assume that f is continuous on $[a, b]$, differentiable on (a, b), and that

$$f'(x) = 0 \quad \text{for all } x \in (a, b).$$

Prove that f is constant on $[a, b]$.

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Question 9 - Solution

Let $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. We will show that

$$f(x_1) = f(x_2).$$

Since f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2), the Mean Value Theorem applies. Thus, there exists a number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By assumption, $f'(c) = 0$, so

$$0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $x_2 - x_1 \neq 0$, it follows that

$$f(x_2) - f(x_1) = 0.$$

Therefore,

$$f(x_2) = f(x_1).$$

Because x_1 and x_2 were arbitrary points in $[a, b]$, we conclude that $f(x)$ has the same value at every point in the interval.

$$\boxed{f \text{ is constant on } [a, b]}$$

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Question 10

Assume that f is differentiable on an open interval I containing a, and that

$$f'(a) = 0 \qquad \text{and} \qquad f''(a) < 0.$$

Prove that f has a strict local maximum at $x = a$.

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Question 10 - Solution

By the definition of the second derivative and $f'(a) = 0$,

$$\lim_{h \to 0} \frac{f'(a+h)}{h} = f''(a).$$

Because this limit is nonzero, the quotient has the same sign as $f''(a)$ for all sufficiently small nonzero h.

Consequently, $f'(a+h) > 0$ for negative h and $f'(a+h) < 0$ for positive h. Thus f is strictly increasing just left of a and strictly decreasing just right of a, by the Mean Value Theorem.

Therefore f has a strict local maximum at a. No continuity assumption on f'' near a is needed.

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Question 1

Assume that f is integrable on the interval $[a, b]$ and that c is a constant. Prove that

$$\int_a^b c\, f(x)\, dx = c \int_a^b f(x)\, dx.$$

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Question 1 - Solution

We prove this property using the definition of the definite integral in terms of Riemann sums.

Let P be a partition of $[a, b]$:

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let $\Delta x_i = x_i - x_{i-1}$. Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

A Riemann sum for f on $[a, b]$ is

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

A Riemann sum for cf on $[a, b]$ is

$$\sum_{i=1}^{n} cf(x_i^*)\,\Delta x_i.$$

Factor out the constant c:

$$\sum_{i=1}^{n} cf(x_i^*)\,\Delta x_i = c\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

Now take the limit as the norm of the partition $\|P\| \to 0$. Since f is integrable on $[a, b]$, the limit of its Riemann sums exists:

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i = \int_a^b f(x)\,dx.$$

Therefore,

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} cf(x_i^*)\,\Delta x_i = c\int_a^b f(x)\,dx.$$

By definition of the definite integral, this limit equals

$$\int_a^b c\,f(x)\,dx.$$

Hence,

$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx.$$

$$\boxed{\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx}$$

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Question 2

Assume that f and g are integrable on the interval $[a, b]$. Prove that

$$\int_a^b \big(f(x) + g(x)\big)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

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Question 2 - Solution

We prove this property using Riemann sums.

Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of $[a, b]$, and let $\Delta x_i = x_i - x_{i-1}$. Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

A Riemann sum for $f + g$ is

$$\sum_{i=1}^{n} \big(f(x_i^*) + g(x_i^*)\big)\Delta x_i.$$

Using algebra, split the sum:

$$\sum_{i=1}^{n} \big(f(x_i^*) + g(x_i^*)\big)\Delta x_i = \sum_{i=1}^{n} f(x_i^*)\Delta x_i + \sum_{i=1}^{n} g(x_i^*)\Delta x_i.$$

Now take the limit as the norm of the partition $\|P\| \to 0$.
Since f and g are integrable on $[a, b]$, the limits of both sums exist:

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \int_a^b f(x)\,dx,$$

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} g(x_i^*)\Delta x_i = \int_a^b g(x)\,dx.$$

Therefore,

$$\int_a^b \big(f(x) + g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

$$\boxed{\int_a^b (f + g) = \int_a^b f + \int_a^b g}$$

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Question 3

Assume that f is integrable on $[a, b]$ and that $a < b$. Prove that

$$\int_a^a f(x)\,dx = 0.$$

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Question 3 - Solution

We use the definition of the definite integral in terms of Riemann sums.

Consider the interval $[a, a]$. Any partition of this interval consists of a single point:

$$a = x_0 = x_1.$$

The width of the subinterval is

$$\Delta x = x_1 - x_0 = a - a = 0.$$

Choose any sample point x^* in the interval $[a, a]$.

A Riemann sum for f over $[a, a]$ is

$$\sum f(x^*)\,\Delta x = f(x^*) \cdot 0 = 0.$$

Since every Riemann sum over $[a, a]$ equals 0, the limit of the Riemann sums exists and equals 0.

By definition of the definite integral,

$$\int_a^a f(x)\,dx = 0.$$

$$\boxed{\int_a^a f(x)\,dx = 0}$$

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Question 4

Assume that f is integrable on $[a, b]$. Prove that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

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Question 4 - Solution

We use the definition of the definite integral in terms of Riemann sums.

Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of the interval $[a, b]$, with subinterval widths

$$\Delta x_i = x_i - x_{i-1}.$$

A Riemann sum for f over $[a, b]$ is

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

Now consider the interval $[b, a]$. Reverse the order of the partition:

$$b = x_n > x_{n-1} > \cdots > x_0 = a.$$

The corresponding subinterval widths are

$$\Delta x_i' = x_{i-1} - x_i = -\Delta x_i.$$

A Riemann sum for f over $[b, a]$ is

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i' = \sum_{i=1}^{n} f(x_i^*)(-\Delta x_i) = -\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

Take the limit as the norm of the partition goes to zero. Since f is integrable on $[a, b]$, the limit exists:

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Rewriting,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

$$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

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Question 5

Assume that f is integrable on $[a, b]$ and that

$$f(x) \geq 0 \quad \text{for all } x \in [a, b].$$

Prove that

$$\int_a^b f(x)\, dx \geq 0.$$

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Question 5 - Solution

We use the definition of the definite integral in terms of Riemann sums.

Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of the interval $[a, b]$, and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

A Riemann sum for f on $[a, b]$ is

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

Since $f(x) \geq 0$ for all $x \in [a, b]$ and $\Delta x_i > 0$, each term in the sum satisfies

$$f(x_i^*)\,\Delta x_i \geq 0.$$

Thus, every Riemann sum for f over $[a, b]$ is nonnegative:

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i \geq 0.$$

Now take the limit as the norm of the partition $\|P\| \to 0$. Since f is integrable, the limit of the Riemann sums exists and equals the definite integral:

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

Because every sum in the limit is nonnegative, the limit itself must be nonnegative.

Therefore,

$$\int_a^b f(x)\,dx \geq 0.$$

$$\boxed{\int_a^b f(x)\,dx \geq 0}$$

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Question 6

Assume that f and g are integrable on $[a, b]$ and that

$$f(x) \leq g(x) \quad \text{for all } x \in [a, b].$$

Prove that

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

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Question 6 - Solution

Define a new function

$$h(x) = g(x) - f(x).$$

Since f and g are integrable on $[a, b]$, the function h is also integrable on $[a, b]$.

From the assumption $f(x) \leq g(x)$ for all $x \in [a, b]$, we have

$$h(x) = g(x) - f(x) \geq 0 \quad \text{for all } x \in [a, b].$$

By the positivity property of integrals,

$$\int_a^b h(x)\,dx \geq 0.$$

Substitute back for $h(x)$:

$$\int_a^b \big(g(x) - f(x)\big)\,dx \geq 0.$$

Use linearity of the integral:

$$\int_a^b g(x)\,dx - \int_a^b f(x)\,dx \geq 0.$$

Rearranging gives

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

$$\boxed{\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx}$$

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Question 7

Assume that f is integrable on $[a, b]$. Prove that

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

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Question 7 - Solution

We begin with a basic inequality involving absolute values.

For all real numbers u, we have

$$-|u| \le u \le |u|.$$

Apply this inequality to $f(x)$:

$$-|f(x)| \le f(x) \le |f(x)| \quad \text{for all } x \in [a,b].$$

Integrate each part of the inequality over $[a,b]$:

$$\int_a^b -|f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx.$$

Simplify the left-hand integral:

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx.$$

This shows that the value of $\int_a^b f(x)\,dx$ lies between $-\int_a^b |f(x)|\,dx$ and $\int_a^b |f(x)|\,dx$.

By definition of absolute value, this implies

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

$$\boxed{\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx}$$

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Question 8

Assume that f is integrable on $[a, b]$ and that c satisfies

$$a < c < b.$$

Prove that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

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Question 8 - Solution

We prove this property using Riemann sums.

Let

$$a = x_0 < x_1 < \cdots < x_k = c < \cdots < x_n = b$$

be a partition of $[a, b]$ that includes the point c.

Let

$$\Delta x_i = x_i - x_{i-1}$$

and choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

A Riemann sum for f over $[a, b]$ is

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

Split this sum at c:

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i = \sum_{i=1}^{k} f(x_i^*)\,\Delta x_i + \sum_{i=k+1}^{n} f(x_i^*)\,\Delta x_i.$$

The first sum is a Riemann sum for f over $[a, c]$, and the second sum is a Riemann sum for f over $[c, b]$.

Now take the limit as the norm of the partition $\|P\| \to 0$.
Since f is integrable on $[a, b]$, it is integrable on both subintervals $[a, c]$ and $[c, b]$.
Thus,

$$\lim_{\|P\|\to 0} \sum_{i=1}^{k} f(x_i^*)\,\Delta x_i = \int_a^c f(x)\,dx,$$

$$\lim_{\|P\|\to 0} \sum_{i=k+1}^{n} f(x_i^*)\,\Delta x_i = \int_c^b f(x)\,dx.$$

Therefore,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

$$\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx}$$

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Question 9

Assume that f is integrable on $[a, b]$ and that there exists a real number m such that

$$f(x) \geq m \quad \text{for all } x \in [a, b].$$

Prove that

$$\int_a^b f(x)\,dx \geq m(b-a).$$

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Question 9 - Solution

Define a new function

$$h(x) = f(x) - m.$$

Since f is integrable on $[a, b]$ and m is a constant, the function h is also integrable on $[a, b]$.

From the assumption $f(x) \geq m$ for all $x \in [a, b]$, we have

$$h(x) = f(x) - m \geq 0 \quad \text{for all } x \in [a, b].$$

By the positivity property of integrals,

$$\int_a^b h(x)\, dx \geq 0.$$

Substitute back for $h(x)$:

$$\int_a^b \big(f(x) - m\big)\, dx \geq 0.$$

Use linearity of the integral:

$$\int_a^b f(x)\, dx - \int_a^b m\, dx \geq 0.$$

Since $\int_a^b m\, dx = m(b - a)$, we obtain

$$\int_a^b f(x)\, dx - m(b - a) \geq 0.$$

Rearranging gives

$$\int_a^b f(x)\, dx \geq m(b - a).$$

$$\boxed{\int_a^b f(x)\, dx \geq m(b - a)}$$

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Question 10

Assume that f is integrable on $[a, b]$ and that there exists a real number M such that

$$f(x) \leq M \quad \text{for all } x \in [a, b].$$

Prove that

$$\int_a^b f(x)\, dx \leq M(b - a).$$

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Question 10 - Solution

Define the function

$$h(x) = M - f(x).$$

Since f is integrable on $[a, b]$ and M is a constant, the function h is also integrable on $[a, b]$.

From the assumption $f(x) \leq M$ for all $x \in [a, b]$, we have

$$h(x) = M - f(x) \geq 0 \quad \text{for all } x \in [a, b].$$

By the positivity property of integrals,

$$\int_a^b h(x)\,dx \geq 0.$$

Substitute back for $h(x)$:

$$\int_a^b \big(M - f(x)\big)\,dx \geq 0.$$

Use linearity of the integral:

$$\int_a^b M\,dx - \int_a^b f(x)\,dx \geq 0.$$

Since

$$\int_a^b M\,dx = M(b-a),$$

we obtain

$$M(b-a) - \int_a^b f(x)\,dx \geq 0.$$

Rearranging gives

$$\int_a^b f(x)\,dx \leq M(b-a).$$

$$\boxed{\int_a^b f(x)\,dx \leq M(b-a)}$$

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Question 1

Assume that f is a continuous, nonnegative function on the interval $[a, b]$. Prove that the area of the region bounded by the graph of $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b f(x)\,dx.$$

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Question 1 - Solution

We interpret area using approximating rectangles.

Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of $[a, b]$, and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

Since $f(x) \geq 0$, the area of the rectangle over $[x_{i-1}, x_i]$ with height $f(x_i^*)$ is

$$\text{Area}_i = f(x_i^*)\,\Delta x_i.$$

The total area of all rectangles is

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

As the partition is refined (that is, as $\|P\| \to 0$), the rectangles better approximate the region under the curve. Because f is continuous on $[a, b]$, it is integrable, and the limit of these sums exists.

Taking the limit gives

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i = \int_a^b f(x)\,dx.$$

By definition, this limit equals the exact area of the region under the curve.

Therefore, the area of the region bounded by $y = f(x)$, the x-axis, and $x = a$, $x = b$ is

$$A = \int_a^b f(x)\,dx.$$

$$\boxed{A = \int_a^b f(x)\,dx}$$

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Question 2

Assume that f is a continuous, nonnegative function on $[a, b]$. Prove that the volume of the solid obtained by revolving the region bounded by $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$ about the x-axis is

$$V = \pi \int_a^b [f(x)]^2 \, dx.$$

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Question 2 - Solution

We approximate the solid of revolution using thin disks.

Partition the interval $[a, b]$ as

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let $\Delta x_i = x_i - x_{i-1}$. Choose a sample point x_i^* in each subinterval.

Over the interval $[x_{i-1}, x_i]$, the graph of $y = f(x)$ generates a disk when revolved about the x-axis.

The radius of this disk is

$$r_i = f(x_i^*).$$

Thus, the volume of the disk is

$$\Delta V_i = \pi r_i^2 \, \Delta x_i = \pi [f(x_i^*)]^2 \, \Delta x_i.$$

The total volume of all disks is approximated by the sum

$$\sum_{i=1}^{n} \pi [f(x_i^*)]^2 \, \Delta x_i.$$

As the partition is refined and $\|P\| \to 0$, this sum approaches the exact volume of the solid. Since f is continuous, the limit exists.

Taking the limit yields

$$V = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \pi [f(x_i^*)]^2 \, \Delta x_i = \pi \int_a^b [f(x)]^2 \, dx.$$

$$\boxed{V = \pi \int_a^b [f(x)]^2 \, dx}$$

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Question 3

Assume that f and g are continuous functions on $[a, b]$ with

$$f(x) \geq g(x) \quad \text{for all } x \in [a, b].$$

Prove that the area of the region bounded by the graphs of $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is

$$A = \int_a^b \big(f(x) - g(x)\big)\, dx.$$

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Question 3 - Solution

We approximate the region using vertical rectangles.

Partition the interval $[a, b]$ as

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

At the point x_i^*, the height of the region between the curves is

$$f(x_i^*) - g(x_i^*).$$

Thus, the area of the corresponding rectangle is

$$\Delta A_i = \big(f(x_i^*) - g(x_i^*)\big)\Delta x_i.$$

The total area of all rectangles is approximated by the sum

$$\sum_{i=1}^{n} \big(f(x_i^*) - g(x_i^*)\big)\Delta x_i.$$

As the partition is refined and $\|P\| \to 0$, these rectangles more accurately approximate the region between the curves. Since f and g are continuous, the limit of the sums exists.

Taking the limit gives

$$A = \lim_{\|P\|\to 0} \sum_{i=1}^{n} \big(f(x_i^*) - g(x_i^*)\big)\Delta x_i = \int_a^b \big(f(x) - g(x)\big)\, dx.$$

$$\boxed{A = \int_a^b \big(f(x) - g(x)\big)\, dx}$$

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Question 4

Assume $0 \leq a < b$ and that f is continuous and nonnegative on $[a, b]$. Prove that the volume of the solid obtained by revolving the region bounded by $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$ about the y-axis is

$$V = 2\pi \int_a^b x\, f(x)\, dx.$$

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Question 4 - Solution

Take a partition $a = x_0 < \cdots < x_n = b$ and let m_i, M_i be the minimum and maximum of f on $[x_{i-1}, x_i]$.

Since $a \geq 0$, the annular cylinders occupy disjoint radial intervals. Their volumes bound the desired volume:

$$\sum_i \pi(x_i^2 - x_{i-1}^2)m_i \ \leq V \ \leq \sum_i \pi(x_i^2 - x_{i-1}^2)M_i.$$

Let $c_i = (x_i + x_{i-1})/2$. Then $\pi(x_i^2 - x_{i-1}^2) = 2\pi c_i \Delta x_i$.

Uniform continuity of f makes $\max_i(M_i - m_i) \to 0$ as the mesh tends to zero, so the difference between the upper and lower bounds tends to zero.

Both bounds approach the Riemann integral of $2\pi x f(x)$. Hence

$$\boxed{V = 2\pi \int_a^b x f(x)\, dx.}$$

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Question 5

Assume that f is a continuous, nonnegative function on $[a, b]$. Prove that the volume of the solid obtained by revolving the region bounded by $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$ about the x-axis is

$$V = \pi \int_a^b [f(x)]^2 \, dx.$$

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Question 5 - Solution

We derive the formula using the idea of approximating the solid with thin disks.

Partition the interval $[a, b]$ into subintervals

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

Over a small interval $[x_{i-1}, x_i]$, the graph of $y = f(x)$ is approximately constant at height $f(x_i^*)$. When this vertical strip is revolved about the x-axis, it forms a thin circular disk.

The radius of the disk is

$$r_i = f(x_i^*).$$

The area of the circular face of the disk is

$$A_i = \pi r_i^2 = \pi[f(x_i^*)]^2.$$

Since the thickness of the disk is Δx_i, its volume is approximately

$$\Delta V_i = \pi[f(x_i^*)]^2 \, \Delta x_i.$$

Adding the volumes of all disks gives an approximation to the total volume:

$$\sum_{i=1}^{n} \pi[f(x_i^*)]^2 \, \Delta x_i.$$

As the partition is refined and the maximum subinterval length approaches zero, these disks more accurately fill the solid. Because f is continuous on $[a, b]$, the limit of the sum exists.

Taking the limit yields

$$V = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \pi[f(x_i^*)]^2 \, \Delta x_i = \pi \int_a^b [f(x)]^2 \, dx.$$

$$\boxed{V = \pi \int_a^b [f(x)]^2 \, dx}$$

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Question 6

Assume that f and g are continuous functions on $[a, b]$ with

$$f(x) \geq g(x) \geq 0 \quad \text{for all } x \in [a, b].$$

Prove that the volume of the solid obtained by revolving the region bounded by $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$ about the x-axis is

$$V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) dx.$$

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Question 6 - Solution

We derive the formula using washers (disks with holes).

Partition the interval $[a, b]$ as

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

Over a small interval near x_i^*, the region between the curves has an outer radius

$$R_i = f(x_i^*)$$

and an inner radius

$$r_i = g(x_i^*).$$

When this vertical strip is revolved about the x-axis, it forms a thin washer.

The area of the outer disk is

$$\pi R_i^2 = \pi[f(x_i^*)]^2,$$

and the area of the inner disk (the hole) is

$$\pi r_i^2 = \pi[g(x_i^*)]^2.$$

Thus, the area of the washer is

$$\pi\big([f(x_i^*)]^2 - [g(x_i^*)]^2\big).$$

Multiplying by the thickness Δx_i, the volume of the washer is

$$\Delta V_i = \pi\big([f(x_i^*)]^2 - [g(x_i^*)]^2\big)\Delta x_i.$$

Adding the volumes of all washers gives an approximation to the total volume:

$$\sum_{i=1}^{n} \pi\big([f(x_i^*)]^2 - [g(x_i^*)]^2\big)\Delta x_i.$$

As the partition is refined and $\|P\| \to 0$, the washers fill the solid more accurately. Since f and g are continuous on $[a, b]$, the limit of the sum exists.

Taking the limit yields

$$V = \lim_{\|P\|\to 0} \sum_{i=1}^{n} \pi\big([f(x_i^*)]^2 - [g(x_i^*)]^2\big)\Delta x_i = \pi \int_a^b \big([f(x)]^2 - [g(x)]^2\big)\, dx.$$

$$\boxed{V = \pi \int_a^b \big([f(x)]^2 - [g(x)]^2\big)\, dx}$$

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Question 7

Prove that the area of a circle of radius R is

$$A = \pi R^2.$$

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Question 7 - Solution

We derive the area formula using integration.

Consider a circle of radius R centered at the origin. Its equation is

$$x^2 + y^2 = R^2.$$

Solving for y gives the upper semicircle:

$$y = \sqrt{R^2 - x^2}.$$

The area of the entire circle is twice the area of the upper semicircle:

$$A = 2\int_{-R}^{R} \sqrt{R^2 - x^2}\, dx.$$

To evaluate the integral, use the substitution

$$x = R\sin\theta, \quad dx = R\cos\theta\, d\theta.$$

When $x = -R$, $\theta = -\frac{\pi}{2}$, and when $x = R$, $\theta = \frac{\pi}{2}$.

Substitute into the integral:

$$\int_{-R}^{R} \sqrt{R^2 - x^2}\, dx = \int_{-\pi/2}^{\pi/2} \sqrt{R^2 - R^2\sin^2\theta}\,(R\cos\theta)\, d\theta.$$

Simplify:

$$\sqrt{R^2(1 - \sin^2\theta)} = R\cos\theta.$$

Thus,

$$\int_{-R}^{R} \sqrt{R^2 - x^2}\, dx = \int_{-\pi/2}^{\pi/2} R^2\cos^2\theta\, d\theta.$$

Use the identity

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}.$$

Then

$$\int_{-\pi/2}^{\pi/2} R^2\cos^2\theta\, d\theta = \frac{R^2}{2}\int_{-\pi/2}^{\pi/2} \big(1 + \cos(2\theta)\big)\, d\theta.$$

Evaluate:

$$\int_{-\pi/2}^{\pi/2} 1\, d\theta = \pi, \qquad \int_{-\pi/2}^{\pi/2} \cos(2\theta)\, d\theta = 0.$$

So,

$$\int_{-R}^{R} \sqrt{R^2 - x^2}\, dx = \frac{R^2}{2}\pi.$$

Finally, multiply by 2 to get the area of the full circle:

$$A = 2 \cdot \frac{R^2}{2}\pi = \pi R^2.$$

$$\boxed{A = \pi R^2}$$

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Question 8

Prove that the volume of a right circular cylinder with radius R and height h is

$$V = \pi R^2 h.$$

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Question 8 - Solution

We derive the volume formula using cross-sectional areas and integration.

Place the cylinder so that its base lies in the xy-plane and its axis is along the x-axis. Then the cylinder extends from $x = 0$ to $x = h$.

At any position x with $0 \le x \le h$, a cross-section perpendicular to the x-axis is a circle of radius R.

The area of this circular cross-section is constant and equal to

$$A(x) = \pi R^2.$$

The volume of the cylinder can be approximated by slicing it into thin slabs of thickness dx. Each slab has volume approximately

$$dV = A(x)\,dx = \pi R^2\,dx.$$

Adding the volumes of all such slabs gives

$$V = \int_0^h \pi R^2\,dx.$$

Since πR^2 is constant, evaluate the integral:

$$V = \pi R^2 \int_0^h dx = \pi R^2 [x]_0^h = \pi R^2 h.$$

$$\boxed{V = \pi R^2 h}$$

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Question 9

Prove that the volume of a right circular cone with base radius R and height h is

$$V = \frac{1}{3}\pi R^2 h.$$

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Question 9 - Solution

We derive the formula using cross-sectional disks and integration.

Place the cone so that its tip is at the origin and its base lies in the plane $x = h$. The axis of the cone lies along the x-axis.

At a distance x from the tip, where $0 \le x \le h$, the radius of the cone is determined by similar triangles.

Since the radius grows linearly from 0 to R over the height h, the radius at position x is

$$r(x) = \frac{R}{h}x.$$

A cross-section perpendicular to the x-axis at position x is a circle of radius $r(x)$. Its area is

$$A(x) = \pi[r(x)]^2 = \pi\left(\frac{R}{h}x\right)^2 = \pi\frac{R^2}{h^2}x^2.$$

An infinitesimal slice of thickness dx has volume

$$dV = A(x)\,dx = \pi\frac{R^2}{h^2}x^2\,dx.$$

Integrate from $x = 0$ to $x = h$ to obtain the total volume:

$$V = \int_0^h \pi\frac{R^2}{h^2}x^2\,dx.$$

Factor out the constants:

$$V = \pi\frac{R^2}{h^2}\int_0^h x^2\,dx.$$

Evaluate the integral:

$$\int_0^h x^2\,dx = \left[\frac{x^3}{3}\right]_0^h = \frac{h^3}{3}.$$

Substitute back:

$$V = \pi\frac{R^2}{h^2}\cdot\frac{h^3}{3} = \frac{1}{3}\pi R^2 h.$$

$$\boxed{V = \frac{1}{3}\pi R^2 h}$$

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Question 10

Prove that the volume of a sphere of radius R is

$$V = \frac{4}{3}\pi R^3.$$

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Question 10 - Solution

We derive the formula using the method of cross-sectional disks.

Place the sphere of radius R so that it is centered at the origin. Its equation is

$$x^2 + y^2 + z^2 = R^2.$$

Fix a value of x with $-R \le x \le R$. A cross-section perpendicular to the x-axis is a circle whose radius depends on x.

Solving the equation of the sphere for $y^2 + z^2$ gives

$$y^2 + z^2 = R^2 - x^2.$$

Thus, the radius of the cross-sectional disk at position x is

$$r(x) = \sqrt{R^2 - x^2}.$$

The area of this circular cross-section is

$$A(x) = \pi[r(x)]^2 = \pi(R^2 - x^2).$$

An infinitesimally thin slice of thickness dx has volume

$$dV = A(x)\,dx = \pi(R^2 - x^2)\,dx.$$

Integrate from $x = -R$ to $x = R$ to obtain the total volume:

$$V = \int_{-R}^{R} \pi(R^2 - x^2)\,dx.$$

Factor out π:

$$V = \pi \int_{-R}^{R} (R^2 - x^2)\,dx.$$

Evaluate the integral:

$$\int_{-R}^{R} R^2\,dx = 2R^3, \qquad \int_{-R}^{R} x^2\,dx = \frac{2R^3}{3}.$$

Thus,

$$\int_{-R}^{R} (R^2 - x^2)\,dx = 2R^3 - \frac{2R^3}{3} = \frac{4R^3}{3}.$$

Substitute back:

$$V = \pi \cdot \frac{4R^3}{3} = \frac{4}{3}\pi R^3.$$

$$\boxed{V = \frac{4}{3}\pi R^3}$$

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Question 1

Prove that

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0.$$

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Question 1 - Solution

We compare the growth rates of the numerator and denominator.

For $x > 0$, both $\ln x$ and x are differentiable. Apply L'Hôpital's Rule to the limit

$$\lim_{x\to\infty} \frac{\ln x}{x}.$$

Differentiate the numerator and denominator:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \qquad \frac{d}{dx}(x) = 1.$$

Thus,

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x}.$$

From basic limits,

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

Therefore,

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0.$$

$$\boxed{0}$$

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Question 2

Prove that

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0 \quad \text{for any positive integer } n.$$

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Question 2 - Solution

We compare the growth rates of the polynomial x^n and the exponential function e^x.

Both x^n and e^x are differentiable for all real x. Apply L'Hôpital's Rule repeatedly to the limit

$$\lim_{x\to\infty} \frac{x^n}{e^x}.$$

Differentiate the numerator and denominator once:

$$\frac{d}{dx}(x^n) = nx^{n-1}, \qquad \frac{d}{dx}(e^x) = e^x.$$

Thus,

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x}.$$

Apply L'Hôpital's Rule again. After n applications, the numerator becomes a constant:

$$\lim_{x\to\infty} \frac{n!}{e^x}.$$

Since e^x grows without bound as $x \to \infty$,

$$\lim_{x\to\infty} \frac{n!}{e^x} = 0.$$

Therefore,

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

$$\boxed{0}$$

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Question 3

Prove that

$$\lim_{x\to\infty} \sqrt{x^2+x} - x = \frac{1}{2}.$$

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Question 3 - Solution

We begin by algebraically simplifying the expression.

Consider

$$\sqrt{x^2+x}-x.$$

Multiply and divide by the conjugate:

$$\sqrt{x^2+x}-x=\frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x}.$$

Simplify the numerator:

$$(x^2+x)-x^2=x.$$

Thus,

$$\sqrt{x^2+x}-x=\frac{x}{\sqrt{x^2+x}+x}.$$

Factor x out of the square root in the denominator:

$$\sqrt{x^2+x}=x\sqrt{1+\frac{1}{x}}.$$

Substitute:

$$\frac{x}{x\sqrt{1+\frac{1}{x}}+x}=\frac{x}{x\left(\sqrt{1+\frac{1}{x}}+1\right)}.$$

Cancel x:

$$\frac{1}{\sqrt{1+\frac{1}{x}}+1}.$$

Now take the limit as $x\to\infty$:

$$\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1}=\frac{1}{\sqrt{1+0}+1}=\frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$

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Question 4

Prove that

$$\lim_{x\to\infty}\left(x-\sqrt{x^2-3x}\right)=\frac{3}{2}.$$

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Question 4 - Solution

We simplify the expression using algebraic manipulation.

Consider

$$x - \sqrt{x^2 - 3x}.$$

Multiply and divide by the conjugate:

$$x - \sqrt{x^2 - 3x} = \frac{(x - \sqrt{x^2 - 3x})(x + \sqrt{x^2 - 3x})}{x + \sqrt{x^2 - 3x}}.$$

Simplify the numerator:

$$x^2 - (x^2 - 3x) = 3x.$$

Thus,

$$x - \sqrt{x^2 - 3x} = \frac{3x}{x + \sqrt{x^2 - 3x}}.$$

Factor x out of the square root in the denominator:

$$\sqrt{x^2 - 3x} = x\sqrt{1 - \frac{3}{x}}.$$

Substitute into the expression:

$$\frac{3x}{x + x\sqrt{1 - \frac{3}{x}}} = \frac{3}{1 + \sqrt{1 - \frac{3}{x}}}.$$

Now take the limit as $x \to \infty$:

$$\lim_{x \to \infty} \frac{3}{1 + \sqrt{1 - \frac{3}{x}}} = \frac{3}{1 + \sqrt{1}} = \frac{3}{2}.$$

$$\boxed{\frac{3}{2}}$$

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Question 5

Prove that

$$\lim_{x\to\infty}\left(\sqrt{x^2+5x}-\sqrt{x^2+x}\right)=2.$$

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Question 5 - Solution

We simplify the expression by factoring out x from each square root.

Consider

$$\sqrt{x^2+5x}-\sqrt{x^2+x}.$$

Factor x^2 inside each square root:

$$\sqrt{x^2\left(1+\frac{5}{x}\right)}-\sqrt{x^2\left(1+\frac{1}{x}\right)}.$$

Since $x>0$ for large x, this becomes

$$x\left(\sqrt{1+\frac{5}{x}}-\sqrt{1+\frac{1}{x}}\right).$$

Multiply and divide by the conjugate:

$$x\cdot\frac{\left(\sqrt{1+\frac{5}{x}}-\sqrt{1+\frac{1}{x}}\right)\left(\sqrt{1+\frac{5}{x}}+\sqrt{1+\frac{1}{x}}\right)}{\sqrt{1+\frac{5}{x}}+\sqrt{1+\frac{1}{x}}}.$$

Simplify the numerator:

$$\left(1+\frac{5}{x}\right)-\left(1+\frac{1}{x}\right)=\frac{4}{x}.$$

Thus the expression becomes

$$x\cdot\frac{\frac{4}{x}}{\sqrt{1+\frac{5}{x}}+\sqrt{1+\frac{1}{x}}}=\frac{4}{\sqrt{1+\frac{5}{x}}+\sqrt{1+\frac{1}{x}}}.$$

Now take the limit as $x\to\infty$:

$$\lim_{x\to\infty}\frac{4}{\sqrt{1+\frac{5}{x}}+\sqrt{1+\frac{1}{x}}}=\frac{4}{1+1}=2.$$

$$\boxed{2}$$

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Question 6

Prove that

$$\lim_{x\to\infty}\left(\sqrt{x^2+4x+1}-x\right)=2.$$

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Question 6 - Solution

Start with the expression

$$\sqrt{x^2+4x+1}-x.$$

Multiply and divide by the conjugate:

$$\sqrt{x^2+4x+1}-x=\frac{(\sqrt{x^2+4x+1}-x)(\sqrt{x^2+4x+1}+x)}{\sqrt{x^2+4x+1}+x}.$$

Simplify the numerator:

$$(x^2+4x+1)-x^2=4x+1.$$

Thus,

$$\sqrt{x^2+4x+1}-x=\frac{4x+1}{\sqrt{x^2+4x+1}+x}.$$

Factor x from the square root in the denominator:

$$\sqrt{x^2+4x+1}=x\sqrt{1+\frac{4}{x}+\frac{1}{x^2}}.$$

Substitute:

$$\frac{4x+1}{x\sqrt{1+\frac{4}{x}+\frac{1}{x^2}}+x}=\frac{4+\frac{1}{x}}{\sqrt{1+\frac{4}{x}+\frac{1}{x^2}}+1}.$$

Take the limit as $x\to\infty$:

$$\lim_{x\to\infty}\frac{4+\frac{1}{x}}{\sqrt{1+\frac{4}{x}+\frac{1}{x^2}}+1}=\frac{4}{1+1}=2.$$

$$\boxed{2}$$

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Question 7

Prove that

$$\lim_{x\to\infty} x\left(\sqrt{x^2+1}-x\right)=\frac{1}{2}.$$

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Question 7 - Solution

Begin with the expression

$$x\big(\sqrt{x^2+1}-x\big).$$

Factor the expression inside the parentheses using a conjugate:

$$\sqrt{x^2+1}-x=\frac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x}=\frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x}=\frac{1}{\sqrt{x^2+1}+x}.$$

Substitute this back:

$$x\big(\sqrt{x^2+1}-x\big)=\frac{x}{\sqrt{x^2+1}+x}.$$

Factor x out of the square root in the denominator:

$$\sqrt{x^2+1}=x\sqrt{1+\frac{1}{x^2}}.$$

Thus,

$$\frac{x}{x\sqrt{1+\frac{1}{x^2}}+x}=\frac{1}{\sqrt{1+\frac{1}{x^2}}+1}.$$

Now take the limit as $x\to\infty$:

$$\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x^2}}+1}=\frac{1}{1+1}=\frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$

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Question 8

Prove that

$$\lim_{x\to\infty} (\ln(x+1) - \ln x) = 0.$$

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Question 8 - Solution

Combine the logarithms using logarithmic properties:

$$\ln(x+1) - \ln x = \ln\left(\frac{x+1}{x}\right).$$

Rewrite the fraction:

$$\frac{x+1}{x} = 1 + \frac{1}{x}.$$

Thus, the expression becomes

$$\ln\left(1 + \frac{1}{x}\right).$$

Now take the limit as $x \to \infty$. Since

$$\lim_{x\to\infty} \frac{1}{x} = 0,$$

and the natural logarithm function is continuous at 1, we obtain

$$\lim_{x\to\infty} \ln\left(1 + \frac{1}{x}\right) = \ln(1) = 0.$$

$$\boxed{0}$$

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Question 9

Prove that

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = 0.$$

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Question 9 - Solution

We compare the growth rates of the logarithmic function and the power function.

Both $\ln x$ and $\sqrt{x}$ are positive and differentiable for $x > 0$. Apply L'Hôpital's Rule to the limit

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}.$$

Differentiate the numerator and denominator:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \qquad \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}.$$

Thus,

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x}.$$

Simplify:

$$\frac{2\sqrt{x}}{x} = \frac{2}{\sqrt{x}}.$$

Since

$$\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0,$$

it follows that

$$\lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

Therefore,

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = 0.$$

$$\boxed{0}$$

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Question 10

Prove that

$$\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x = \frac{1}{e}.$$

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Question 10 - Solution

Let

$$L = \lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x.$$

Take the natural logarithm:

$$\ln L = \lim_{x\to\infty} x \ln\left(\frac{x}{x+1}\right).$$

Rewrite the logarithm:

$$\ln\left(\frac{x}{x+1}\right) = \ln\left(\frac{1}{1+\frac{1}{x}}\right) = -\ln\left(1+\frac{1}{x}\right).$$

Thus,

$$\ln L = -\lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right).$$

Consider the limit

$$\lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right).$$

Rewrite it as a quotient:

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}.$$

This is an indeterminate form $\frac{0}{0}$, so apply L'Hôpital's Rule.
Differentiate numerator and denominator:

$$\frac{d}{dx}\ln\left(1+\frac{1}{x}\right) = \frac{-1}{x(x+1)}, \qquad \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

Thus,

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{-1}{x(x+1)}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{x^2}{x(x+1)} = \lim_{x\to\infty} \frac{x}{x+1} = 1.$$

Therefore,

$$\ln L = -1.$$

Exponentiating both sides:

$$L = e^{-1} = \frac{1}{e}.$$

$$\boxed{\frac{1}{e}}$$

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Question 1

Prove that for any positive integer n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

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Question 1 - Solution

We prove the formula using mathematical induction.

Base Case:
For $n = 1$,

$$\sum_{k=1}^{1} k = 1, \qquad \frac{1(1+1)}{2} = 1.$$

Thus, the formula holds for $n = 1$.

Inductive Hypothesis:
Assume that for some positive integer n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Inductive Step:
Consider the sum up to $n + 1$:

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1).$$

Using the inductive hypothesis,

$$\sum_{k=1}^{n+1} k = \frac{n(n+1)}{2} + (n+1).$$

Factor out $(n + 1)$:

$$\sum_{k=1}^{n+1} k = (n+1)\left(\frac{n}{2} + 1\right) = (n+1)\frac{n+2}{2}.$$

Thus,

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}.$$

This matches the given formula with n replaced by $n + 1$.

By mathematical induction, the formula holds for all positive integers n.

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$$

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Question 2

Prove that for any positive integer n,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

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Question 2 - Solution

We prove the formula using mathematical induction.

Base Case:
For $n = 1$,

$$\sum_{k=1}^{1} k^2 = 1^2 = 1,$$

and

$$\frac{1(1+1)(2\cdot 1+1)}{6} = \frac{6}{6} = 1.$$

Thus, the formula holds for $n = 1$.

Inductive Hypothesis:
Assume that for some positive integer n,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Inductive Step:
Consider the sum up to $n+1$:

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2.$$

Using the inductive hypothesis,

$$\sum_{k=1}^{n+1} k^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2.$$

Factor out $(n+1)$:

$$\sum_{k=1}^{n+1} k^2 = (n+1)\left(\frac{n(2n+1)}{6} + (n+1)\right).$$

Combine terms inside the parentheses:

$$\frac{n(2n+1)+6(n+1)}{6} = \frac{2n^2+7n+6}{6} = \frac{(2n+3)(n+2)}{6}.$$

Thus,

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}.$$

This matches the given formula with n replaced by $n+1$.

By mathematical induction, the formula holds for all positive integers n.

$$\boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}$$

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Question 3

Prove that for any positive integer n,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

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Question 3 - Solution

We prove the formula using mathematical induction.

Base Case:
For $n = 1$,

$$\sum_{k=1}^{1} k^3 = 1^3 = 1, \qquad \left(\frac{1(1+1)}{2}\right)^2 = 1.$$

Thus, the formula holds for $n = 1$.

Inductive Hypothesis:
Assume that for some positive integer n,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Inductive Step:
Consider the sum up to $n + 1$:

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3.$$

Using the inductive hypothesis,

$$\sum_{k=1}^{n+1} k^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3.$$

Factor out $(n + 1)^2$:

$$\sum_{k=1}^{n+1} k^3 = (n+1)^2 \left(\frac{n^2}{4} + n + 1\right).$$

Combine terms:

$$\frac{n^2 + 4n + 4}{4} = \frac{(n+2)^2}{4}.$$

Thus,

$$\sum_{k=1}^{n+1} k^3 = (n+1)^2 \cdot \frac{(n+2)^2}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2.$$

This matches the given formula with n replaced by $n + 1$.

By mathematical induction, the formula holds for all positive integers n.

$$\boxed{\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2}$$

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Question 4

Prove that for any positive integer n,

$$\sum_{k=1}^{n}(2k-1) = n^2.$$

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Question 4 - Solution

We prove the identity using mathematical induction.

Base Case:
For $n = 1$,

$$\sum_{k=1}^{1}(2k-1) = 2(1) - 1 = 1,$$

and

$$1^2 = 1.$$

Thus, the statement holds for $n = 1$.

Inductive Hypothesis:
Assume that for some positive integer n,

$$\sum_{k=1}^{n}(2k-1) = n^2.$$

Inductive Step:
Consider the sum up to $n+1$:

$$\sum_{k=1}^{n+1}(2k-1) = \sum_{k=1}^{n}(2k-1) + \big(2(n+1)-1\big).$$

Using the inductive hypothesis,

$$\sum_{k=1}^{n+1}(2k-1) = n^2 + (2n+1).$$

Simplify:

$$n^2 + 2n + 1 = (n+1)^2.$$

Thus,

$$\sum_{k=1}^{n+1}(2k-1) = (n+1)^2.$$

This matches the given formula with n replaced by $n+1$.

By mathematical induction, the identity holds for all positive integers n.

$$\boxed{\sum_{k=1}^{n}(2k-1) = n^2}$$

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Question 5

Prove that for any positive integer n,

$$\sum_{k=1}^{n}\big(3k+2\big) = \frac{3n(n+1)}{2} + 2n.$$

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Question 5 - Solution

We prove the identity using properties of summations.

Start with the given sum:

$$\sum_{k=1}^{n}(3k+2).$$

Split the sum using linearity:

$$\sum_{k=1}^{n}(3k+2)=\sum_{k=1}^{n}3k+\sum_{k=1}^{n}2.$$

Factor out constants:

$$\sum_{k=1}^{n}3k=3\sum_{k=1}^{n}k, \qquad \sum_{k=1}^{n}2=2\sum_{k=1}^{n}1.$$

Use known summation formulas:

$$\sum_{k=1}^{n}k=\frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n}1=n.$$

Substitute these results:

$$\sum_{k=1}^{n}(3k+2)=3\cdot\frac{n(n+1)}{2}+2n.$$

Thus,

$$\sum_{k=1}^{n}(3k+2)=\frac{3n(n+1)}{2}+2n.$$

$$\boxed{\sum_{k=1}^{n}(3k+2)=\frac{3n(n+1)}{2}+2n}$$

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Question 6

Prove that for any positive integer n,

$$\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \frac{n}{n+1}.$$

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Question 6 - Solution

Write out the first few terms of the sum:

$$\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right).$$

Notice that consecutive terms cancel:

$$-\frac{1}{2}+\frac{1}{2}=0,\quad -\frac{1}{3}+\frac{1}{3}=0,\quad \ldots$$

After all cancellations, only the first and last terms remain:

$$1-\frac{1}{n+1}.$$

Simplify:

$$1-\frac{1}{n+1}=\frac{n}{n+1}.$$

$$\boxed{\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\frac{n}{n+1}}$$

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Question 7

Prove that for any positive integer n,

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}.$$

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Question 7 - Solution

Begin by rewriting the general term using partial fractions.

For each $k \geq 1$,

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Substitute this into the sum:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right).$$

Write out the first few terms:

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

All intermediate terms cancel, leaving

$$1 - \frac{1}{n+1}.$$

Simplify:

$$1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

$$\boxed{\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}}$$

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Question 8

Prove that for any positive integer n and real number $r \neq 1$,

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}.$$

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Question 8 - Solution

Let

$$S = \sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n.$$

Multiply both sides by r:

$$rS = r + r^2 + r^3 + \cdots + r^{n+1}.$$

Subtract the second equation from the first:

$$S - rS = (1 + r + r^2 + \cdots + r^n) - (r + r^2 + \cdots + r^{n+1}).$$

All intermediate terms cancel, leaving

$$S - rS = 1 - r^{n+1}.$$

Factor the left-hand side:

$$(1 - r)S = 1 - r^{n+1}.$$

Since $r \neq 1$, divide both sides by $1 - r$:

$$S = \frac{1 - r^{n+1}}{1 - r}.$$

Thus,

$$\boxed{\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}}$$

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Question 9

Prove that for every positive integer n,

$$\sum_{k=1}^{n} \frac{k}{k+1} = n - \sum_{k=1}^{n} \frac{1}{k+1}.$$

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Question 9 - Solution

For every $k \geq 1$,

$$\frac{k}{k+1} = 1 - \frac{1}{k+1}.$$

Summing all n terms gives

$$\boxed{\sum_{k=1}^{n} \frac{k}{k+1} = n - \sum_{k=1}^{n} \frac{1}{k+1}.}$$

Equivalently, the result is $n + 1 - H_{n+1}$, where $H_{n+1} = \sum_{j=1}^{n+1} 1/j$.
For $n = 1$, both sides of the corrected identity equal $1/2$.

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Question 10

Prove that for any positive integer n,

$$\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1} = 1 - \frac{1}{n+1}.$$

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Question 10 - Solution

Write both sums explicitly:

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

$$\sum_{k=1}^{n} \frac{1}{k+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n+1}.$$

Subtract the second sum from the first:

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right).$$

All common terms cancel:

$$\frac{1}{2} - \frac{1}{2} = 0, \quad \frac{1}{3} - \frac{1}{3} = 0, \quad \ldots$$

Only the first term of the first sum and the last term of the second sum remain:

$$1 - \frac{1}{n+1}.$$

Thus,

$$\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1} = 1 - \frac{1}{n+1}.$$

$$\boxed{1 - \frac{1}{n+1}}$$

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Question 1

Suppose that F and G are antiderivatives of the same function f on an interval I. Prove that there exists a constant C such that

$$F(x) = G(x) + C \quad \text{for all } x \in I.$$

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Question 1 - Solution

Since F and G are both antiderivatives of f on I, we have

$$F'(x) = f(x) \quad \text{and} \quad G'(x) = f(x) \quad \text{for all } x \in I.$$

Consider the function

$$H(x) = F(x) - G(x).$$

Differentiate $H(x)$:

$$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0 \quad \text{for all } x \in I.$$

Thus, the derivative of H is zero on the interval I.

From the result that a function with zero derivative on an interval is constant, there exists a real number C such that

$$H(x) = C \quad \text{for all } x \in I.$$

Substituting back for $H(x)$ gives

$$F(x) - G(x) = C.$$

Rewriting,

$$F(x) = G(x) + C.$$

$$\boxed{F(x) = G(x) + C}$$

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Question 2

Suppose that $F'(x) = 0$ for all x in an interval I. Prove that F is constant on I.

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Question 2 - Solution

Let $x_1, x_2 \in I$ with $x_1 < x_2$. We will show that

$$F(x_1) = F(x_2).$$

Since $F'(x) = 0$ for all $x \in I$, the function F is differentiable on I and continuous on the closed interval $[x_1, x_2]$.

By the Mean Value Theorem, there exists a number $c \in (x_1, x_2)$ such that

$$F'(c) = \frac{F(x_2) - F(x_1)}{x_2 - x_1}.$$

But $F'(c) = 0$ by assumption, so

$$0 = \frac{F(x_2) - F(x_1)}{x_2 - x_1}.$$

Since $x_2 - x_1 \neq 0$, it follows that

$$F(x_2) - F(x_1) = 0.$$

Thus,

$$F(x_2) = F(x_1).$$

Because x_1 and x_2 were arbitrary points in I, the function F has the same value at every point of I.

$$\boxed{F \text{ is constant on } I}$$

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Question 3

Let f be a continuous function on an interval I. Suppose that

$$\int f(x)\,dx = F(x) + C_1 \quad \text{and} \quad \int f(x)\,dx = G(x) + C_2$$

are two indefinite integrals of f. Prove that

$$F(x) - G(x) = \text{constant}.$$

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Question 3 - Solution

Since both expressions represent antiderivatives of f, we have

$$F'(x) = f(x) \quad \text{and} \quad G'(x) = f(x) \quad \text{for all } x \in I.$$

Consider the difference of the two functions:

$$H(x) = F(x) - G(x).$$

Differentiate $H(x)$:

$$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0 \quad \text{for all } x \in I.$$

Thus, the derivative of H is zero on the interval I.

From the Mean Value Theorem, a function whose derivative is zero everywhere on an interval must be constant. Therefore, there exists a real number K such that

$$H(x) = K \quad \text{for all } x \in I.$$

Substituting back for $H(x)$ gives

$$F(x) - G(x) = K.$$

$$\boxed{F(x) - G(x) = \text{constant}}$$

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Question 4

Suppose that f is continuous on an interval I and that

$$F'(x) = f(x) \quad \text{for all } x \in I.$$

Show that every solution of the differential equation

$$y' = f(x)$$

has the form

$$y = F(x) + C$$

for some constant C.

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Question 4 - Solution

Let y be any solution of the differential equation

$$y' = f(x)$$

on the interval I.

Since $F'(x) = f(x)$ and $y'(x) = f(x)$ for all $x \in I$, we have

$$y'(x) - F'(x) = 0 \quad \text{for all } x \in I.$$

Consider the function

$$H(x) = y(x) - F(x).$$

Differentiate $H(x)$:

$$H'(x) = y'(x) - F'(x) = 0 \quad \text{for all } x \in I.$$

Thus, $H'(x) = 0$ on I.

By the result that a function with zero derivative on an interval is constant, there exists a real number C such that

$$H(x) = C \quad \text{for all } x \in I.$$

Substituting back gives

$$y(x) - F(x) = C.$$

Rewriting,

$$y = F(x) + C.$$

$$\boxed{y = F(x) + C}$$

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Question 5

Suppose that f and g are continuous on a nondegenerate interval I, that $a \in I$, and that

$$\int_a^x f(t)\,dt = \int_a^x g(t)\,dt \quad \text{for all } x \in I.$$

Prove that

$$f(x) = g(x) \quad \text{for all } x \in I.$$

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Question 5 - Solution

Set $F(x) = \int_a^x f(t)\,dt$ and $G(x) = \int_a^x g(t)\,dt$.

The hypothesis gives $F = G$ on I. At every interior point x of I, continuity of both integrands allows the Fundamental Theorem of Calculus:

$$f(x) = F'(x) = G'(x) = g(x).$$

At an included endpoint, take limits from the interior and use continuity of f and g. Thus

$$\boxed{f(x) = g(x) \quad \text{for every } x \in I.}$$

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Question 6

Let f be continuous on an interval I, and fix one antiderivative F. Write the general antiderivative as

$$y(x) = F(x) + C.$$

Show that the condition

$$y(x_0) = y_0, \qquad x_0 \in I,$$

uniquely determines C.

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Question 6 - Solution

Fix an antiderivative F of f on I. Every other antiderivative has the form $y(x) = F(x) + C$.

Imposing $y(x_0) = y_0$ yields $F(x_0) + C = y_0$, so

$$\boxed{C = y_0 - F(x_0).}$$

This value is unique because F, x_0, and y_0 are fixed.

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Question 7

Suppose that f is continuous on an interval I and that

$$\int_a^x f(t)\,dt = F(x) + C$$

for some constant C. Show that choosing a different lower limit $b \in I$ changes the antiderivative only by a constant.

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Question 7 - Solution

Consider the two functions

$$F_1(x) = \int_a^x f(t)\,dt, \qquad F_2(x) = \int_b^x f(t)\,dt.$$

Both F_1 and F_2 are antiderivatives of f on I by the Fundamental Theorem of Calculus.

Evaluate the difference between them:

$$F_1(x) - F_2(x) = \int_a^x f(t)\,dt - \int_b^x f(t)\,dt.$$

Using properties of definite integrals,

$$\int_a^x f(t)\,dt - \int_b^x f(t)\,dt = \int_a^b f(t)\,dt.$$

The right-hand side is a fixed real number that does not depend on x.

Therefore,

$$F_1(x) - F_2(x) = \text{constant}.$$

This shows that changing the lower limit of integration alters the antiderivative only by an additive constant.

$$\boxed{\int_a^x f(t)\,dt = \int_b^x f(t)\,dt + \text{constant}}$$

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Question 8

Suppose that f is continuous on an interval I and that

$$\int f(x)\,dx = F(x) + C.$$

Prove that if

$$F'(x) = 0 \quad \text{for all } x \in I,$$

then

$$\int f(x)\,dx = C.$$

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Question 8 - Solution

Since

$$\int f(x)\,dx = F(x) + C,$$

the function $F(x)$ is an antiderivative of f.

Given that

$$F'(x) = 0 \quad \text{for all } x \in I,$$

it follows from the Mean Value Theorem that F is constant on the interval I.

Thus, there exists a real number K such that

$$F(x) = K \quad \text{for all } x \in I.$$

Substitute this into the expression for the indefinite integral:

$$\int f(x)\,dx = K + C.$$

Since $K + C$ is itself a constant, we may write

$$\int f(x)\,dx = C'.$$

Therefore, if the derivative of an antiderivative is zero on an interval, the indefinite integral reduces to a constant.

$$\boxed{\int f(x)\,dx = C}$$

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Question 9

Suppose that f is continuous on an interval I and that

$$\int f(x)\,dx = F(x) + C.$$

Show that if two antiderivatives of f agree at a single point, then they are identical on the entire interval.

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Question 9 - Solution

Let F and G be two antiderivatives of f on the interval I. Then

$$F'(x) = f(x) \quad \text{and} \quad G'(x) = f(x) \quad \text{for all } x \in I.$$

Assume that there exists a point $x_0 \in I$ such that

$$F(x_0) = G(x_0).$$

Consider the function

$$H(x) = F(x) - G(x).$$

Differentiate $H(x)$:

$$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0 \quad \text{for all } x \in I.$$

Thus, H has zero derivative on I, so H is constant on I. That is,

$$H(x) = K \quad \text{for all } x \in I.$$

Evaluate H at x_0:

$$H(x_0) = F(x_0) - G(x_0) = 0.$$

Hence, $K = 0$, and therefore

$$F(x) - G(x) = 0 \quad \text{for all } x \in I.$$

This implies

$$F(x) = G(x) \quad \text{for all } x \in I.$$

$$\boxed{F(x) \equiv G(x)}$$

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Question 10

Suppose that f is continuous on an interval I and that

$$\int f(x)\,dx = F(x) + C.$$

Show that the family of curves

$$y = F(x) + C$$

consists of vertical translations of a single curve.

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Question 10 - Solution

Consider two members of the family of antiderivatives:

$$y_1 = F(x) + C_1, \qquad y_2 = F(x) + C_2,$$

where C_1 and C_2 are constants.

Subtract the equations:

$$y_2 - y_1 = (F(x) + C_2) - (F(x) + C_1) = C_2 - C_1.$$

The difference $C_2 - C_1$ is a constant independent of x. Thus, for every value of x, the vertical distance between the two curves is the same.

This means that changing the constant of integration does not alter the shape of the graph, but only shifts it upward or downward by a fixed amount.

Therefore, the family of functions

$$y = F(x) + C$$

represents all vertical translations of a single curve.

Different values of C produce vertical shifts of the same graph

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